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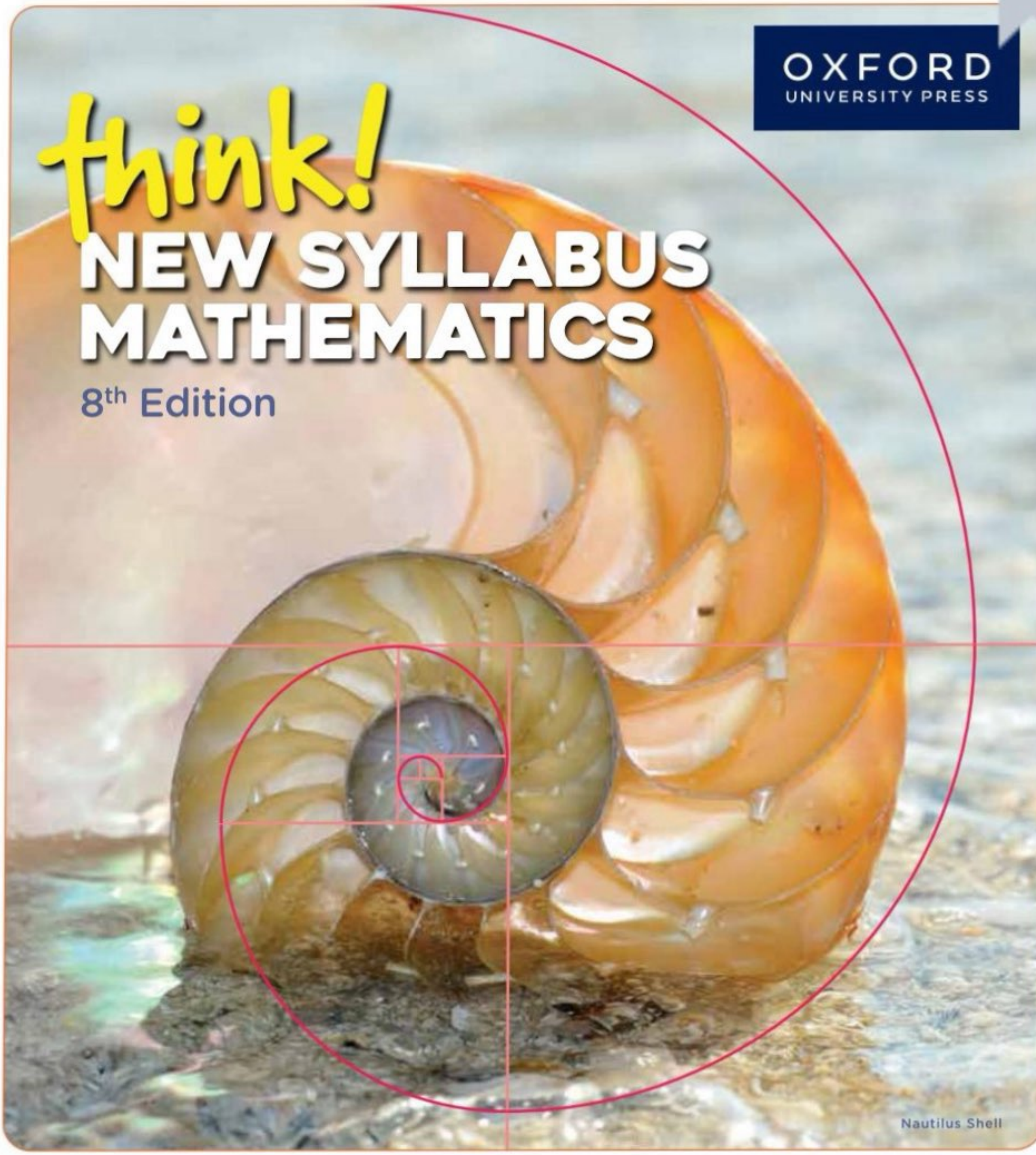
# think! NEW SYLLABUS MATHEMATICS

8th Edition



Nautilus Shell

For
Cambridge O Level and Cambridge IGCSE Mathematics

# 2

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# PREFACE

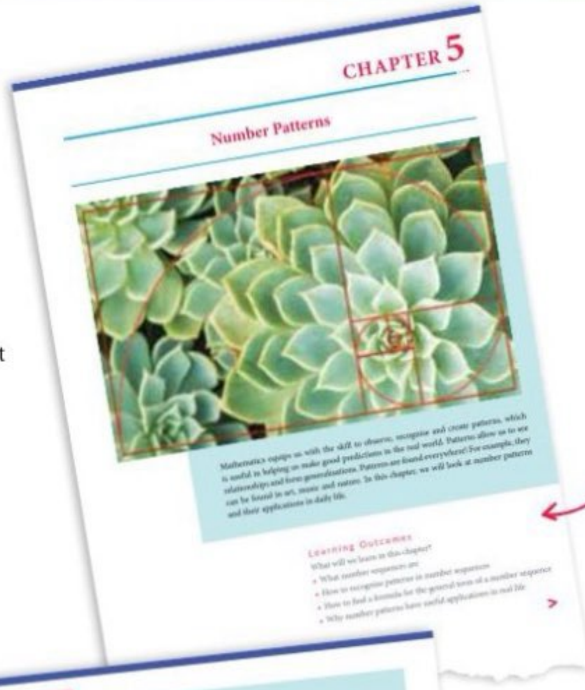
**think! Mathematics** is a series of textbooks specially designed to provide students valuable **learning experiences** by engaging their minds and hearts as they learn mathematics.

The features of this textbook series reflect the important shifts towards the development of 21st century competencies and a greater appreciation of mathematics, as articulated in the Singapore mathematics curriculum and other international curricula. Every chapter begins with a Chapter Opener and an Introductory Problem to motivate the development of the key concepts in the topic. The Chapter Opener gives a coherent overview of the **big ideas** that will frame the study of the topic, while the Introductory Problem positions problem solving at the heart of learning mathematics. Two key considerations guide the development of every chapter – seeing mathematics as a tool and as a discipline. Opportunities to engage in Investigation, Class Discussion, Thinking Time, Journal Writing and Performance Tasks are woven throughout the textbook to enhance students' learning experiences. Stories, songs, videos and puzzles serve to arouse interest and pique curiosity. Real-life examples serve to influence students to appreciate the beauty and usefulness of mathematics in their surroundings.

Underpinning the writing of this textbook series is the belief that all students can learn and appreciate mathematics. Worked Examples are carefully selected, questions in the Reflection section prompt students to reflect on their learning, and problems are of varying difficulty levels to ensure a high baseline of mastery, and to stretch students with special interest in mathematics. The use of ICT helps students to visualise and manipulate mathematical objects with ease, hence promoting interactivity.

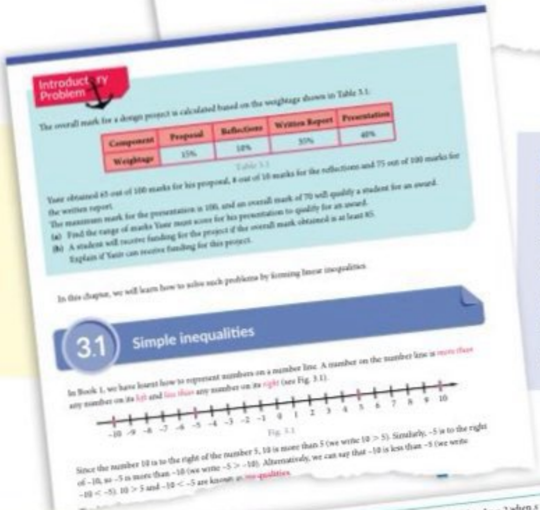
We hope you will enjoy the subject as we embark on this exciting journey together to develop important mathematical dispositions that will certainly see you through beyond the examinations, to appreciate mathematics as an important tool in life, and as a discipline of the mind.

# KEY FEATURES



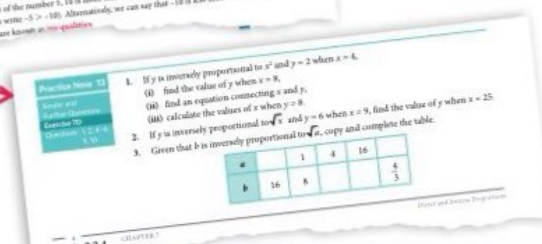

**Chapter Opener**
gives students an overview of the topic. It includes *rationales* for learning the chapter to arouse students' *interest* and **big ideas** that *connect* the concepts within the chapter or with other chapters.

**Learning Outcomes**
help students to be aware of what they are about to study so as to *monitor* their progress.



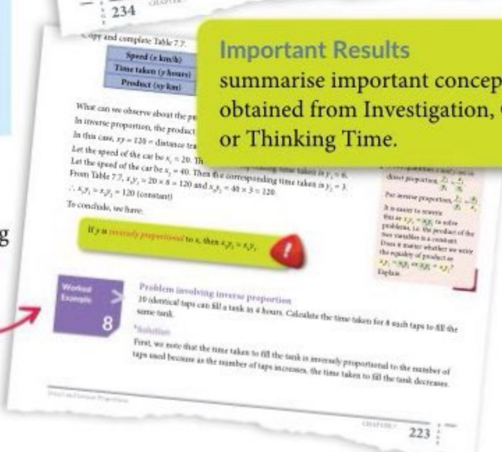

**Introductory Problem**
provides students with a more specific *motivation* to learn the topic, using a problem that helps develop a concept, or an application problem that students will revisit after they have gained necessary knowledge from the chapter.

**Recap**
revisits relevant prerequisites at the beginning of the chapter or at appropriate junctures so that students are *ready* to learn new knowledge built on their existing schema.




**Practise Now**
consists of questions that help students achieve *mastery* of procedural *skills*. Puzzles are sometimes used for consolidation to make practice *motivating* and fun.

**Similar and Further Questions**
follow after Practise Now to help teachers select appropriate questions for students' self-practice.




**Important Results**
summarise important concepts or formulae obtained from Investigation, Class Discussion or Thinking Time.

**Worked Example**
shows students how to present their working clearly when solving related *problems*. In more challenging worked examples, **Pólya's Problem Solving Model** is used to help students learn how to address a problem.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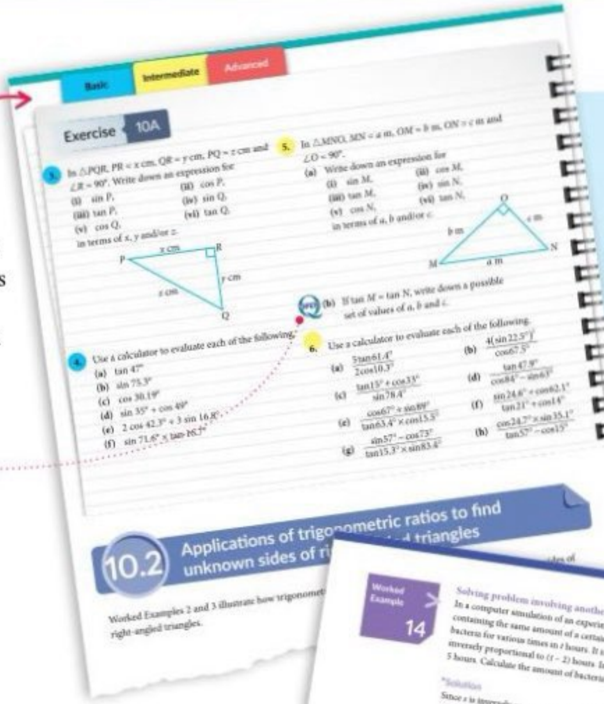
## Exercise

questions are classified into three levels of difficulty – **Basic**, **Intermediate** and **Advanced**. Questions at the Basic level are usually short-answer items to test basic concepts and skills. The Intermediate level contains more structured questions, while the Advanced level involves applications and higher order thinking skills.

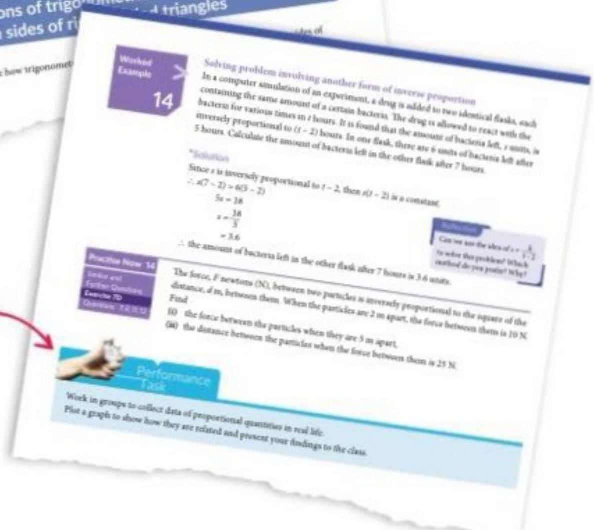


## Explanation Questions

require students to communicate their explanations in writing and are spread throughout the textbook.

## Open-ended Problems

are mathematics problems with more than one correct answer. Solving such problems expose students to real-world problems.

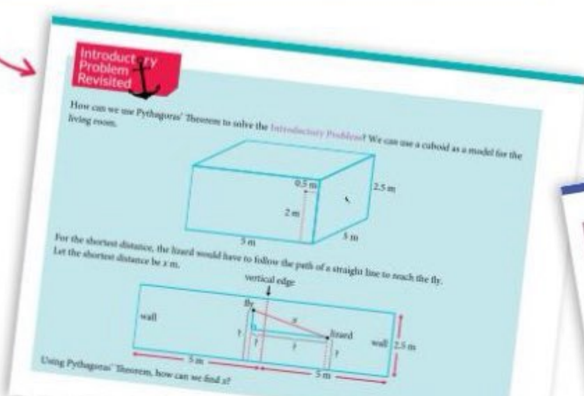


## Performance Task

consists of mini-projects designed to develop research and presentation skills of students, through writing a report and/or giving an oral presentation.

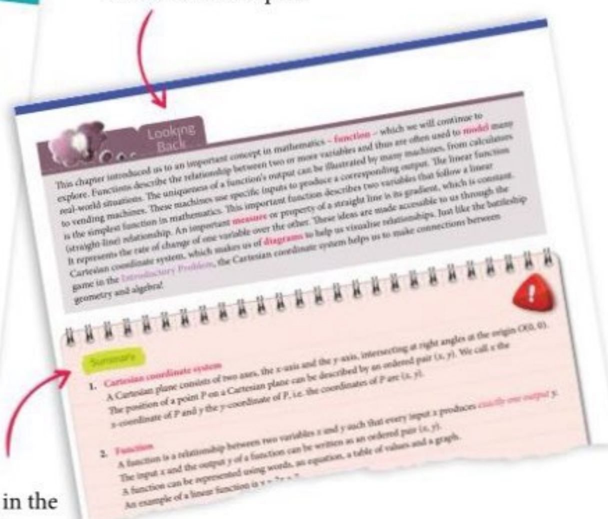
## Introductory Problem Revisited

revisits an application-based Introductory Problem later in the chapter. This is absent if the Introductory Problem leads directly to the development of a concept.



## Looking Back

complements the Chapter Opener and helps students internalise the **big ideas** that they have learnt in the chapter.



## Summary

*compounds* the key concepts taught in the chapter in a succinct manner. Questions are included to help students *reflect* on their learning.

### Investigation

Guided investigation provides students the relevant *learning experiences* to explore and discover important mathematical *concepts*. It usually takes the **Concrete-Pictorial-Abstract (C-P-A)** approach to help students construct their knowledge meaningfully. The connections between concrete experiences (manipulative or examples), different pictorial representations and symbolic representations are explicitly made. Some investigations may also involve the use of **Information and Communication Technology (ICT)**.

### Class Discussion

Questions are provided to **engage** students in discussion, with the teacher acting as the facilitator. Class discussions provide students the relevant *learning experiences* to think and *reason* mathematically, enhance their oral *communication* skills, and learn new *concepts* and *skills*.

### Thinking Time

Key questions are included at appropriate junctures to provide students the relevant *learning experiences* to think critically on their own before sharing their thoughts with their classmates. Mathematical fallacies are sometimes included to check and test students' understanding.

### Journal Writing

Journal writing provides opportunities for students to *reflect* on their learning and to *communicate* mathematically in writing. It can also be used as a formative assessment for the teacher to provide feedback for their students.

### Reflection

Students are usually required to reflect on what they have learnt at the end of each section so as to *monitor* and *regulate* their own learning. The reflection questions provided can be generic prompts or specific to the topics in the section or chapter, to check if students have understood the key ideas.

# MARGINAL NOTES

**Big Idea**

This provides additional details of the big idea mentioned in the main text.

**Recall**

Unlike the key feature 'Recap' in the main text, this contains just-in-time recall of prerequisite knowledge that students have already learnt.

**Attention**

This contains important information that students should know.

**Information**

This includes information that may be of interest to students.

**Reflection**

This guides students to think about different methods used to solve a problem.

**Problem-solving Tip**

This guides students on how to approach a problem in Worked Examples or Practise Now.

**Internet Resources**

This guides students to search the Internet for valuable information or interesting online games for their independent and self-directed learning.

**Just For Fun**

This contains puzzles, fascinating facts and interesting stories about mathematics as enrichment for students.

# CONTENTS

## CHAPTER 1



## CHAPTER 2



## CHAPTER 5



## CHAPTER 3



## CHAPTER 6




## CHAPTER 4

# CHAPTER 7




# CHAPTER 10



# CHAPTER 8



# CHAPTER 11




# CHAPTER 9



# CHAPTER 12

# CHAPTER 1

## Linear Functions and Graphs

What do our SNIC numbers, car plate numbers, vending machines, calculators and keyboards have in common? All of them show a relation between an input and a unique corresponding output. This special relation is known as a **function**, an important concept in mathematics. In 1673, the famous mathematician Leibniz used the word "function" to represent any quantity varying from point to point on a curve, and eventually used it to represent quantities that depend on a variable. However, it was the invention of the Cartesian coordinate system in the 17th century by René Descartes that pushed the boundaries of mathematics by connecting geometry to algebra, and placed functions at the heart of mathematics. In this chapter, we will be introduced to the concept of function, learn about the connections between geometry and algebra (enabled by the Cartesian coordinate system), and explore one of the simplest geometrical objects — the straight line.

### Learning Outcomes

What will we learn in this chapter?

- What linear functions and graphs are
- How to plot points on graph paper and draw the graph of a linear function
- How to find the gradient of a straight line and state its $y$-intercept
- Why linear functions and graphs have useful applications in real-world contexts

## Introductory Problem

In this two-player battleship game, each player is to place five ships on a square grid, as shown in Fig. 1.1, without letting his/her opponent see the grid. Players will then take turns to call out a square on the grid. Part of a ship is hit when a square that it occupies is called out. Otherwise, it will be a miss. The objective is to sink all the ships on your opponent's board.

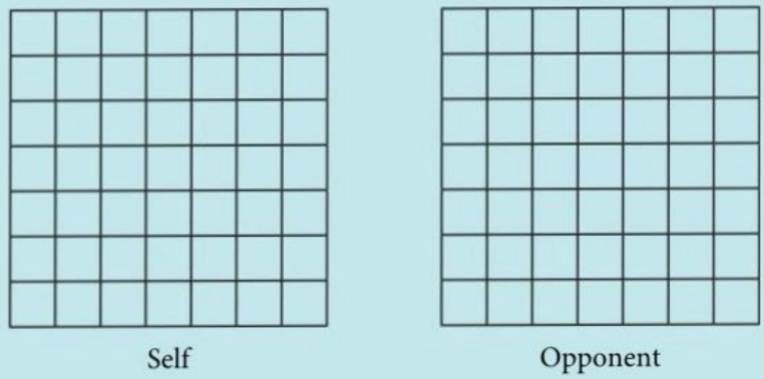



Fig. 1.1

Follow the instructions below.

**A.** On the grid labelled 'Self', arrange the following five ships, where each ▢ occupies one square grid:

- An aircraft carrier P P P P P
- A battleship Q Q Q Q
- A cruiser R R R
- A submarine S S S
- A destroyer T T

The ships must be placed horizontally or vertically; none of the ships can be placed diagonally. Note that players are not allowed to see each other's grid.

**B.** Take turns to try to sink your opponent's ships. A ship is sunk when all the spaces it occupies have been called out.

**C.** A player wins the game when all his/her opponent's ships have been sunk.

1. What did you have to do to the grids in Fig. 1.1 in order to play the game?
2. Think of a way to uniquely locate any square on the grid.
3. How many ways can you think of? Which is better? What makes you say that?

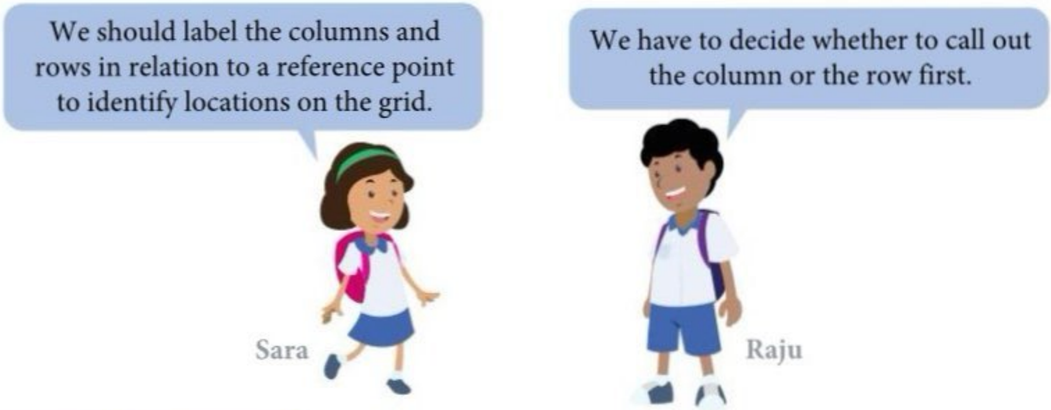
In this chapter, we will learn about a system to locate the position of a point on a plane, and to describe a line on a plane using the equation of a function.

**Attention**

A plane is a flat two-dimensional surface. It has a length and a breadth, but no thickness.

# 1.1 Cartesian coordinates

From the Introductory Problem, we gain two important insights.




## Class Discussion — Cartesian coordinate system

Fig. 1.2 shows the same square grid used in the Introductory Problem. The columns and rows are now numbered from 1 to 7 in relation to the reference point $O$.

1. Square $P$ is located along Column 2 and Row 4, so we label its position as (2, 4).
   Why must we use two numbers to label the square $P$? Is one number sufficient to describe the exact position of $P$? Explain.
2. Square $Q$ is located along Column 4 and Row 2, so we label its position as (4, 2).
   Is the order in which the two numbers are written important? Do (2, 4) and (4, 2) indicate the same position? Explain.
3. The pair of numbers (2, 4) is called an **ordered pair**. Why do you think it is called an ordered pair?
4. Label the positions of the squares $R$ and $S$.
5. The position of square $T$ is (3, 5). Label $T$ in Fig. 1.2.
6. What other examples in real-world contexts use ordered pairs to pinpoint a location?

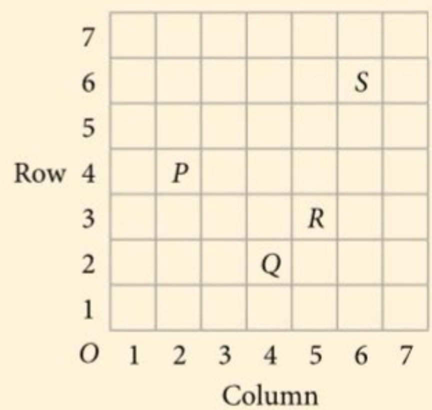



Fig. 1.2

How can we make use of the above concept to locate the position of any point on a plane? Fig. 1.3 shows the same square grid with green horizontal and vertical lines drawn through the centres of the boxes. We indicate each point by marking out the centre of each of the squares $P$, $Q$, $R$ and $S$ with a cross.

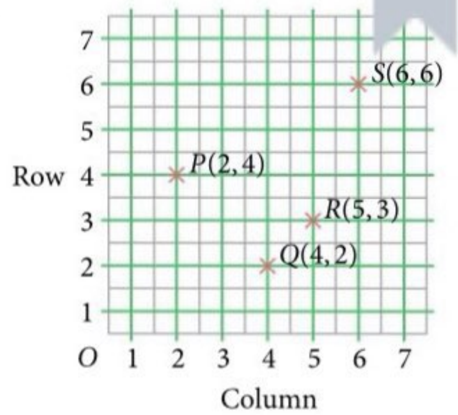



Fig. 1.3

We can now extend this system to locate any point on a given plane. Fig. 1.4 shows a **Cartesian** plane consisting of two number lines intersecting at right angles at the reference point $O$, called the **origin**. The horizontal and vertical axes are called the ***x*-axis** and the ***y*-axis** respectively.

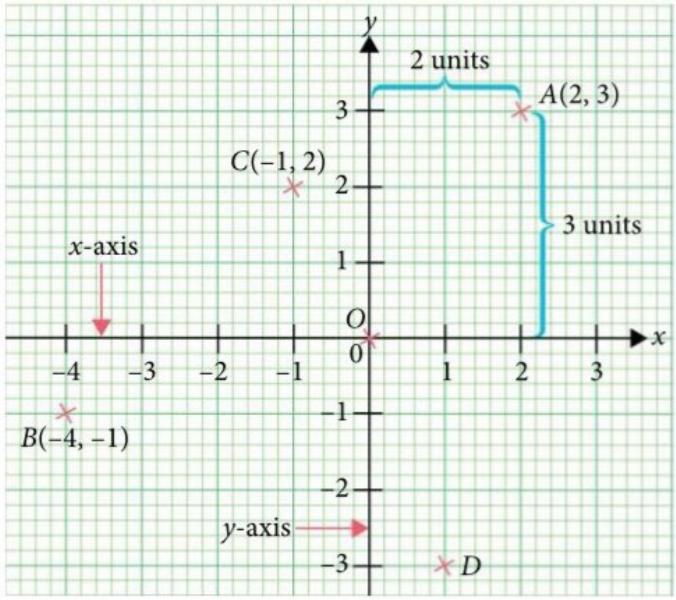



Fig. 1.4

**Big Idea**

**Diagrams**

The invention of the Cartesian coordinate system opened a new way to communicate the location of points — the building blocks of other geometric objects — and represent relationships between quantities using a simple diagram. As many of these relationships can be represented using algebraic expressions, the use of Cartesian coordinates made it possible to build *connections* between algebra and geometry.

In Fig. 1.4, point $A$ is located

2 units from the origin along the *positive x-axis* → (2, 3) ← 3 units from the origin along the *positive y-axis*

These are the **coordinates** of $A$:

- 2 is the *x-coordinate*, and
- 3 is the *y-coordinate* of $A$.

Similarly, the position $B$ is given by the coordinates

4 units from the origin along the *negative x-axis* → (−4, – 1) ← 1 unit from the origin along the *negative y-axis*

**Cartesian coordinate system**

We call $x$ the *$x$-coordinate* of $P$ and $y$ the *$y$-coordinate* of $P$, i.e. the **coordinates** of $P$ are $(x, y)$.

## Journal Writing

1. Explain why the position of the point $C$ is given by the coordinates $(-1, 2)$.
2. Describe how you would find the coordinates of the point $D$.
3. What are the coordinates of the origin $O$? Why?
4. What is the difference between the two terms 'ordered pair' and 'coordinates'?

The coordinate system described above was invented by René Des**cartes** (1596-1650) when he apparently tried to outline the path of a fly crawling along crisscrossed beams on the ceiling while he lay on his bed. This is why it is called the **Cartes**ian coordinate system.

**Practise Now 1A**

Similar and Further Questions
**Exercise 1A**
Questions 1, 2, 5(a)–(e), 6, 7, 9

Plot the points $P(1, 2)$, $Q(-2, 3)$, $R(-2, -2)$ and $S(3, -1)$ on the Cartesian plane in Fig. 1.4 on page 4.

## Reflection

1. What have I previously learnt about number lines that could have helped me learn the coordinate system?
2. Given a point with coordinates $(1, 2)$, how can I remember which value refers to the $x$-coordinate and to the $y$-coordinate?

# 1.2 Functions

In the previous section, we have learnt how to use the coordinates $(x, y)$ to describe the position of a point in a plane. In this section, we will study the relationship between the two variables $x$ and $y$.

## Investigation — Function machine

Let us explore the relationship between two quantities or variables $x$ and $y$. The relationship is called a **function** and can be thought of as a *function machine*. A function machine follows a rule to perform an operation on an **input $x$** to obtain an **output $y$**.

**Part 1**

Fig. 1.5 shows a function machine where the *rule* of the function is to 'add 3' to any integer input $x$ to produce an integer output $y$.

> **Attention**
> We can represent a function using words, e.g. 'add 3'.


Fig. 1.5

For example, if we input $x = 2$, the output will be $y = 2 + 3 = 5$.

> **Attention**
> We can represent a function using an equation.

1. Write down an *equation* to show the relationship between the input $x$ and the output $y$: $y =$ ______
2. Write down the output $y$ for each of the following inputs $x$.
   **(a)** Input $x = 4 \rightarrow$ Output $y =$ ______
   **(b)** Input $x = -7 \rightarrow$ Output $y =$ ______
3. Write down the input $x$ for each of the following outputs $y$.
   **(a)** Input $x =$ ______ $\rightarrow$ Output $y = 9$
   **(b)** Input $x =$ ______ $\rightarrow$ Output $y = 0$
4. The above data can be represented by Table 1.1. Copy and complete the *table*.

> **Attention**
> We can represent a function using a table.

| $x$ | $-7$ | | 2 | 4 | |
|---|---|---|---|---|---|
| $y$ | | 0 | 5 | | 9 |

Table 1.1

5. Fig. 1.6 shows the point (2, 5) plotted on a *graph* paper. Plot the rest of the points based on Table 1.1. Do *not* connect the points because the specified inputs for this function are integers.

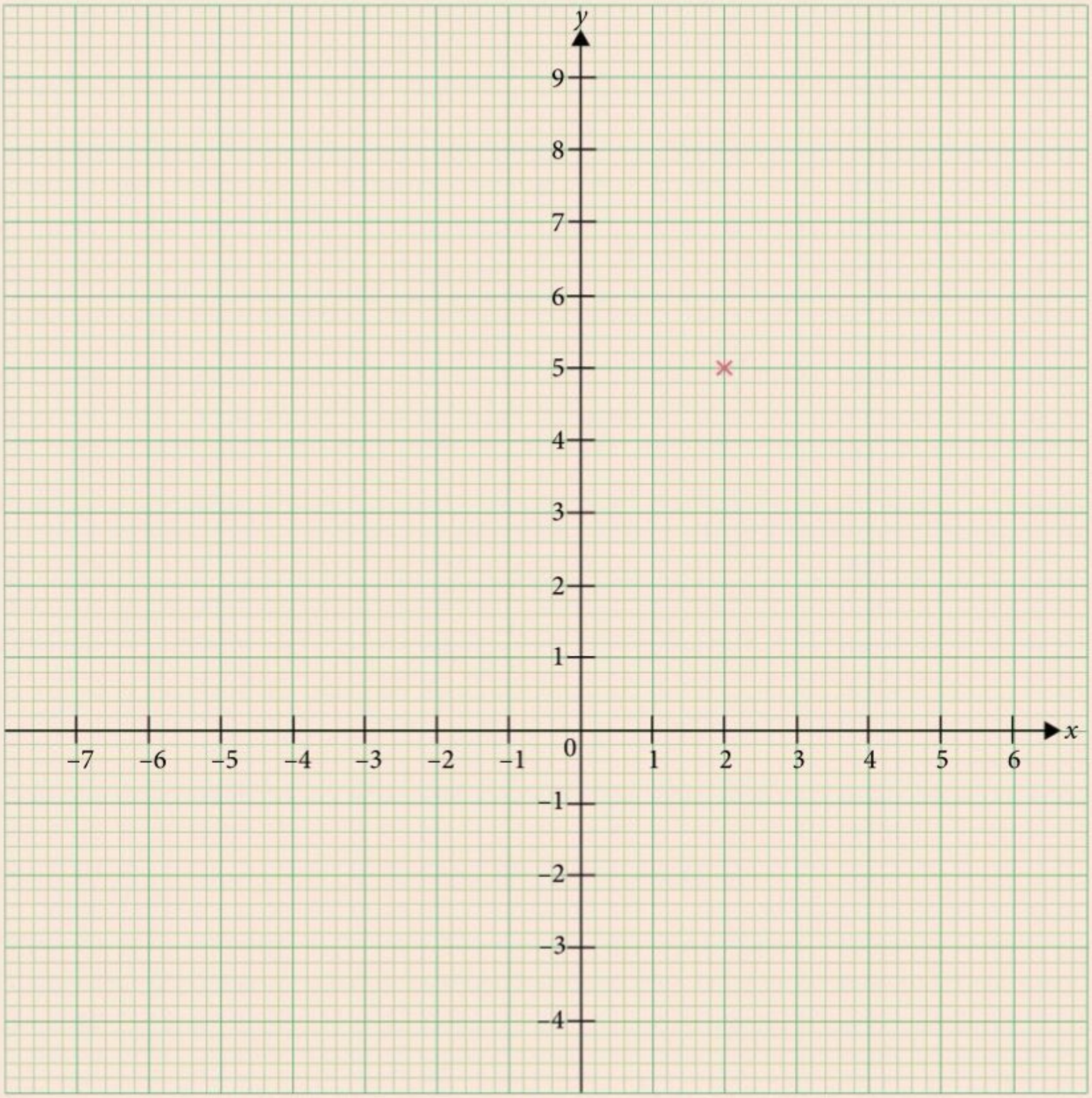



Fig. 1.6

6. Based on Table 1.1 and Fig. 1.6, state the number of outputs $y$ for each input $x$.

**Part 2**

Now, consider the function where the *rule* is 'add 3' again, but this time, instead of integers, the input $x$ and the output $y$ can be any real number.

7. Is this new function the same function as the function in **Part 1**? Why or why not?
8. Is the equation of this new function the same as the equation of the function in **Part 1**? Explain.
9. How will the graph of this new function look like? Explain.

**Part 3**

Fig. 1.7 shows another function machine where the *rule* of the function is to 'multiply by (–2)' to any integer input $x$ before 'subtracting 1' from the result, to produce an integer output $y$.

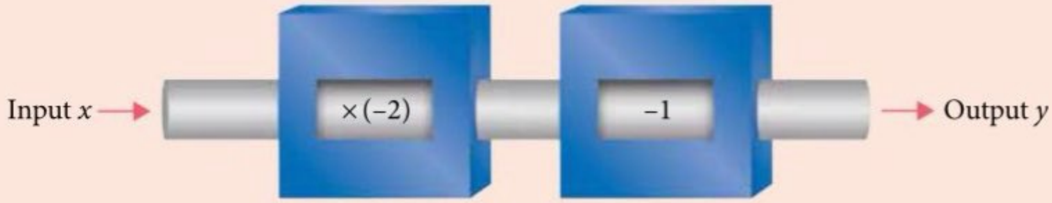



Fig. 1.7

For example, if we input $x = 3$, the output will be $y = 3 \times (-2) - 1 = -7$.

**10.** Representation of a function using an *equation*:

Write down the equation of the function: $y =$

**11.** Representation of a function using a *table*:

Copy and complete Table 1.2 to show the corresponding output values for the input values.

| $x$ | –1 | | | 2 | 3 |
|---|---|---|---|---|---|
| $y$ | | 0 | –1 | | –7 |

Table 1.2

**12.** Representation of a function using a *graph*:

In Fig. 1.8, the point (3, –7) is shown. Plot the rest of the points based on Table 1.2.

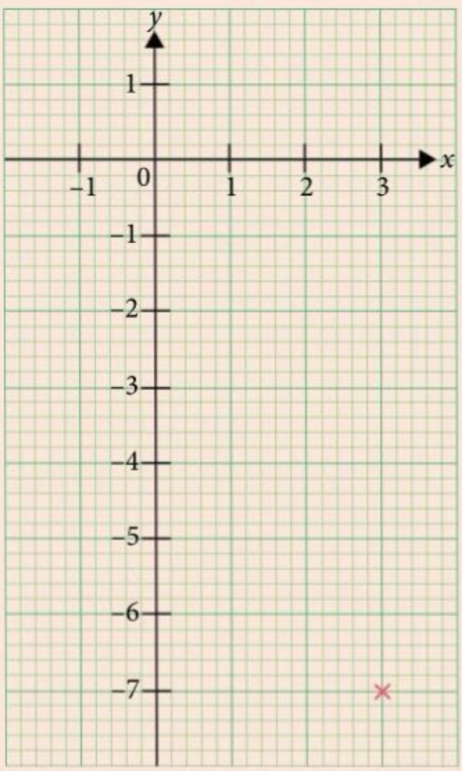



Fig. 1.8

**Internet Resources**

Search the Internet for any interactive 'Function Machine' where you can key in an input for the machine to give you an output. Then, try to guess the equation of the function.

**13.** Based on Table 1.2 and Fig. 1.8, state the number of output(s) $y$ for each input $x$.

From the Investigation on pages 6–8, we observe the following:

**Function**

A function is a relationship between two variables $x$ and $y$ such that every specified input $x$ produces *exactly one output* $y$.

- The input $x$ and the output $y$ of a function can be written as an ordered pair $(x, y)$.
- A function can be represented using words, an equation, a table of values and a graph.

**Attention**

The word 'graph' could mean the entire graph paper including the axes, or it could just mean the graph of the function, e.g. the straight line $y = x + 3$.

It is important to specify the input $x$ to describe a function completely. Different inputs produce different graphs, even if the functions are the same:

| Input | | Graph |
|---|---|---|
| Integers | $\longrightarrow$ | *Discrete points* |
| Real numbers | $\longrightarrow$ | *Straight lines* |

When the inputs are not specified (e.g. as integers only), we can assume that they can be any real number.

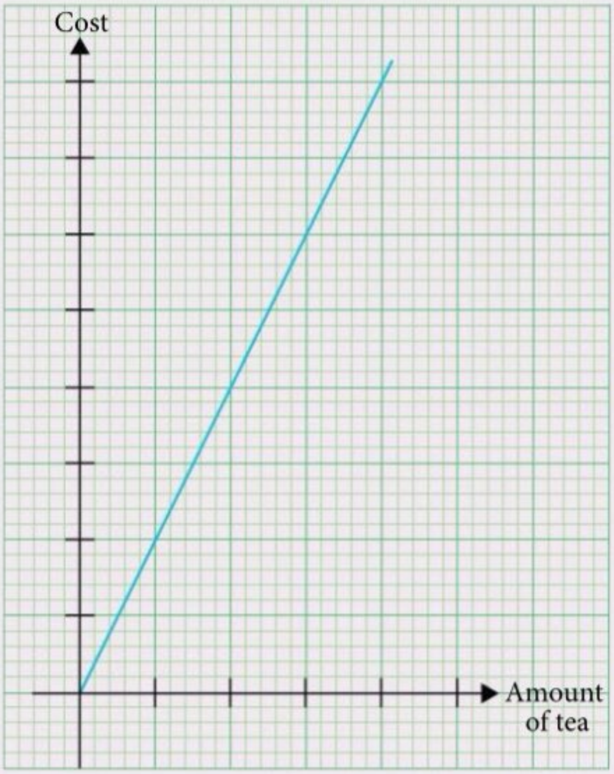
## Thinking Time

1. An input value of $x = 4$ has two output values $y = 2$ and $y = -2$.
   Is the relationship between $x$ and $y$ a function? Explain.
2. An input value of $x = 4$ does not have any output value $y$.
   Is the relationship between $x$ and $y$ a function? Explain.
3. An input value of $x = 4$ has an output value of $y = 16$.
   Another input value of $x = -4$ has the same output value of $y = 16$.
   Is the relationship between $x$ and $y$ a function? Explain.
4. Fig. 1.9 shows two advertisements.
   Match each of the graphs on page 10 to each advertisement. Explain your choice.

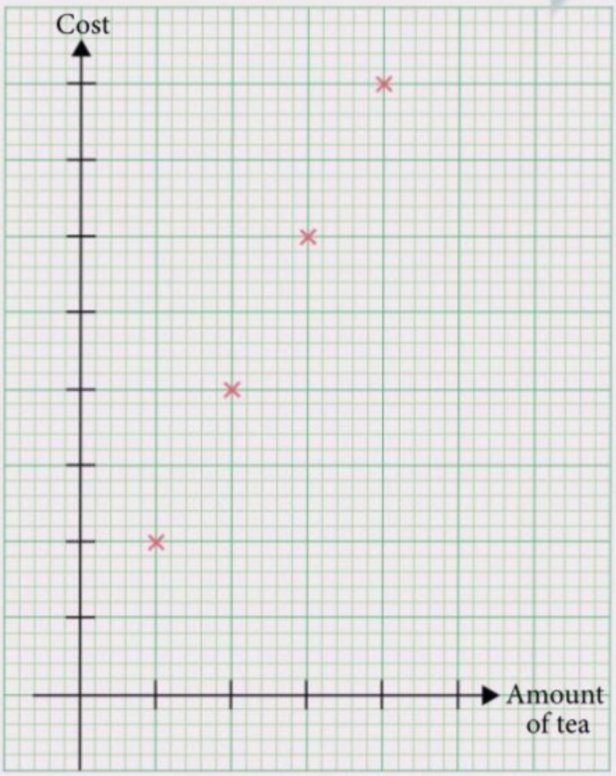

Tea bags
$2 per box

Tea leaves
$2 per kg

Fig. 1.9




Graph A

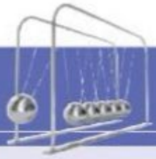



Graph B

**Practise Now 1B**

Similar and Further Questions
**Exercise 1A**
Questions 3, 4, 8(a), (b)

1. The equation of a function is $y = 2x - 3$. Find
   **(i)** the value of $y$ when $x = 4$,
   **(ii)** the value of $x$ when $y = -5$.
2. The equation of a function is $y = -\frac{1}{3}x - \frac{2}{5}$. Find
   **(i)** the value of $y$ when $x = 0$,
   **(ii)** the value of $x$ when $y = -\frac{2}{3}$.

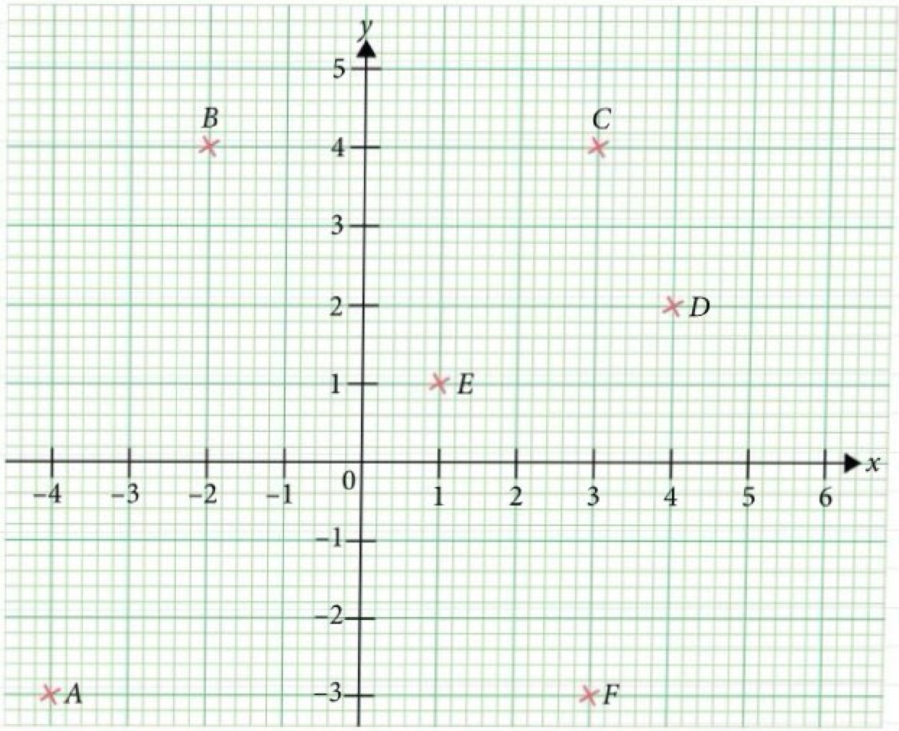


## Reflection

1. Based on my understanding,
   **(a)** what is a function?
   **(b)** what does a graph of a function represent?
2. What have I learnt in this section that I am still unclear of?

## Exercise 1A

1. Write down the coordinates of each point shown in the figure.




2. Plot the points $P(2, 5)$, $Q(1, 2)$, $R(-2, -1)$, $S(6, -2)$, $T(3, -2)$ and $U(-1, 2)$ on the Cartesian plane in Question 1.

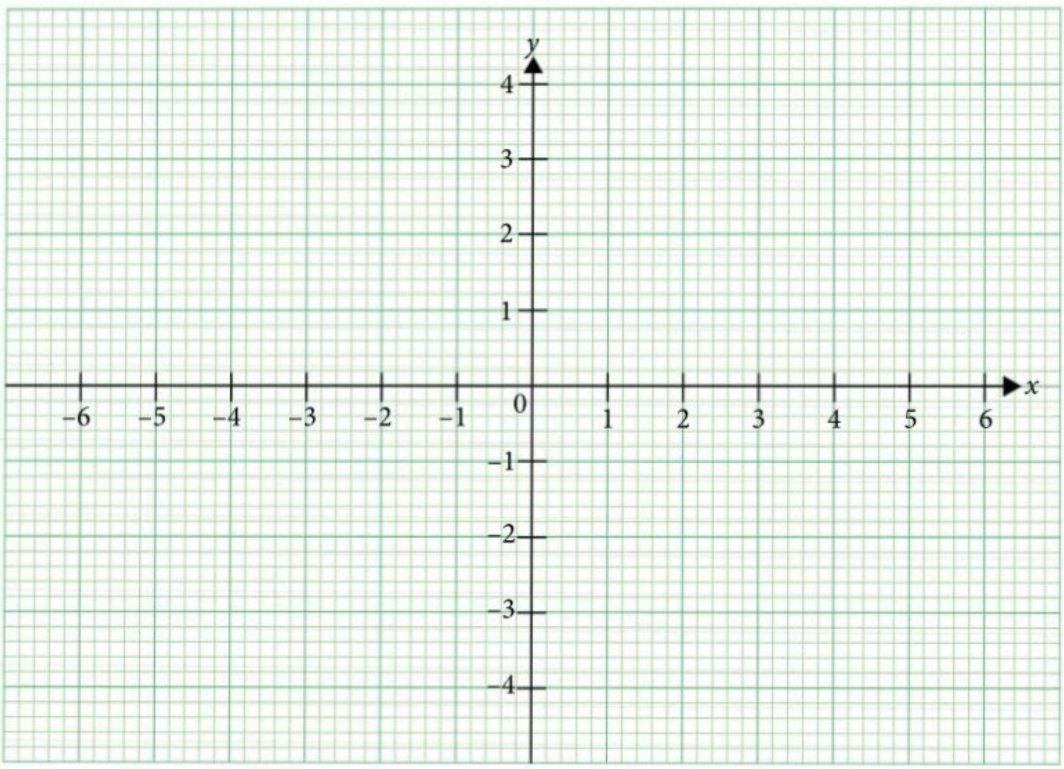
3. The equation of a function is $y = 4x + 5$. Find the value of $y$ when
   **(i)** $x = 3$, **(ii)** $x = -2$.

4. The equation of a function is $y = 25 - 3x$. Find the value of $x$ when
   **(i)** $y = 34$, **(ii)** $y = -5$.

Basic | Intermediate | Advanced

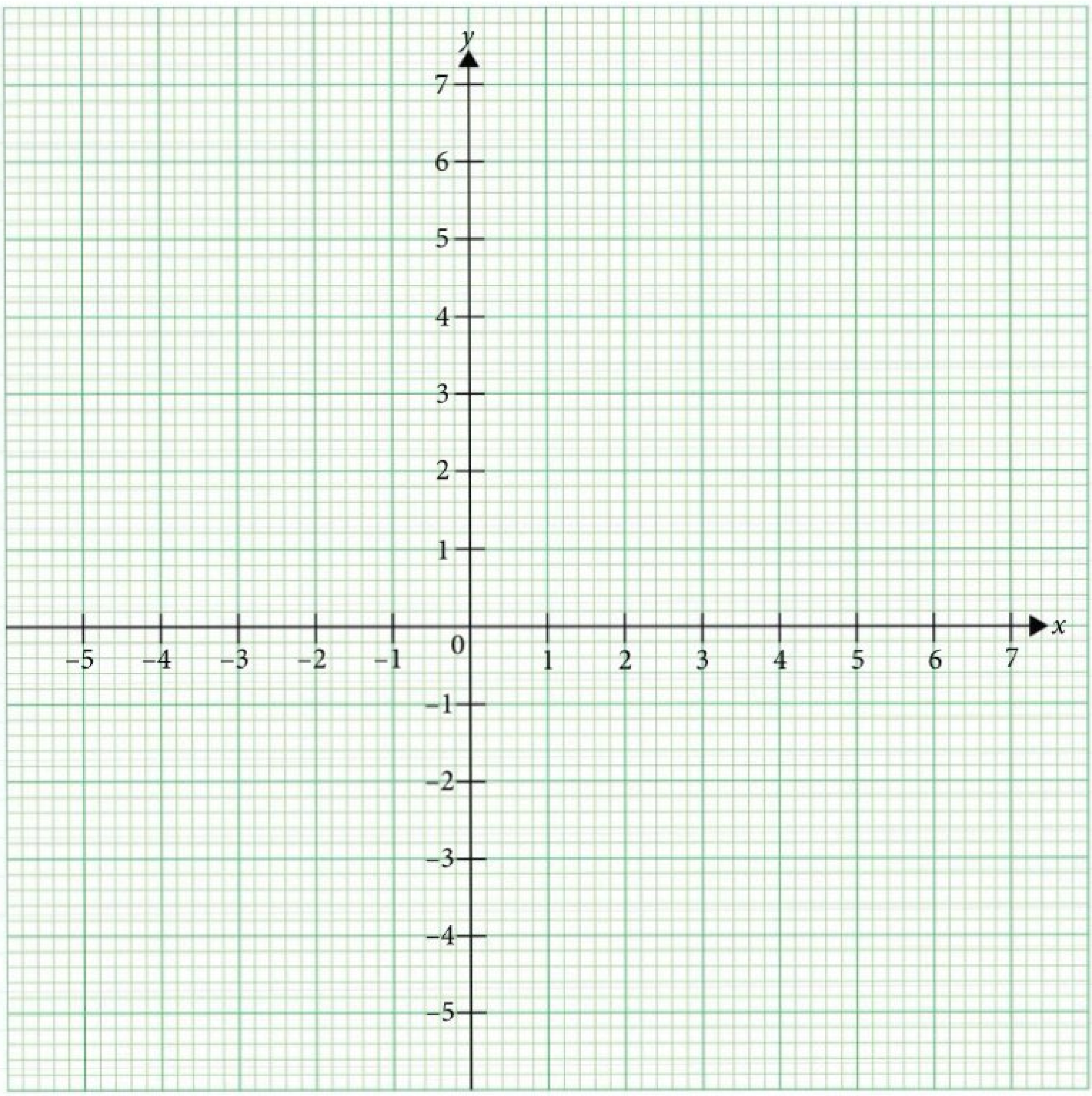
## Exercise 1A

5. Plot each set of the given points on the Cartesian plane below. Join the points (in order) with straight lines and identify each geometrical shape obtained.
   (a) (6, 4), (−6, 4), (−6, −4), (6, −4)
   (b) (0, 4), (−5, 0), (0, −4), (5, 0)
   (c) (0, 0), (0, 4), (4, 2)
   (d) (1, 0), (0, 3), (−1, 4), (−5, −2)
   (e) (5, 2), (−1, 3), (−1, −3), (5, −2)



## Exercise 1A

6. The vertices of a right-angled triangle are $A(1, 0)$, $B(7, 0)$ and $C(1, 6)$. Plot the points $A$, $B$ and $C$ on the Cartesian plane below. Hence, find the area of $\triangle ABC$.




7. Plot each of the following points on the Cartesian plane in Question 6.
$(3, -5)$, $(2, -3)$, $(1, -1)$, $(0, 1)$, $(-1, 3)$, $(-2, 5)$, $(-3, 7)$
Do you notice that the points lie in a special pattern? Describe the pattern.

8. The equation of a function is $y = \frac{2}{3}x + \frac{1}{3}$. Find
   **(a)** the value of $y$ when
   **(i)** $x = -3$, **(ii)** $x = 1\frac{1}{2}$,
   **(b)** the value of $x$ when
   **(i)** $y = 1$, **(ii)** $y = -\frac{1}{6}$.

9. Two of the vertices of a triangle $ABC$ are $A(1, 1)$ and $B(5, 5)$. The area of $\triangle ABC$ is 12 units² and the $y$-coordinate of the point $C$ is 1. By plotting the points $A$ and $B$ on the Cartesian plane in Question 6, determine the possible $x$-coordinates of $C$.

# 1.3 Linear functions

## A. Graph of linear function

In the previous section, we have seen how we can represent a function by drawing its graph. In this section, we will learn how to draw the graph of a linear function given its equation.

**Worked Example 1**

### Drawing graph of linear function

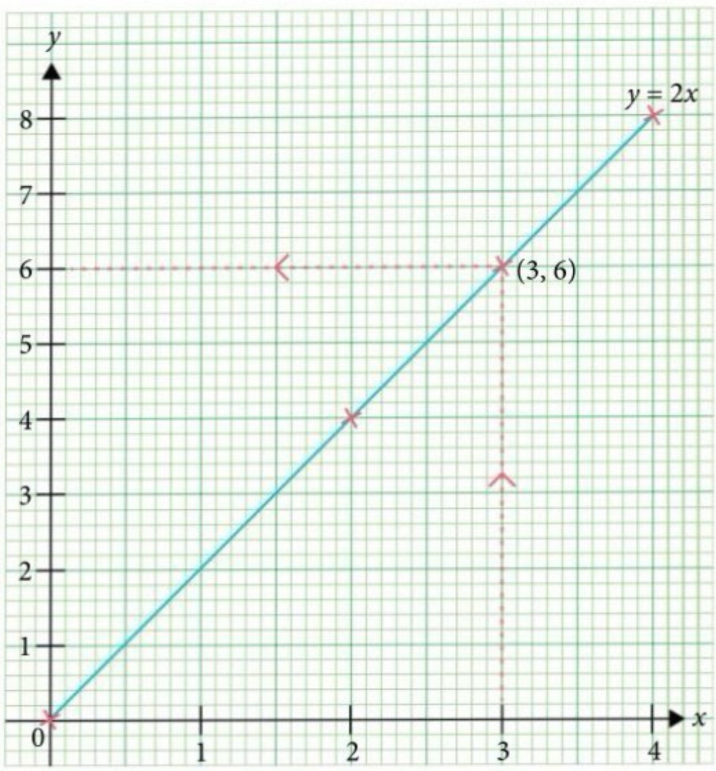
**(i)** On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the $x$-axis and 1 cm to represent 1 unit on the $y$-axis, draw the graph of the function $y = 2x$ for values of $x$ from 0 to 4.

**(ii)** The point $(3, p)$ lies on the graph in part **(i)**. Find the value of $p$.

**Solution**

**(i)**

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y = 2x$ | 0 | 4 | 8 |




**(ii)** From the graph in part **(i)**, when $x = 3$, $p = y = 6$.

**Problem-solving Tip**

**Table of values**

First, set up a table of values for $x$ and $y$. We need to plot 3 points to draw the graph of a linear function: the two endpoints since we need to plot from $x = 0$ to 4, and another point somewhere in the middle. A straight line can be drawn by plotting only 2 points, but we use the 3rd point to check for mistakes.

**Problem-solving Tip**

**Graph**

The table of values tells us that the range of values is 0 to 4 for the $x$-axis, and 0 to 8 for the $y$-axis. Use a pencil to draw the axes with these values at the end, for the given scales. Plot the 3 points and draw a straight line passing through all the 3 points: the line must end at the two endpoints of the given range of values of $x$. Label the line with the equation $y = 2x$.

Practise Now 1C

Similar and Further Questions
**Exercise 1B**
Questions 1–3, 8–10

1. **(i)** On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the $x$-axis and 1 cm to represent 1 unit on the $y$-axis, draw the graph of the function $y = 2x + 1$ for values of $x$ from 0 to 4.
   **(ii)** The point $(q, 6)$ lies on the graph in part **(i)**. Find the value of $q$.
2. On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the $x$-axis and 1 cm to represent 2 units on the $y$-axis, draw the graphs of the functions $y = 6x$ and $y = 2 - 4x$ for values of $x$ from $-2$ to 2.

## Class Discussion — Graphs of linear functions

1. Li Ting says that the straight line in Worked Example 1 consists of only 3 points. Do you agree? Explain.
2. The coordinates of the points $A$ and $B$ are $(1, 2)$ and $(3, 7)$ respectively. How do you determine whether these two points lie on the straight line in Worked Example 1?
3. The line drawn in Worked Example 1 is said to be the graph of the *linear* function $y = 2x$ because the graph is a *straight line*. Nadia says that the graphs of the functions $y = x + 3$ and $y = -2x - 1$ are linear. Do you agree with her? Explain your answer.

## B. Equation of straight line

From the above Class Discussion, we see that the equation of a straight line can be written in the form $y = mx + c$, where $m$ and $c$ are constants. What happens to the line if we change the value of $m$ and of $c$?

## Investigation — Equation of straight line

Let us find out how the graph of a straight line in the form $y = mx + c$ changes when either $m$ or $c$ varies. Go to www.sl-education.com/tmsoupp2/pg15 or scan the QR code on the right and open the template 'Equation of a straight line'.

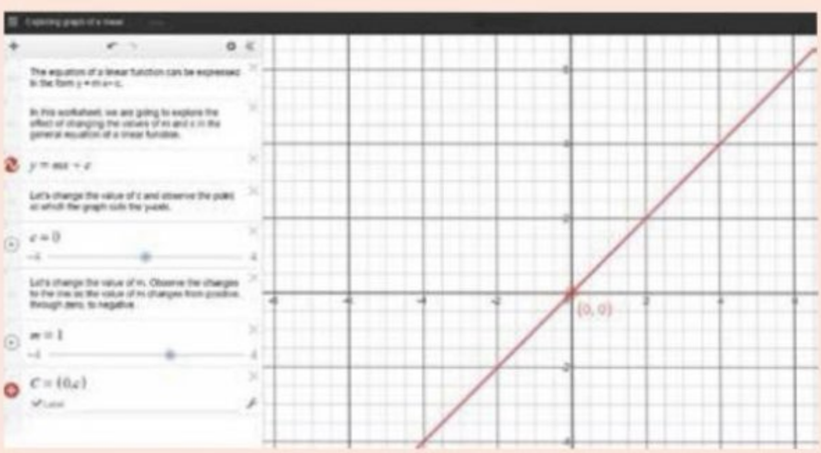



Fig. 1.10

1. Change the value of $c$ from $-4$ to 4 by moving the slider. What happens to the line? State the coordinates of the point where the line cuts the $y$-axis.

2. Change the value of $m$ from 0 to 4 using the slider. What happens to the line?
3. Change the value of $m$ from 0 to $-4$ using the slider. What happens to the line?
4. What is the difference between a line with a positive value for $m$ and a line with a negative value for $m$?

From the above Investigation, we observe that the value of $c$ is the $y$-coordinate of the point where the line cuts the $y$-axis. Hence, $c$ is called the **$y$-intercept**.

We also see that the slope of the line changes when we change the value of $m$. $m$ is the **gradient**, which is a measure of the slope of the line.

**Equation of straight line**

The equation of a straight line is $y = mx + c$, where the constant $m$ is the *gradient* of the line and the constant $c$ is the *$y$-intercept*.

**Worked Example 2**

**Equation of straight line**

**(a)** The equation of a straight line is $y = 2x - 1$. State its gradient and $y$-intercept.
**(b)** A line has gradient $-3$ and $y$-intercept 4. State its equation.

**Solution**

**(a)** Line: $y = 2x - 1$
Gradient = 2; $y$-intercept = $-1$

**(b)** The equation of the straight line is $y = -3x + 4$.

**Practise Now 2**

Similar and Further Questions
**Exercise 1B**
Questions 4(a)–(f), 5(a)–(f)

1. State the gradient and $y$-intercept of each of the following lines.
   **(a)** $y = 4x - 3$ **(b)** $y = -x + 7$ **(c)** $y = x + \frac{5}{2}$
   **(d)** $y = 7.16 - 0.5x$ **(e)** $y = 6x$ **(f)** $y = 6$

**Attention**

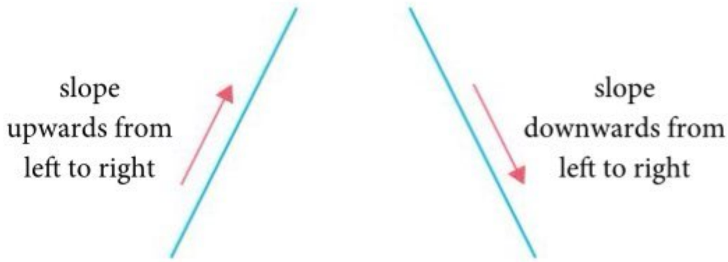
The gradients of some lines are 0. We will learn more about such lines later in Section 1.3E.

2. State the equation of each of the following lines given its gradient and $y$-intercept.
   **(a)** gradient = 3; $y$-intercept = 5 **(b)** gradient = $-7$; $y$-intercept = $-2$
   **(c)** gradient = 1; $y$-intercept = $-\frac{2}{3}$ **(d)** gradient = $-1$; $y$-intercept = 7.69
   **(e)** gradient = $-\frac{1}{2}$; $y$-intercept = 0 **(f)** gradient = 0; $y$-intercept = $-\frac{1}{2}$

## C. Calculation of gradient of straight line

In this section, we will learn how to calculate the gradient of a straight line by looking at its graph.

From the Investigation on pages 15 and 16, we have discovered that the gradient of a line can be *positive* or *negative*.




**(a)** Positive gradient **(b)** Negative gradient

Fig. 1.11

> **Attention**
>
> In Fig. 1.11**(a)**, the line slopes downwards from right to left but its gradient is positive. Thus it is important to specify *'from left to right'*.

**Sign of gradient of straight line**

If a line slopes *upwards* from *left to right*, its gradient is *positive*.

If a line slopes *downwards* from *left to right*, its gradient is *negative*.

> **Recall**
>
> The absolute value of a number $x$ is the non-negative value of $x$, regardless of its sign.

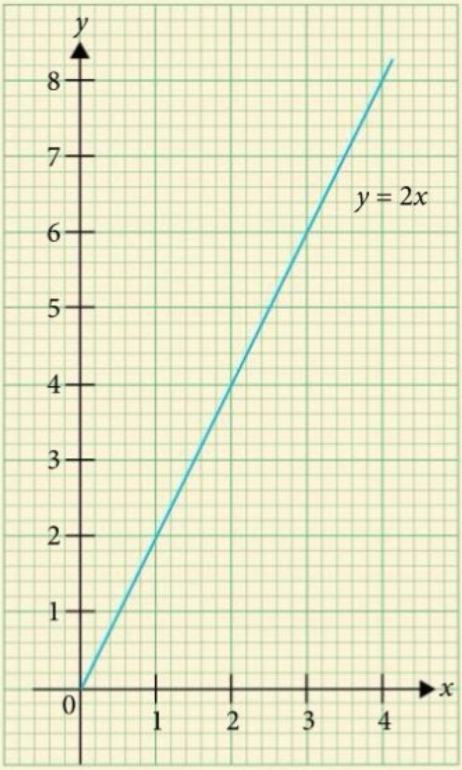
The Investigation on pages 15 and 16 also showed us that as the *absolute value* of the gradient $m$ *increases*, the steepness of the line *increases*. For example, a line with a gradient of 3 is steeper than a line with a gradient of 2 since $3 > 2$; and a line with a gradient of $-3$ is also steeper than a line with a gradient of $-2$ (comparing absolute values, $3 > 2$).

> **Attention**
>
> We *cannot* say that as the gradient $m$ increases, the steepness of the line increases because when $m$ is negative, e.g. when $m$ increases from $-3$ to $-2$, the steepness of the line actually decreases.

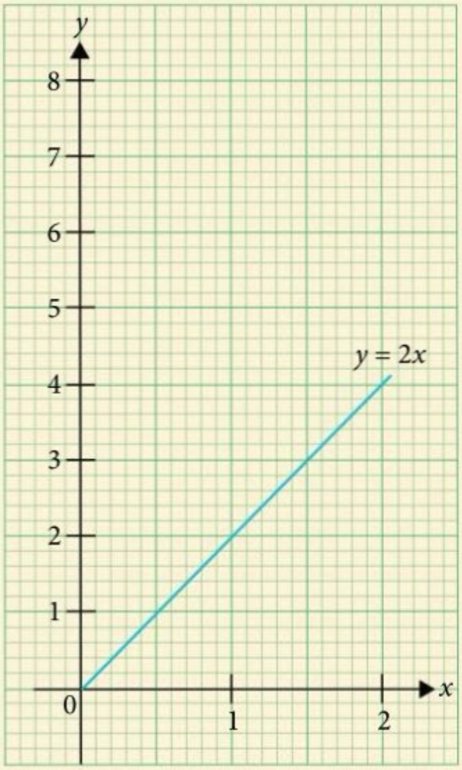
## Class Discussion

### Gradient of straight line

1. Both graphs in Fig. 1.12 below show the graph of $y = 2x$. Why do they look different?




Graph A




Graph B

Fig. 1.12

2. Which of the two lines is steeper or has the larger gradient? Explain.
3. What does a gradient of 2 mean in both graphs, i.e. how do you find the gradient of the line in both graphs?

From the above Class Discussion, we learn that the use of *different scales* for both axes in Graph B will change how the graph looks. Hence, the gradient of the line in Graph B might not appear to be 2, as expected. The use of different scales also makes it difficult to compare their steepness.

**Information**

In Fig. 1.12, Graph A uses the same scale for both axes: this is called a homogeneous coordinate system (homo = same).
Graph B uses different scales for the axes: this is called a non-homogeneous coordinate system.

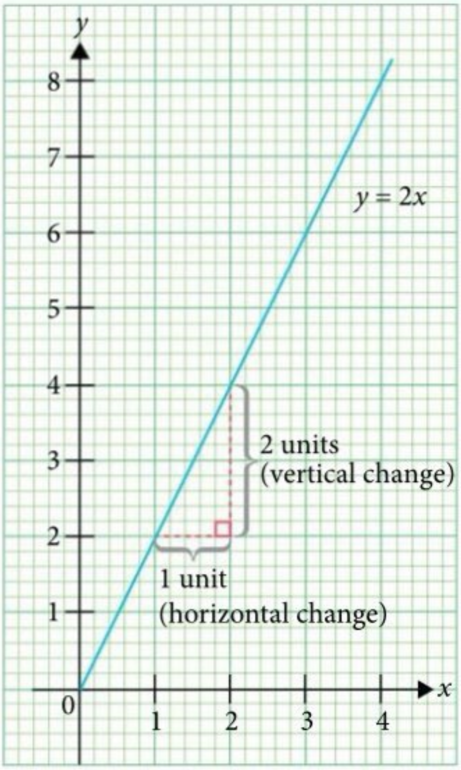
Regardless of the scales used, we can still determine the gradient of a line from its graph. The gradient of both graphs in the Class Discussion on page 18 is 2, as indicated in the equation of the line $y = 2x$. Notice that the vertical change is 2 units for every 1 unit of horizontal change for both graphs, as shown in Fig. 1.13. Therefore, when measuring the horizontal change in Graph B, we do not use the actual length of 2 cm, but the *units indicated on the scale*, i.e. 1 unit.

Big Idea

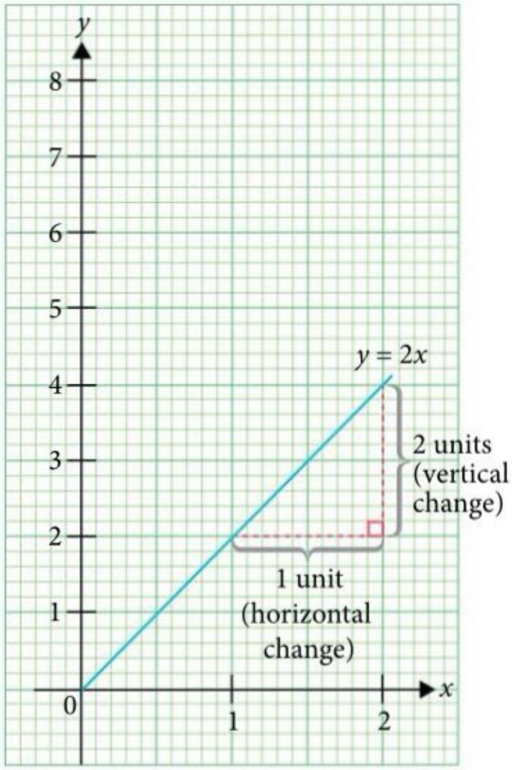
**Measures**

Gradient is a measure of the steepness of a slope, which allows us to analyse how steep a slope is and to compare the *steepness* of two or more slopes. Here, we use a ratio to measure steepness. Hence, it has no units. Why?

Other measures of the steepness of a slope include the angle of inclination between the slope and the horizontal line.




Graph A




Graph B

Fig. 1.13

Thus, the definition of gradient is:

**Gradient of straight line**

The gradient of a straight line is a measure of its steepness:

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}} \text{ or } \frac{\text{rise}}{\text{run}},$$

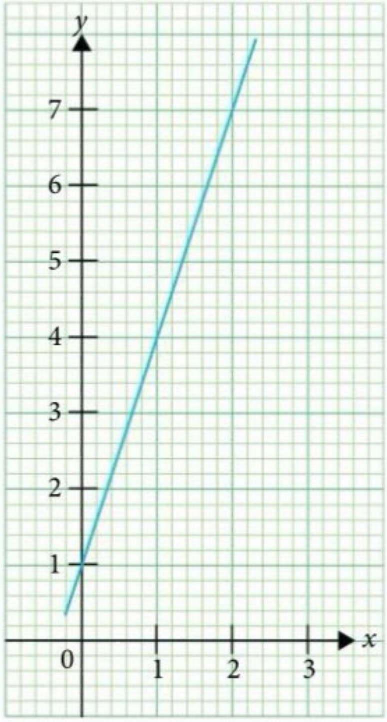
where the change refers to the change in the number of units.

Worked Example 3

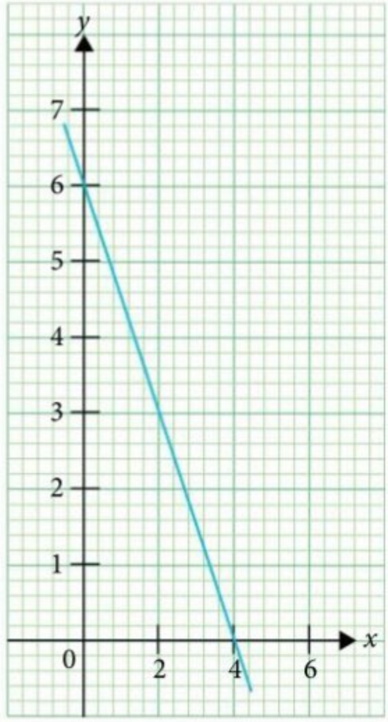
## Finding gradients of straight lines

Find the gradient of each of the following lines.

(a)

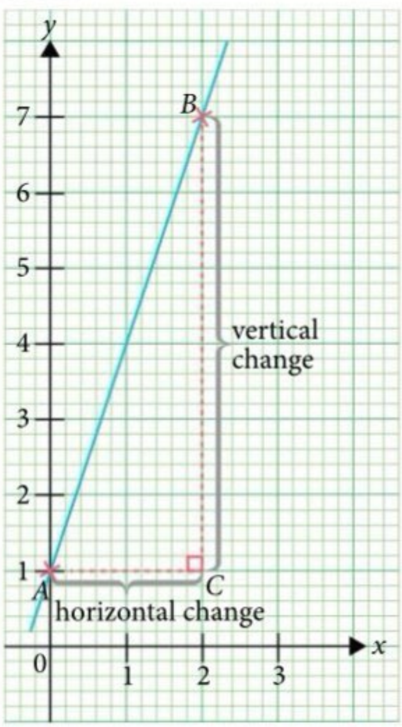



(b)




### Solution

(a)




Vertical change (or rise)

$= 7 - 1$

$= 6$

Horizontal change (or run)

$= 2 - 0$

$= 2$

$\therefore$ gradient $= \dfrac{\text{rise}}{\text{run}}$

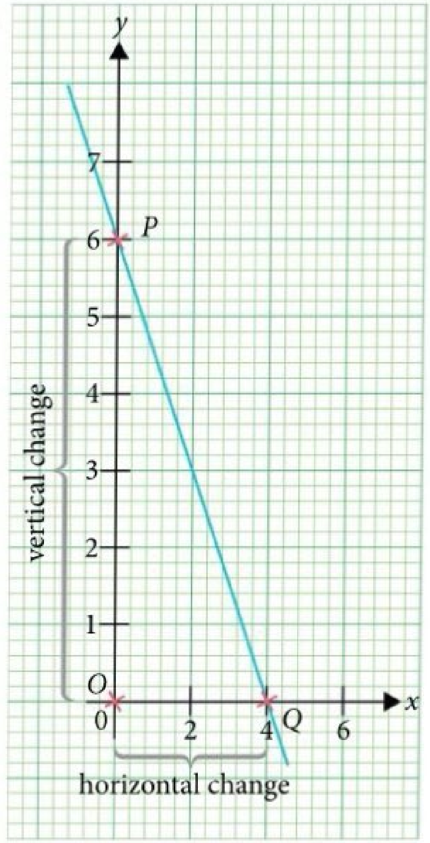
$= \dfrac{6}{2}$

$= 3$

**Problem-solving Tip**

Choose two points on the line with coordinates that can be read easily ($A$ and $B$). Then draw dotted lines to form the right-angled triangle $ABC$. To find the vertical change, we can either calculate $7 - 1 = 6$, or *count the number of units* from 1 to 7. Similarly, for the horizontal change.

(b)




Vertical change (or rise)
$= -6$

Horizontal change (or run)
$= 4$

$\therefore$ gradient $= \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{-6}{4}$

$= -\dfrac{3}{2}$

**Problem-solving Tip**

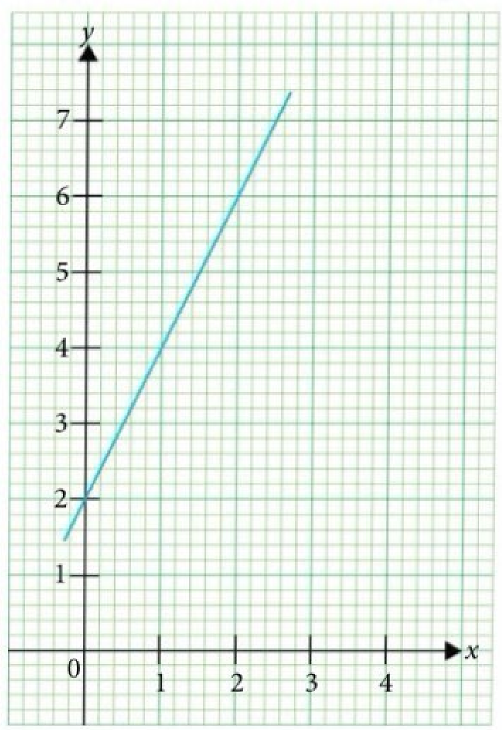
Choose two points on the line with coordinates that can be read easily ($P$ and $Q$). Because the line slopes *downwards* from left to right, the vertical change is *negative*. Thus vertical change $= -6$. Using this method, the horizontal change will always be positive as it goes from left to right. To find the horizontal change, we can *still count the number of units* from 0 to 4 (notice difference in scale for $x$-axis).

**Practise Now 3**

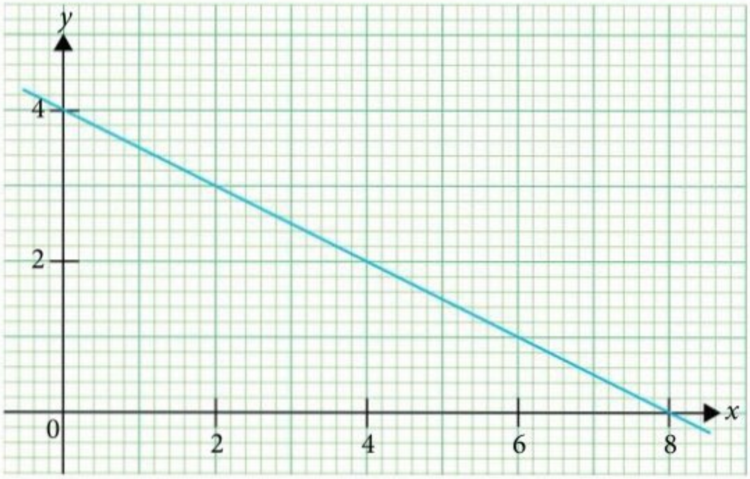
Similar and Further Questions
**Exercise 1B**
Questions 6(a)–(h)

Find the gradient of each of the following lines.

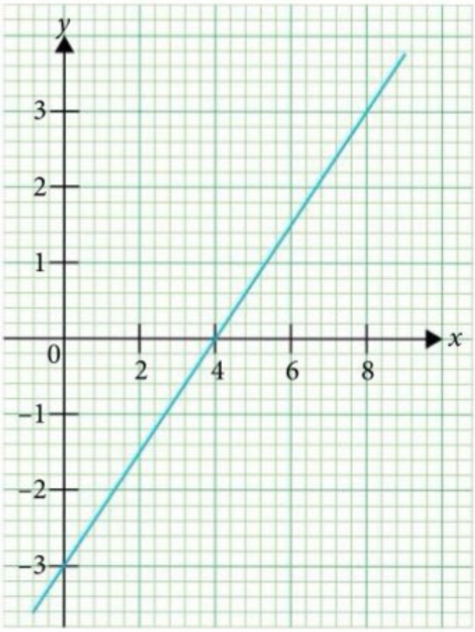
(a)



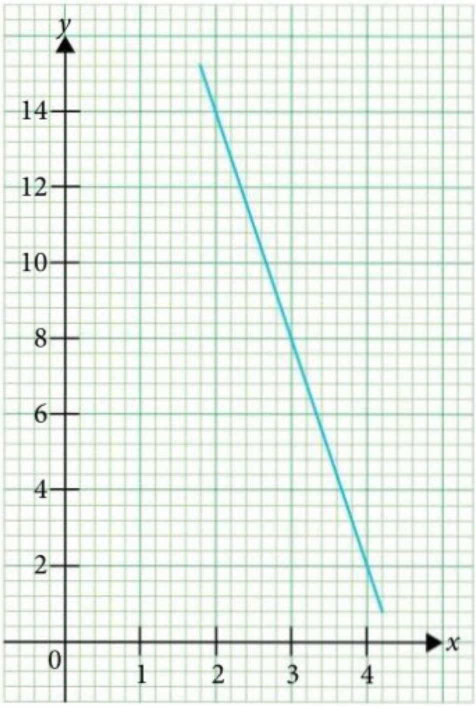
(b)




(c)




(d)




## Thinking Time

Consider the points $D(1, 4)$ and $E(1.5, 5.5)$ on the line in Worked Example 3(**a**) on page 20.

1. Find the gradient of $DE$.
2. Is gradient of $DE$ = gradient of $AB$? Explain.
3. Can you choose any two points on a line to find its gradient? Why or why not?

From the above Thinking Time, we observe that we can choose any two points on a line to find its gradient because the gradient of a straight line is a **constant**. However, we should still choose the points with coordinates that can be easily read.

**Recall**

The equation of a straight line is $y = mx + c$, where $m$ (gradient) and $c$ ($y$-intercept) are constants.

## D. Sense of magnitude of gradient

How steep is a line with gradient 1? Let us discuss this using real-life examples of a straight road. Straight lines can be used to **model** slopes of straight roads in real life, where a straight road (from the side view) can be seen as a straight line. This can then be modelled using a linear **function**.

> **Big Idea**
>
> **Functions and Models**
> Functions are important because they can be used to model many real-world situations. In Section 1.4, we will examine some real-world situations that linear functions can be used to model.

### Class Discussion — Sense of magnitude of steepness

Look at the roads around you. Some roads are steeper than others and hence more difficult to walk up.

1. Can the gradient of a road be negative?
2. **(i)** How steep is a road with a gradient of 1? Fig. 1.14 shows a line with a gradient of 1. Measure the *angle of inclination* between the line and the horizontal dotted line.

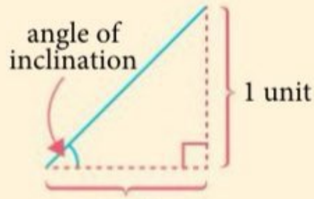
   
   

   Fig. 1.14

   **(ii)** How steep is a road with a gradient of 2? Draw a line with a gradient of 2, labelling the vertical change and horizontal change clearly.
   Measure the angle of inclination.

   **(iii)** Repeat Question 2(**ii**) for a road with a gradient of $\frac{1}{2}$.
3. **(i)** Do you consider a road with a gradient of 1 steep or gentle? Are the gradients of most roads in Singapore greater than or less than 1?

   **(ii)** Do you consider a road with a gradient of $\frac{1}{2}$ steep or gentle? Are the gradients of most roads in Singapore greater than or less than $\frac{1}{2}$?
4. The Guinness World Records states that the steepest road in the world is Baldwin Street in the city of Dunedin in New Zealand. The gradient of the road at its steepest portion is about 0.35.
   With reference to the steepest road in the world, do you think the gradients of most roads in Pakistan are greater than or less than $\frac{1}{2}$?

## E. Horizontal line

### Investigation — Gradient of horizontal line

Fig. 1.15 shows a horizontal line.

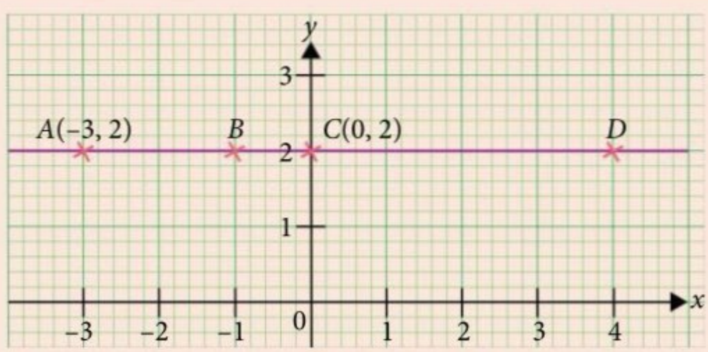



Fig. 1.15

1. There are 4 points on the line. The coordinates of $A$ and $C$ are given. Write down the coordinates of $B$ and $D$.
2. (i) In the line segment $AC$, vertical change (rise) = ____ and horizontal change (run) = ____.
   (ii) In the line segment $BD$, vertical change (rise) = ____ and horizontal change (run) = ____.
3. What can you conclude about the gradient of a horizontal line?

From the above Investigation, we learn that the gradient of a horizontal line is **0** because the vertical change (rise) is always 0 for any horizontal change (run), i.e. a horizontal line has *no slope*.

## F. Vertical line

### Investigation — Gradient of vertical line

Fig. 1.16 shows a vertical line.

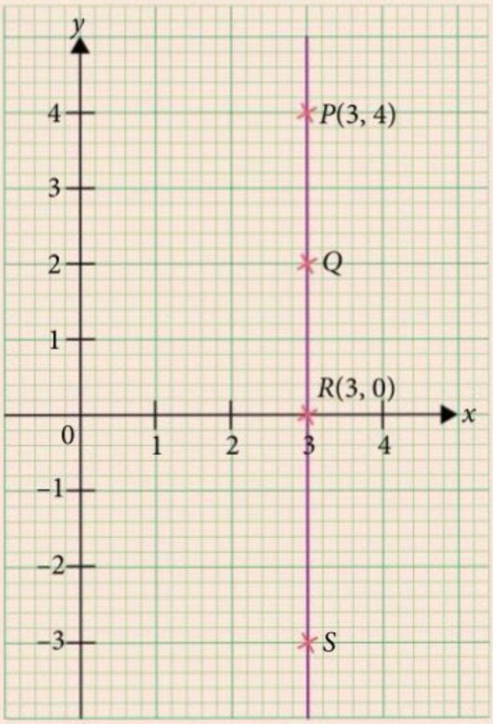



Fig. 1.16

1. There are 4 points on the line. The coordinates of $P$ and $R$ are given. Write down the coordinates of $Q$ and $S$.
2. (i) In the line segment $PR$, vertical change (rise) = ____ and horizontal change (run) = ____.
   (ii) In the line segment $QS$, vertical change (rise) = ____ and horizontal change (run) = ____.
3. What can you conclude about the gradient of a vertical line?

Similar and Further Questions
**Exercise 1B**
Questions 7, 11, 12

From the Investigation on page 24, we learn that the gradient of a vertical line is *undefined* because the ho[...]l change (run) is always 0. In other words, a vertical line is so steep that we cannot specify a value for its gradient.

**Gradient of horizontal line and of vertical line**

The gradient of a horizontal line is **0** while the gradient of a vertical line is **undefined**.

Advanced Intermediate Basic

## Exercise 1B

**1.** **(i)** On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the $x$-axis and 1 cm to represent 1 unit on the $y$-axis, draw the graph of each of the following functions for values of $x$ from 0 to 4.

**(a)** $y = 2x + 8$ **(b)** $y = 2x + 2$

**(c)** $y = 2x - 3$ **(d)** $y = 2x - 6$

**(ii)** What do you notice about the lines you have drawn in part **(i)**?

**2.** **(i)** On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the $x$-axis and 1 cm to represent 2 units on the $y$-axis, draw the graph of each of the following functions for values of $x$ from $-4$ to 4.

**(a)** $y = 3x + 7$ **(b)** $y = 3x + 5$

**(c)** $y = 3x - 3$ **(d)** $y = 3x - 6$

**(ii)** What do you notice about the lines you have drawn in part **(i)**?

**3.** **(i)** On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the $x$-axis and 1 cm to represent 2 units on the $y$-axis, draw the graph of each of the following functions for values of $x$ from $-4$ to 4.

**(a)** $y = -2x + 5$ **(b)** $y = -2x + 3$

**(c)** $y = -2x - 4$ **(d)** $y = -2x - 7$

**(ii)** What do you notice about the lines you have drawn in part **(i)**?

**4.** State the gradient and $y$-intercept of each of the following lines.

**(a)** $y = 3x + 7$ **(b)** $y = -x - 1$

**(c)** $y = 6x + 6$ **(d)** $y = \frac{10}{3} - 4x$

**(e)** $y = 0.2x$ **(f)** $y = -11$

**5.** State the equation of each of the following lines given its gradient and $y$-intercept.

**(a)** gradient = 2; $y$-intercept = 4

**(b)** gradient = $-2$; $y$-intercept = $-4$

**(c)** gradient = 1; $y$-intercept = $-\frac{1}{5}$

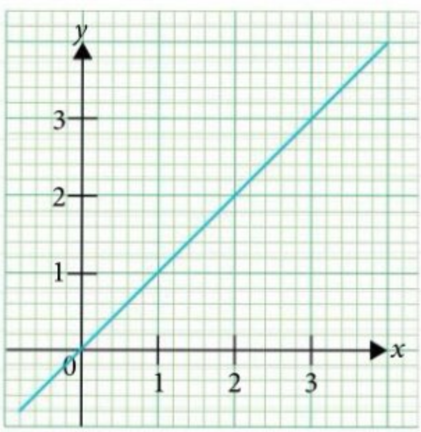
**(d)** gradient = $-1$; $y$-intercept = 3.78

**(e)** gradient = $-\frac{2}{3}$; $y$-intercept = 0

**(f)** gradient = 0; $y$-intercept = $-\frac{2}{3}$

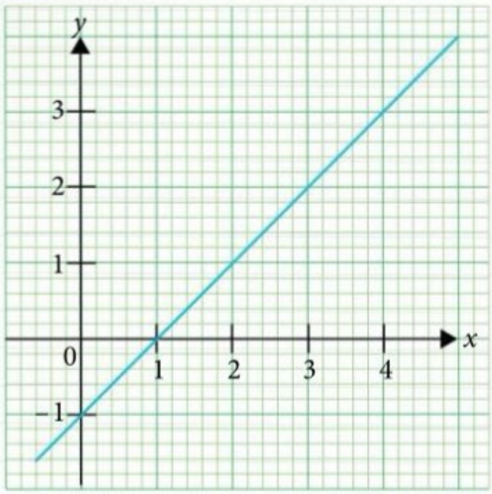
**6.** Given that the equation of the line representing each of the following linear graphs is in the form $y = mx + c$, find the gradient $m$ and state the $y$-intercept $c$.

**(a)**

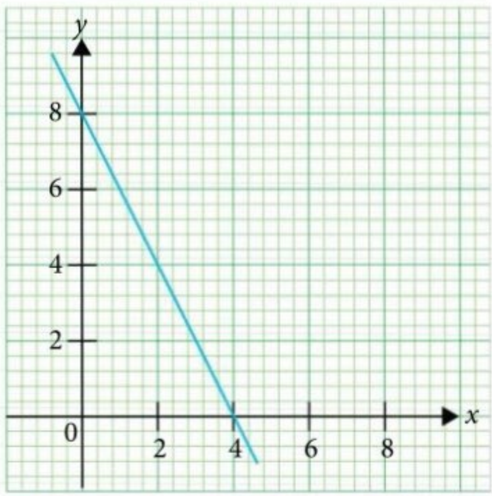


## Exercise 1B

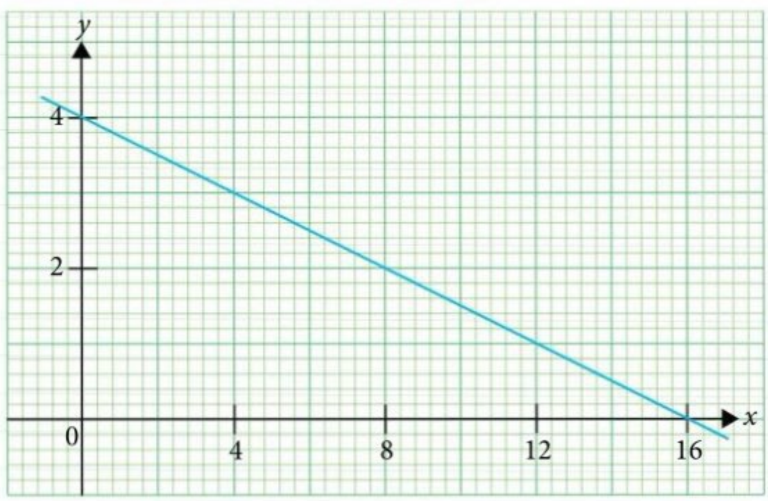
(b)



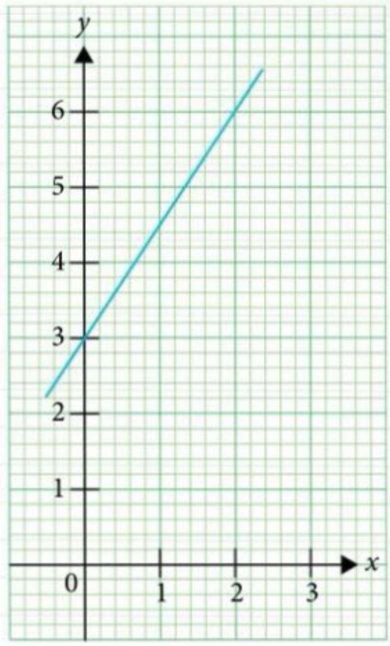

(c)



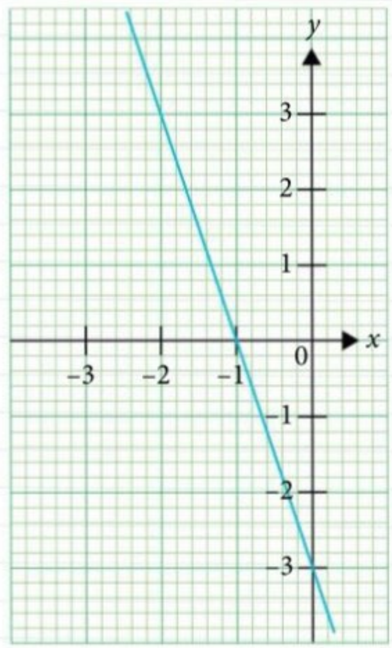

(d)




(e)

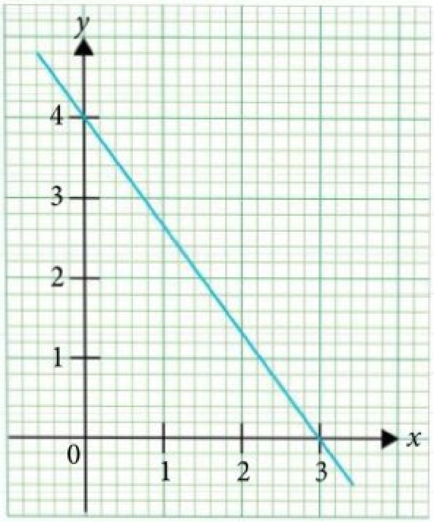



(f)

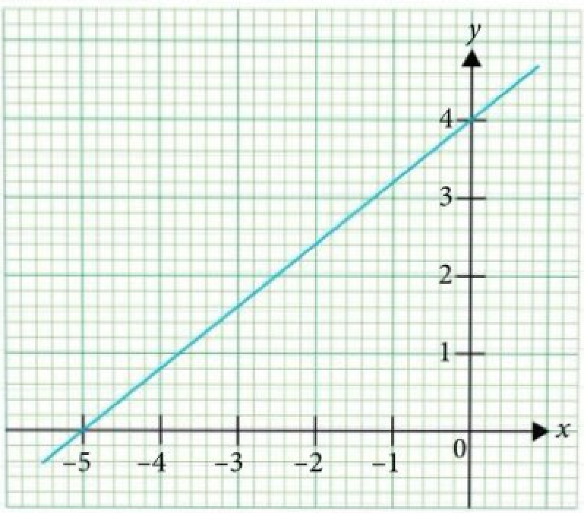


## Exercise 1B

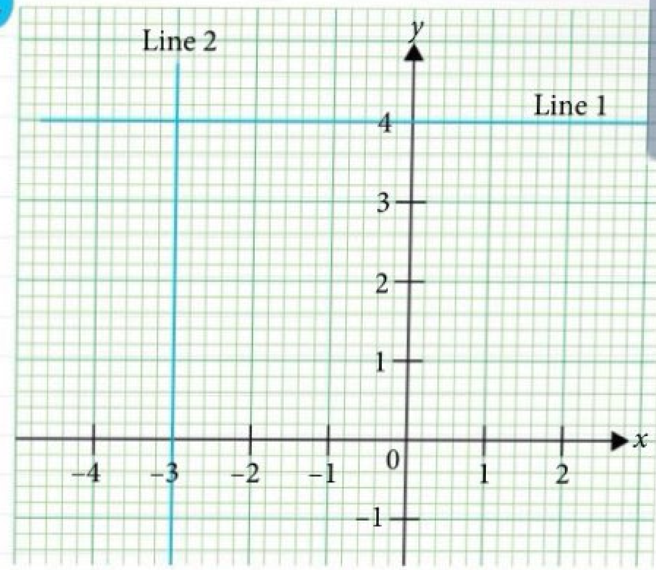
(g)




(h)




**7.**




Write down the gradient of each of the given lines.

**8.** **(i)** On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the $x$-axis and 1 cm to represent 2 units on the $y$-axis, draw the graph of each of the following functions for values of $x$ from $-4$ to 4.

**(a)** $y = -4x + 8$ **(b)** $y = -4x + 2$

**(c)** $y = -4x - 3$ **(d)** $y = -4x - 6$

OPEN **(ii)** Write down another set of four linear functions whose graphs are parallel to each other.

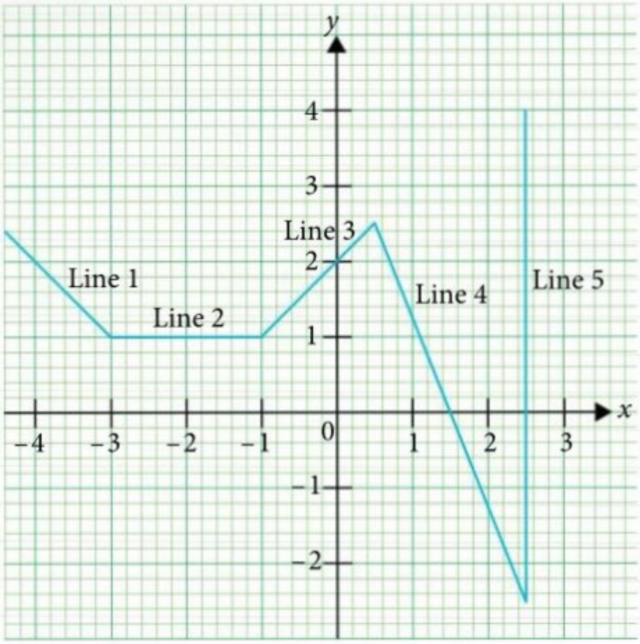
**9.** **(i)** On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the $x$-axis and 1 cm to represent 1 unit on the $y$-axis, draw the graph of the function $y = 6 - 3x$ for values of $x$ from $-3$ to 3.

**(ii)** The points $(a, 0)$, $(-2, b)$ and $(c, 1.5)$ lie on the graph in part **(i)**. Find the values of $a$, $b$ and $c$.

**10.** On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the $x$-axis and 1 cm to represent 1 unit on the $y$-axis, draw the graphs of the functions $y = 2x + 4$ and $y = 2 - 3x$ for values of $x$ from $-2$ to 2.

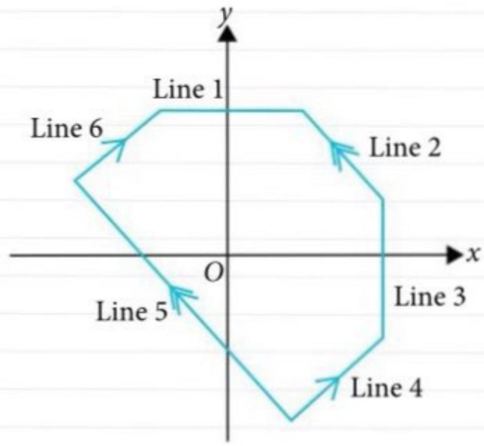
## Exercise 1B

**11.** The figure shows five line segments.




**(i)** Find the gradient of each of the line segments.
**(ii)** Find the equations of Lines 1, 3 and 4.

**12.** In the figure, Line 1 is parallel to the $x$-axis and Line 3 is parallel to the $y$-axis. Line 2 is parallel to Line 5 and Line 4 is parallel to Line 6. If the gradients of Line 5 and Line 6 are $-3$ and $\frac{1}{2}$ respectively, write down the gradients of Lines 1, 2, 3 and 4.




# 1.4 Applications of linear graphs in real-world contexts

Linear functions and graphs are used in many daily situations. In this section, we will examine how linear functions can be used to model some real-world situations.

**Class Discussion** Graphical representation of height of flag

Which graph in Fig. 1.17 best represents the height of a flag during the flag hoisting?

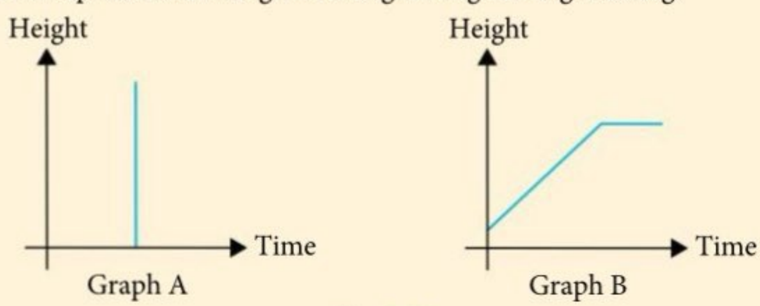



Fig. 1.17

In Fig. 1.17, the vertical line in Graph A might look like a flagpole but it is *not* the correct answer. A graph is not a visual representation of a situation or phenomenon but an **abstract representation** of the relationship between two variables. Therefore, the correct answer is Graph B, which shows that the height of the flag increases as time increases, and plateaus when it reaches the maximum height.

Worked Example 4

### Taxi fare

The flag-down fare of a taxi in Islamabad is PKR 200.

**(i)** Given that a passenger is charged PKR 70 for each kilometre the taxi travels, find the amount of money the passenger has to pay if the taxi covers a distance of

**(a)** 2 km, **(b)** 6 km, **(c)** 10 km.

**(ii)** Given that PKR $y$ represents the amount of money a passenger has to pay if the taxi travels $x$ km, copy and complete the table.

| $x$ | 2 | 6 | 10 |
|---|---|---|---|
| $y$ | | | |

**(iii)** On a sheet of graph paper, using a scale of 1 cm to represent 2 km on the horizontal axis and 1 cm to represent PKR 100 on the vertical axis, plot the pairs of values of $(x, y)$.

**(iv)** Write down the equation of the straight line.

**(v)** How much does a passenger have to pay for a taxi ride of

**(a)** 8 km, **(b)** 17 km?

### Solution

**(i)** **(a)** Fare for 2 km
= PKR 200 + 2 × PKR 70
= PKR 340

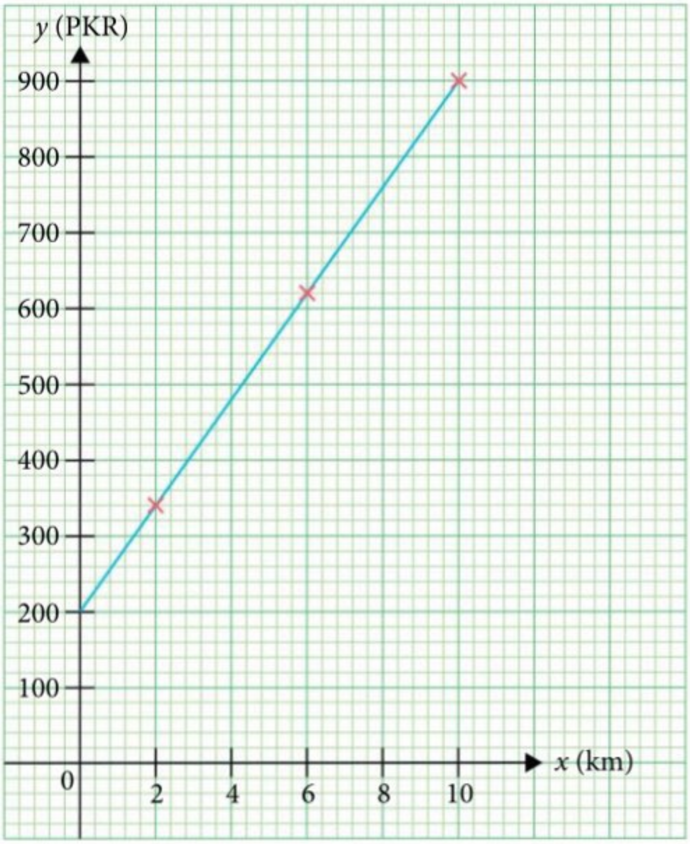
**(b)** Fare for 6 km
= PKR 200 + 6 × PKR 70
= PKR 620

**(c)** Fare for 10 km
= PKR 200 + 10 × PKR 70
= PKR 900

**(ii)**

| $x$ | 2 | 6 | 10 |
|---|---|---|---|
| $y$ | 340 | 620 | 900 |

(iii)




(iv) From the graph, gradient $= \dfrac{\text{rise}}{\text{run}}$

$$= \frac{700}{10}$$

$$= 70$$

$y$-intercept $= 200$

$\therefore$ the equation of the straight line is $y = 70x + 200$.

(v) (a) **Method 1:**

Fare for 8 km = PKR 200 + 8 × PKR 70
= PKR 760

**Method 2:**

Substitute $x = 8$ into $y = 70x + 200$:

$y = 70(8) + 200$
$= 760$

$\therefore$ the fare for 8 km is PKR 760.

**Method 3:**

From the graph, when $x = 8$, $y = 760$.

$\therefore$ the fare for 8 km is PKR 760.

**Reflection**

How are Methods 1 and 2 related? Which method do you prefer?

(b) **Method 1:**

Fare for 17 km = PKR 200 + 17 × PKR 70
= PKR 1390

**Method 2:**

Substitute $x = 17$ into $y = 70x + 200$:

$y = 70(17) + 200$

$= 1390$

$\therefore$ the fare for 17 km is PKR 1390.

**Attention**

For **(b)**, we cannot use Method 1 like in **(a)**, as the graph is only plotted for values of $x$ up till 10.

**Practise Now 4**

Similar and Further Questions
**Exercise 1C**
Questions 1, 2, 4, 5

A telecommunications service provider charges \$30 for a mobile plan offering 2 GB of data monthly.

**(i)** Given that a customer is charged \$12 for every GB of data used that exceeds 2 GB, find the amount that a customer will be charged if he uses a monthly data of

**(a)** 3 GB, **(b)** 5 GB, **(c)** 8 GB.

**(ii)** Given that \$$y$ represents the amount a customer is charged if he uses $x$ GB of data, copy and complete the table.

| $x$ | 3 | 5 | 8 |
|---|---|---|---|
| $y$ | | | |

**(iii)** On a sheet of graph paper, using a scale of 2 cm to represent 1 GB on the horizontal axis and 1 cm to represent \$5 on the vertical axis, plot the pairs of values of $(x, y)$.

**(iv)** Write down the equation of the straight line for $x > 2$.

**(v)** How much will a customer be charged if he uses a monthly data of

**(a)** 7 GB, **(b)** 13 GB?

**Worked Example 5**

## Conversion of units

Two units of measurement for temperature are degree Fahrenheit (°F) and degree Celsius (°C). The table below shows the corresponding $y$ values in °F after conversion from $x$ °C.

| $x$ | 0 | 5 | 20 |
|---|---|---|---|
| $y$ | 32 | 41 | 68 |

**(i)** On a sheet of graph paper, plot the pairs of values of $(x, y)$. Use a scale of 2 cm to represent 5 °C on the $x$-axis and 2 cm to represent 10 °F on the $y$-axis.

**(ii)** Write down the equation of the straight line.

**(iii)** Convert

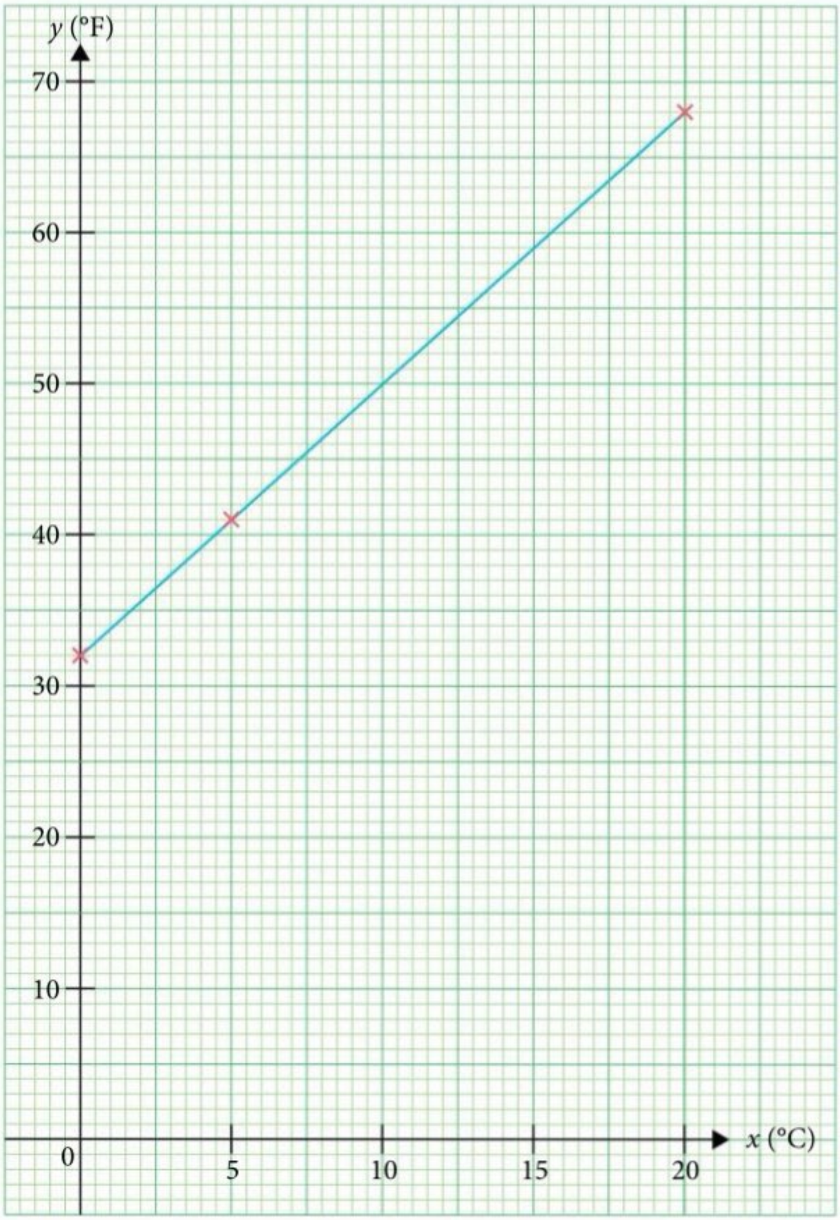
**(a)** 10 °C to °F,

**(b)** 212 °F to °C,

**(c)** 12 °C to °F.

**Solution**

**(i)**




**(ii)** From the graph, gradient $= \dfrac{\text{rise}}{\text{run}}$

$$= \frac{68 - 32}{20}$$

$$= 1.8$$

$y$-intercept $= 32$

$\therefore$ the equation of the straight line is $y = 1.8x + 32$.

**(iii) (a)** **Method 1:**

Substitute $x = 10$ into $y = 1.8x + 32$:

$y = 1.8(10) + 32$

$= 50$

$\therefore$ the temperature is 50 °F.

**Attention**

Graphs such as the one shown in this worked example, which can be used to convert a value from one unit to another, are known as **conversion graphs**. Another example of a conversion graph is one that shows the exchange rate between two currencies.

**Method 2:**

From the graph, when $x = 10$, $y = 50$.

$\therefore$ the temperature is 50 °F.

**(b)** Substitute $y = 212$ into $y = 1.8x + 32$:

$$212 = 1.8x + 32$$
$$212 - 32 = 1.8x + 32 - 32$$
$$1.8x = 180$$
$$1.8x \div 1.8 = 180 \div 1.8$$
$$x = 100$$

$\therefore$ the temperature is 100 °C.

**(c)** Substitute $x = 12$ into $y = 1.8x + 32$:

$y = 1.8(12) + 32$

$= 53.6$

$\therefore$ the temperature is 53.6 °F.

**Attention**

For **(c)**, we are not able to read off the graph to 0.1 °F. In this case, it is preferable to use the equation of the graph or the conversion rate if we know it.

**Practise Now 5**

Similar and Further Questions
**Exercise 1C**
Questions 3, 6

On a particular day, a money changer exchanges $x$ South Korean won (KRW) for $y$ Pakistani rupees (PKR). The table shows some values of $x$ and the corresponding values of $y$.

| $x$ (PKR) | 100 | 200 | 400 |
|---|---|---|---|
| $y$ (KRW) | 420 | | 1680 |

**(i)** Complete the table.

**(ii)** On a sheet of graph paper, using a scale of 2 cm to represent PKR 50 on the $x$-axis and 2 cm to represent KRW 100 on the $y$-axis, plot the points given in the table and draw a straight line passing through them.

**(iii)** Write down the equation of the straight line. Hence, state the currency exchange rate on that day.

**(iv)** Determine

**(a)** the amount of South Korean won received in exchange for PKR 300,

**(b)** the amount in Pakistani rupees received in exchange for KRW 50,

**(c)** the amount of South Korean won received in exchange for PKR 270.

**Worked Example 6**

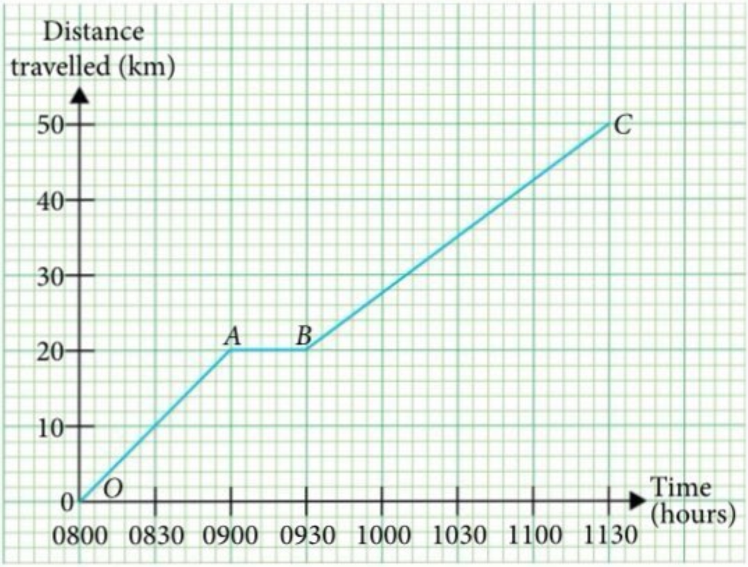
## Distance-time graph

The travel graph shows a journey taken by a cyclist. He started his 50-km journey at 0800 hours. At 0900 hours, he spent half an hour replacing a punctured tyre. He then continued his journey and reached his destination at 1130 hours.

**(i)** How far did the cyclist travel before his bicycle tyre was punctured?

**(ii)** Find the gradient of each of the following line segments, stating clearly what each gradient represents.

**(a)** $OA$ **(b)** $AB$ **(c)** $BC$




**Big Idea**

**Models**

The graph is used to model the journey of the cyclist. In this case, the model is a simplification of the situation because in reality, $OA$ and $BC$ may not be linear. Can you explain why?

### Solution

**(i)** 20 km

**(ii)** **(a)** Gradient of $OA = \dfrac{20 \text{ km}}{1 \text{ h}}$

$= 20$ km/h

This means that the average speed of the cyclist from $O$ to $A$ was 20 km/h.

**(b)** Gradient of $AB = 0$

This means that the average speed of the cyclist from $A$ to $B$ was 0 km/h, i.e. he was stationary.

**(c)** Gradient of $BC = \dfrac{30 \text{ km}}{2 \text{ h}}$

$= 15$ km/h

This means that the average speed of the cyclist from $B$ to $C$ was 15 km/h.

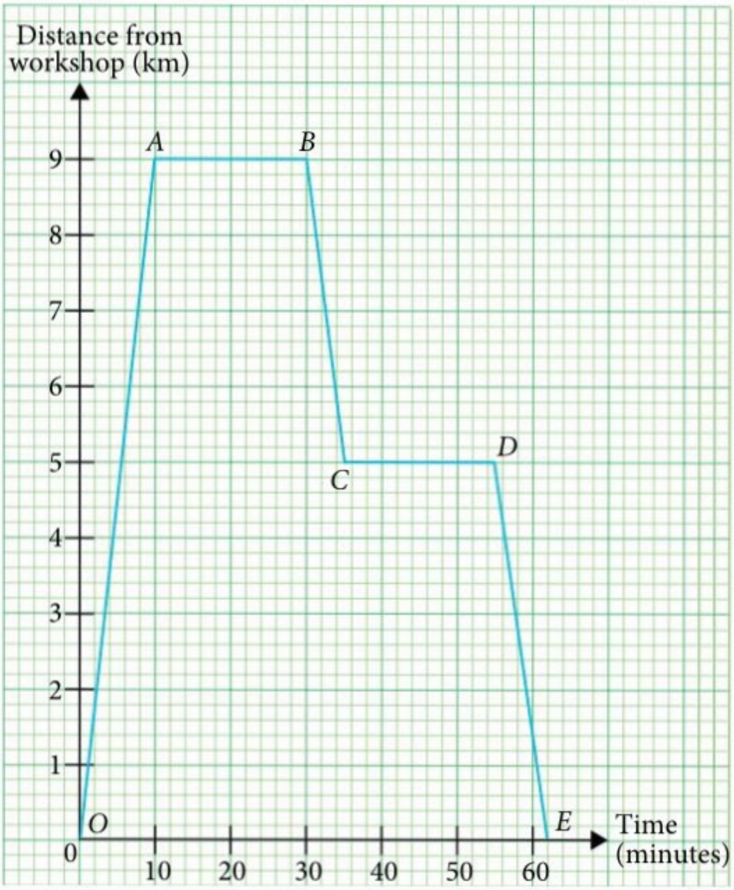
**Attention**

The gradient of a line in a distance-time graph is a measure of the speed of the object. Note that in this context, the gradient has a unit of measure.

**Practise Now 6**

Similar and Further Questions
**Exercise 1C** Questions 7, 8

A technician in a computer firm drove from his workshop to repair a customer's computer. On his way back, he stopped to repair another customer's computer. The distance-time graph shows his entire journey.




**(i)** How long did he take to repair each computer?

**(ii)** How far from his workshop was his first customer?

**(iii)** Find the gradient of each of the following line segments, stating clearly what each gradient represents.

**(a)** $OA$ **(b)** $AB$ **(c)** $BC$ **(d)** $CD$ **(e)** $DE$

Basic Intermediate Advanced

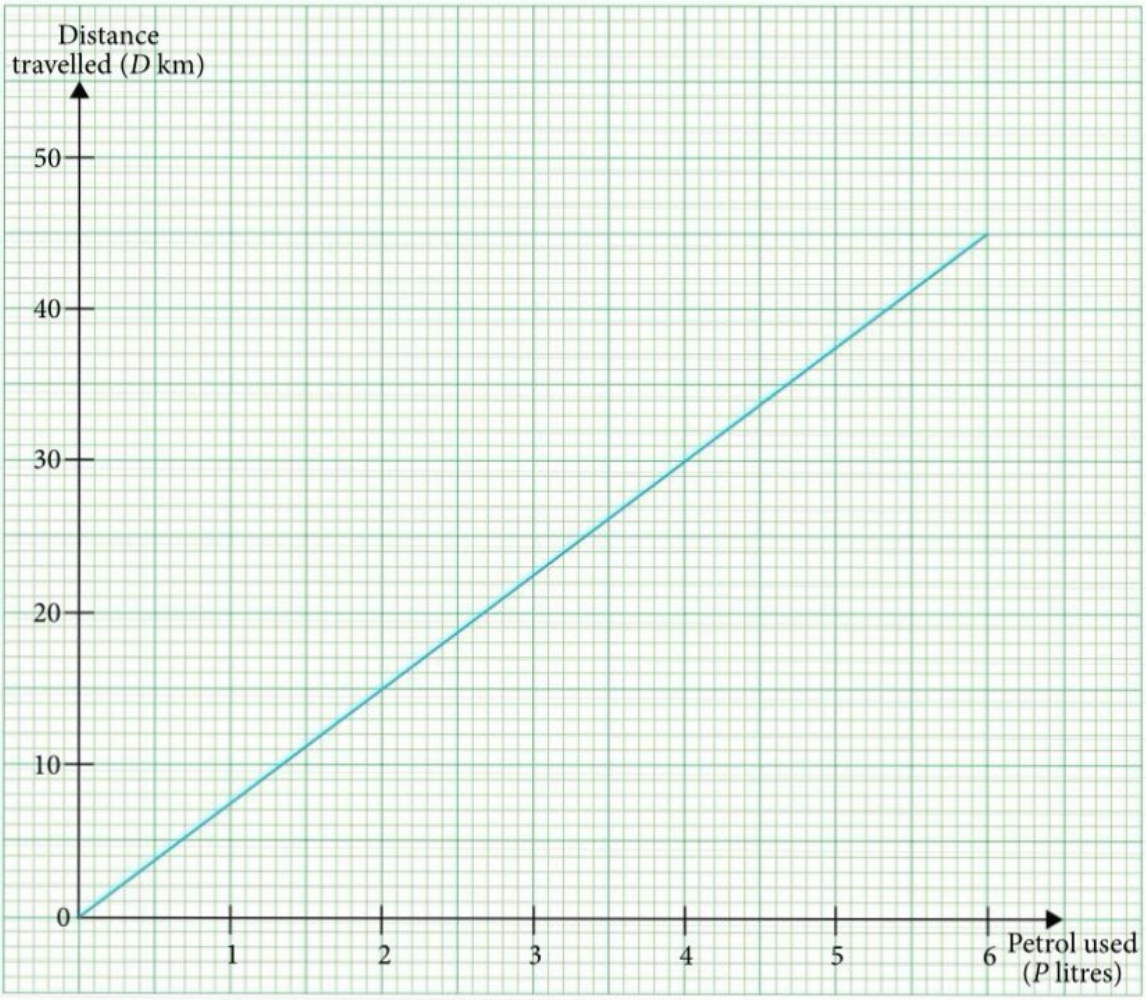
## Exercise 1C

1. An online retailer charges a basic service fee of \$3 for any purchase made.
   (i) A fee of \$2 per kg is charged for the delivery of packages. Find the amount a customer will have to pay if the package weighs
   (a) 0.5 kg, (b) 4 kg, (c) 9 kg.
   (ii) Given that \$$y$ represents the amount a customer pays for a package that weighs $x$ kg, copy and complete the table.

| $x$ | 0.5 | 4 | 9 |
|---|---|---|---|
| $y$ | | | |

   (iii) On a sheet of graph paper, using a scale of 2 cm to represent 1 kg on the horizontal axis and 1 cm to represent \$1 on the vertical axis, plot the pairs of values of $(x, y)$ and draw a straight line passing through them.
   (iv) Write down the equation of the straight line.
   (v) How much will a customer have to pay for a package that weighs
   (a) 7.5 kg, (b) 18.5 kg?

2. The graph shows the distance, $D$ km, travelled for every $P$ litre of petrol used.




Use the graph to find
   (i) how far the car can travel if it has
   (a) 2 litres of petrol, (b) 5.2 litres of petrol,
   (ii) the cost of petrol required to travel 30 km, given that 1 litre of petrol costs \$2.90.

## Exercise 1C

**3.** To attract customers, a shop launches a loyalty program that awards a customer with points for each purchase made. Under this program, 1 point is awarded for every dollar spent. The points can then be redeemed for a price reduction on the next purchase.

Nadia wants to find out the conversion rate between loyalty points and price reduction. She observes the number of points, $p$, each of three customers redeems and the corresponding price reduction, $\$d$, received, and records her findings in the table.

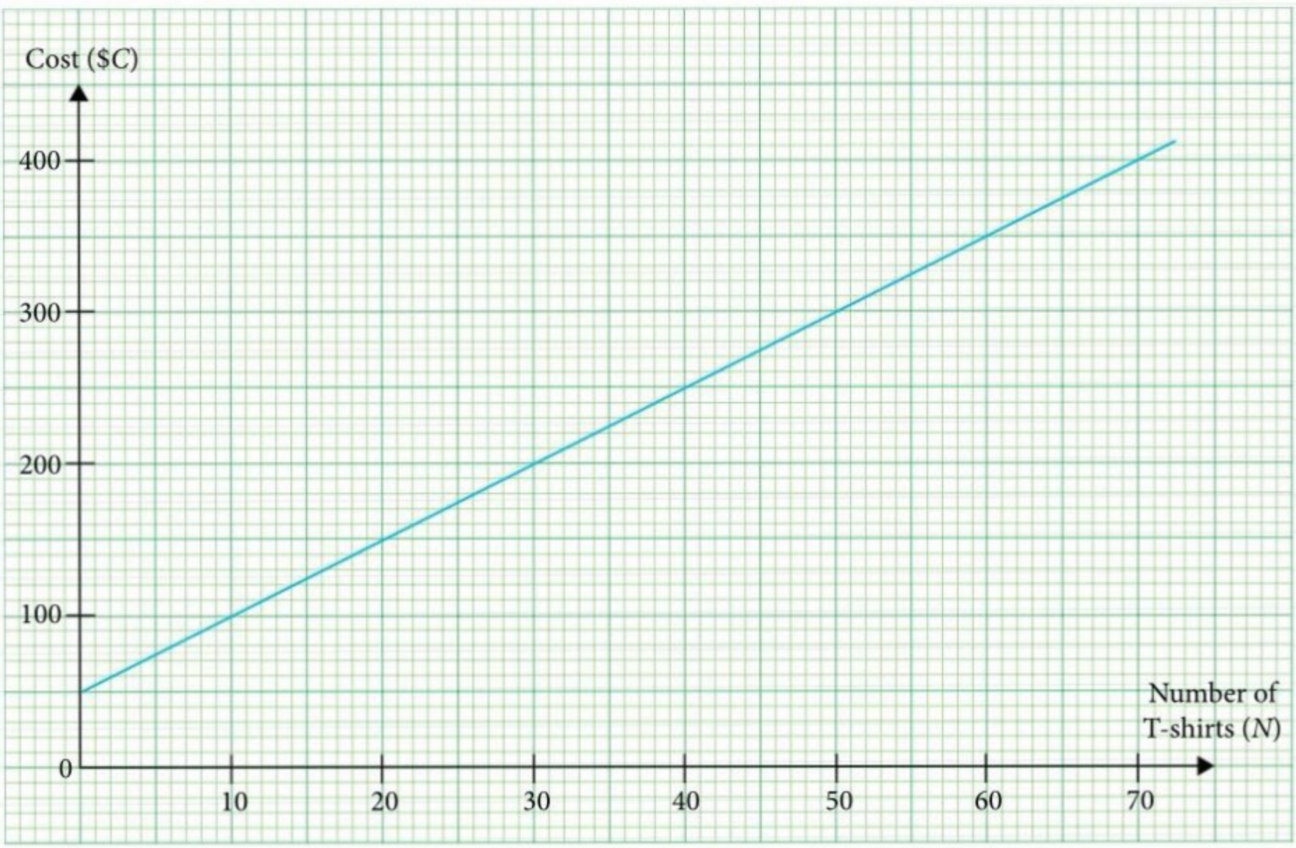
| $p$ | 10 | 43 | 75 |
|---|---|---|---|
| $d$ | 0.20 | 0.86 | 1.50 |

**(i)** On a sheet of graph paper, using a scale of 2 cm to represent 10 points on the $x$-axis and 2 cm to represent \$0.20 on the $y$-axis, plot the points given in the table.

**(ii)** Find the equation of the straight line.

**(iii)** A customer redeems 210 points for a purchase of \$8.90. Calculate the amount he pays in cash.

**4.** David works for a customisable T-shirt printing shop and is tasked to plot a graph that displays the cost of printing T-shirts.




**(i)** Copy and complete the table.

| $N$ | 10 | 30 | 50 | 70 |
|---|---|---|---|---|
| $C$ | | | | |

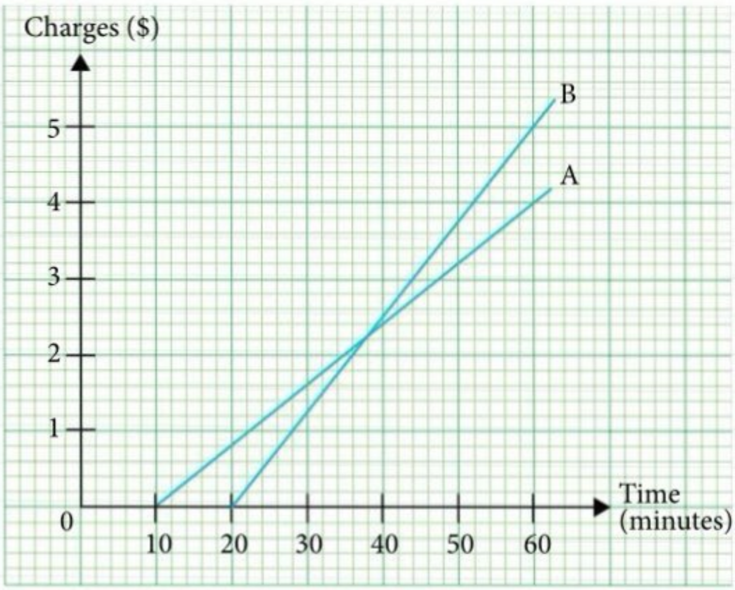
## Exercise 1C

**(ii)** David notices that the cost per T-shirt decreases when more T-shirts are printed and is puzzled by the observation from the graph that '0 T-shirts cost $50'. Provide a possible explanation to this problem.

**(iii)** Find the cost of printing 68 T-shirts.

**(iv)** A charitable organisation has a budget of $410 to print T-shirts for a charity event. How should David advise on the number of T-shirts that he can print for them?

**5.** Two mobile phone companies, A and B, offer plans with a talk time rate as shown in the graph.




**(i)** How much does Company A charge for 20 minutes of talk time?

**(ii)** How much does Company B charge for 50 minutes of talk time?

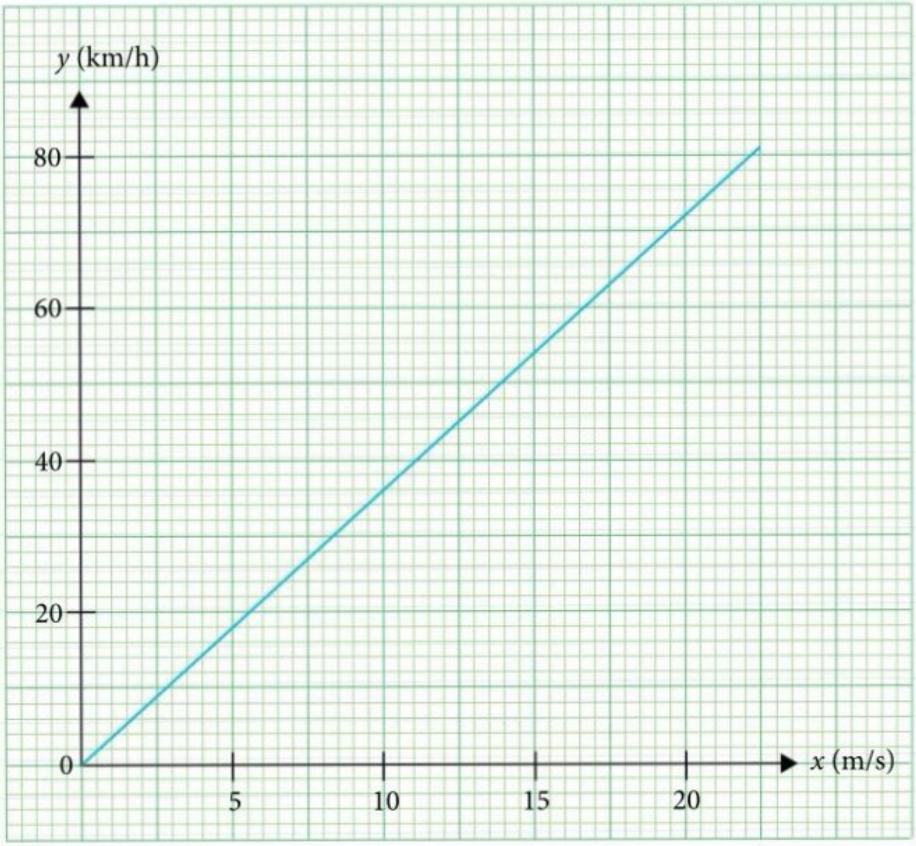
**(iii)** If Raju uses less than 30 minutes of talk time per month, which company offers him a better price? Explain your answer.

**(iv)** Which company has a greater rate of increase in charges? Explain your answer.

**(v)** If Albert wants to pay only $4 per month for a talk time plan, which company should he choose? Explain your answer.

## Exercise 1C

6. The graph below shows the relationship between values measured in m/s and in km/h.




(i) Given that $y$ km/h = $x$ m/s, complete the table below.

| $x$ | 5 | 10 | 20 |
|---|---|---|---|
| $y$ | | | |

(ii) Find the equation of the graph. Hence explain how

(a) a value in m/s can be converted to km/h,

(b) a value in km/h can be converted to m/s.

(iii) The speed limit on some roads in Pakistan is 80 km/h. Explain whether a car travelling at 25 m/s exceeds this speed limit.

Basic Intermediate Advanced

## Exercise 1C

**7.** The graph shows Ken's journey when he visited a friend in Town $C$. During the journey, he stopped at a cafeteria, after which he continued to drive to Town $C$.


**(i)** At what time did he leave home?

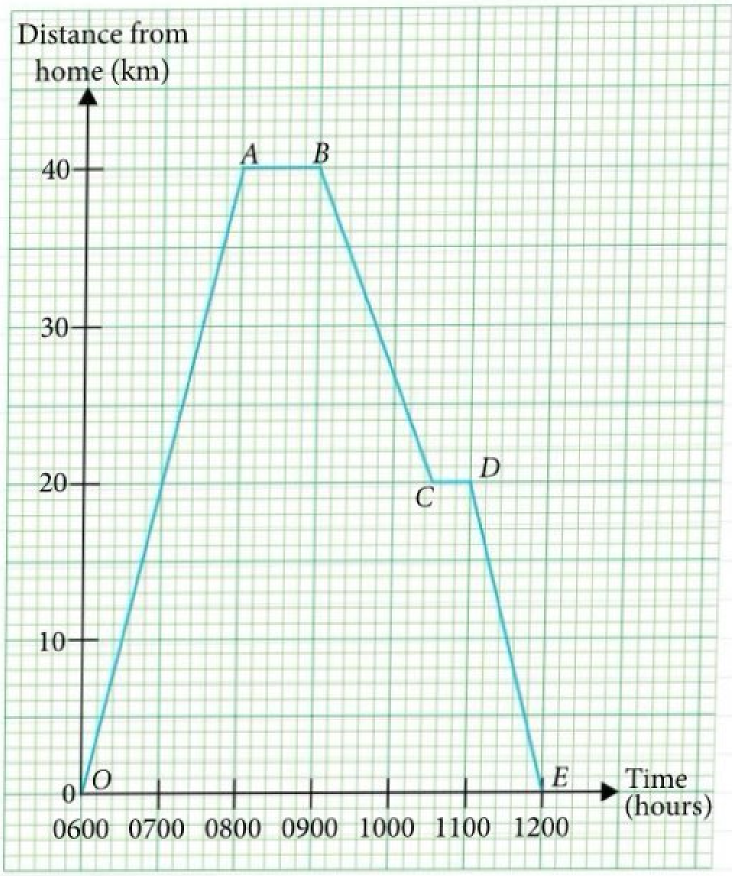
**(ii)** How far did he travel before he reached the cafeteria?

**(iii)** Find the gradient of each of the following line segments, stating clearly what each gradient represents.

**(a)** $OA$ **(b)** $AB$ **(c)** $BC$

## Exercise 1C

8. Waseem cycled from home to a town. On his way back, he stopped at a restaurant to have his breakfast. The distance-time graph shows his entire journey.




**(i)** How far was the town from Waseem's home?

**(ii)** Find the total time he stayed at the town and at the restaurant.

**(iii)** Find the gradient of each of the following line segments, stating clearly what each gradient represents.

**(a)** $OA$ **(b)** $BC$ **(c)** $DE$

**(iv)** What is his speed at the following times?

**(a)** 0700 hours **(b)** 0830 hours **(c)** 1015 hours

## Looking Back

This chapter introduced us to an important concept in mathematics – **function** – which we will continue to explore. Functions describe the relationship between two or more variables and thus are often used to **model** many real-world situations. The uniqueness of a function's output can be illustrated by many machines, from calculators to vending machines. These machines use specific inputs to produce a corresponding output. The linear function is the simplest function in mathematics. This important function describes two variables that follow a linear (straight-line) relationship. An important **measure** or property of a straight line is its gradient, which is constant. It represents the rate of change of one variable over the other. These ideas are made accessible to us through the Cartesian coordinate system, which makes us of **diagrams** to help us visualise relationships. Just like the battleship game in the Introductory Problem, the Cartesian coordinate system helps us to make connections between geometry and algebra!

## Summary

1. **Cartesian coordinate system**
   A Cartesian plane consists of two axes, the $x$-axis and the $y$-axis, intersecting at right angles at the origin $O(0, 0)$. The position of a point $P$ on a Cartesian plane can be described by an ordered pair $(x, y)$. We call $x$ the $x$-coordinate of $P$ and $y$ the $y$-coordinate of $P$, i.e. the coordinates of $P$ are $(x, y)$.

2. **Function**
   A function is a relationship between two variables $x$ and $y$ such that every input $x$ produces *exactly one output* $y$.
   The input $x$ and the output $y$ of a function can be written as an ordered pair $(x, y)$.
   A function can be represented using words, an equation, a table of values and a graph.
   An example of a linear function is $y = 2x + 3$.
   - Give two other examples of linear functions.

3. **Equation of straight line**
   The equation of a straight line is $y = mx + c$, where the constant $m$ is the *gradient* of the line and the constant $c$ is the *y-intercept*.

4. **Gradient of straight line**
   The gradient of a straight line is a measure of its steepness:

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}} \text{ or } \frac{\text{rise}}{\text{run}}.$$

   If a line slopes *upwards* from *left to right*, its gradient is *positive*.
   If a line slopes *downwards* from *left to right*, its gradient is *negative*.

Positive gradient Negative gradient

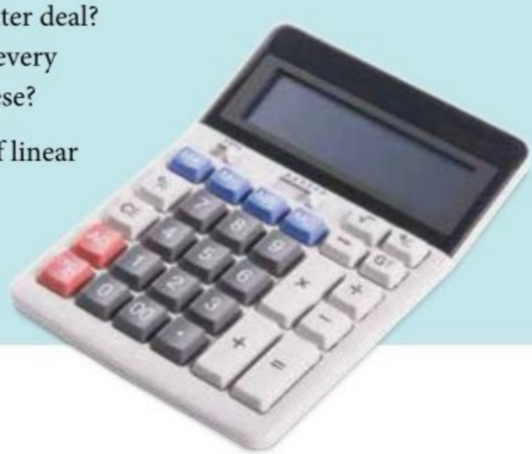
5. **Distance-time graph**
   The gradient of a line in a distance-time graph is a measure of the speed of the object.
   - Give another real-life example of a graph of a linear function.

CHAPTER 2

# Linear Graphs and Simultaneous Linear Equations

Given two mobile data plans, how can you decide which is a better deal? Would you pay $50 for unlimited data or would you pay $5 for every gigabyte (GB) of data used? How can we solve problems like these?

In this chapter, we will **model** situations like these using pairs of linear functions that relate two variables through equations or graphs. More specifically, we will solve problems to find the values which satisfy two conditions at the same time.



## Learning Outcomes

What will we learn in this chapter?

- What the equation of a horizontal line and of a vertical line is
- How to draw graphs of linear equations in the form $ax + by = k$
- How to solve simultaneous linear equations in two variables using
  - the graphical method
  - the elimination method
  - the substitution method
- Why formulating a pair of linear equations in two variables has useful real-life applications

## Introductory Problem

Cheryl has more money than Albert. If Cheryl gives \$40 to Albert, they will have the same amount. If Albert gives \$80 to Cheryl, Cheryl will then have four times as much as Albert. How much money does each of them have?

You might have encountered word problems like the **Introductory Problem** in primary school. Most likely, you used one of the following strategies:

**(a)** listing values in a table to guess and check;
**(b)** visualising the problem using the model method; or
**(c)** attempting to write equations to represent the relationships between the amounts of money that Cheryl and Albert have.

The 'guess and check' method may not be efficient in most cases and it may be challenging to draw the models for this kind of word problems. If you had tried to write algebraic equations to represent the amount of money that Cheryl and Albert have, you would probably have ended up with two distinct equations involving two variables, say $c$ and $d$, which represent the amount of money that Cheryl and Albert have respectively. To solve the problem is to find the values of $c$ and $d$ such that both equations are satisfied simultaneously or at the same time. At this point in time, we have not learnt how to solve this kind of equations. In this chapter, we will apply what we have learnt about a linear function and its graph, and extend the techniques of solving a linear equation to formulate and solve a pair of simultaneous linear equations in two variables.

## 2.1 Equations of straight lines

We have learnt that the equation of a straight line is in the form:

$$y = \underbrace{m}_{\text{gradient of line}}x + \underbrace{c}_{y\text{-intercept}},$$

where $m$ and $c$ are constants.

> **Big Idea**
>
> **Functions**
> The function $y = mx + c$ relates the input $x$ to the output $y$.

For example, the equation of the linear function $y = 2x + 1$ describes a straight line that has gradient 2 and $y$-intercept 1.

We have also learnt how to determine and create pairs of coordinates $(x, y)$ from the equation of a function that relates $y$ to $x$. The graph of the function is produced by plotting these coordinate pairs. This graph represents all possible $(x, y)$ coordinates described by the equation relating $y$ to $x$. Hence, the coordinates of any point on the graph will satisfy the equation of the function.

For instance, the graph of $y = 2x + 1$ shown in Fig. 2.1 is drawn using the following pairs of coordinates:

| $x$ | $y = 2x + 1$ | $(x, y)$ |
|---|---|---|
| 0 | $2(0) + 1 = 1$ | $(0, 1)$ |
| 2 | $2(2) + 1 = 5$ | $(2, 5)$ |
| 3.5 | $2(3.5) + 1 = 8$ | $(3.5, 8)$ |

Table 2.1

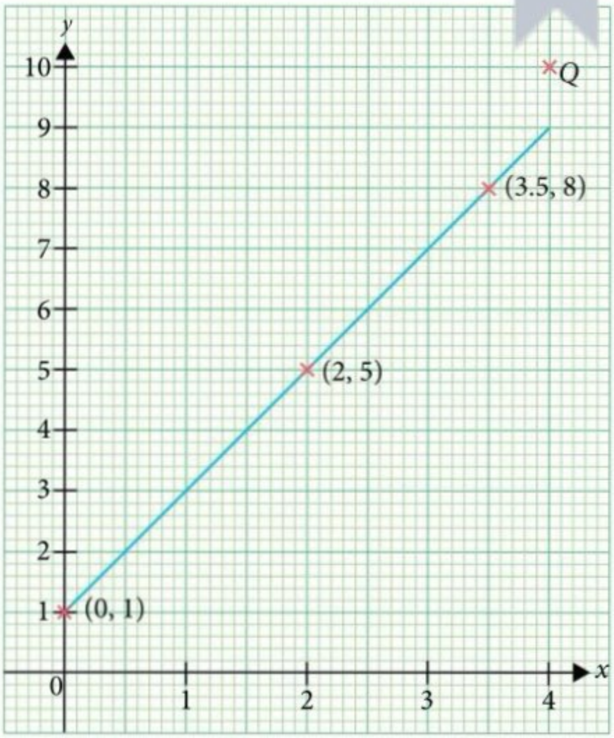



Fig. 2.1

If we substitute the $x$- and $y$-coordinates of point $Q(4, 10)$ into the equation,

LHS $= 10$

RHS $= 2(4) + 1$

$= 9 \neq$ LHS.

The coordinates of $Q$ do not satisfy the equation $y = 2x + 1$, and as can be seen in Fig. 2.1, $Q$ does not lie on the line.

We have seen how to represent a sloping line using an equation. Let us now look at two special cases: horizontal and vertical lines.

What are the equations of a horizontal line and a vertical line?

## A. Equation of horizontal lines

### Investigation — Equation of a horizontal line

Fig. 2.2 shows a horizontal line.

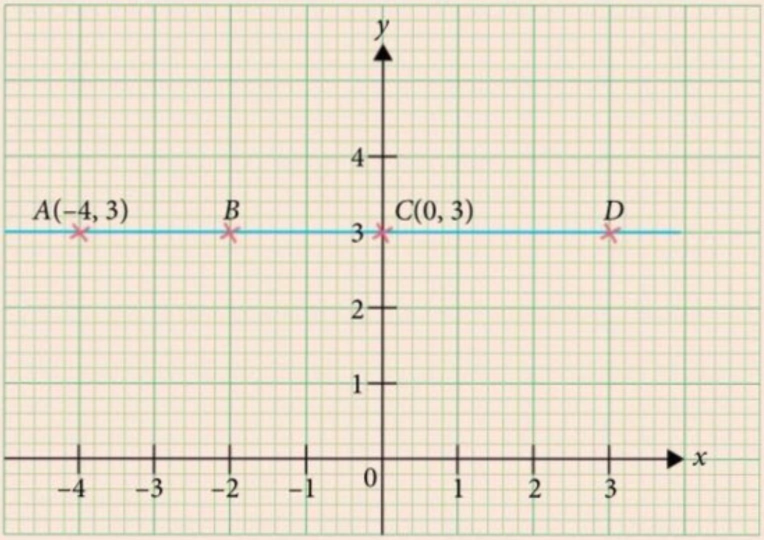



Fig. 2.2

1. What is the gradient of the horizontal line?
2. $A$, $B$, $C$ and $D$ lie on the line. Write down the coordinates of $B$ and of $D$.
3. What do you notice about the $y$-coordinates of all the four points on the horizontal line?
4. What can you say about a point $(k, 3)$, where $k$ is a real number?
5. What do you think the equation of the horizontal line is?

From the above Investigation, since the gradient $m$ of a horizontal line is 0,

> the equation of a horizontal line is
> $y = c$.

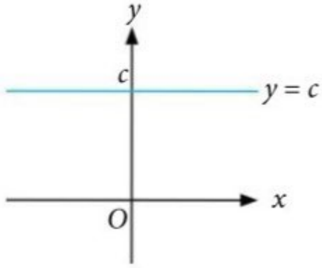


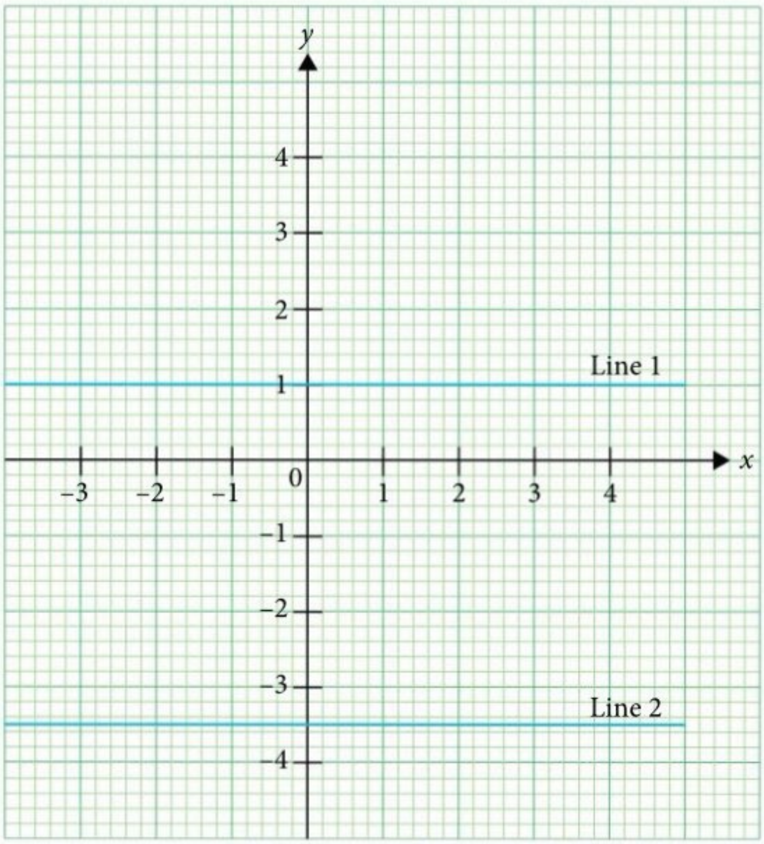
Fig. 2.3

We can show why the equation of a horizontal line is $y = c$:
The horizontal line is a special case of a linear function, where the gradient $m$ is 0.
Recall that the equation of a horizontal line is given by:

$$y = mx + c$$
$$y = (0)x + c$$
$$y = c$$

**Practise Now 1A**

Similar and Further Questions
**Exercise 2A**
Question 1(a), (b)




**(a)** Write down the equation of each of the given horizontal lines.
**(b)** On the graph, draw each of the lines with the following equations.
**(i)** $y = 2$ **(ii)** $y = 0$
Describe the lines.

## B. Equation of vertical lines

What about a vertical line?

**Information**

A vertical line is not the graph of a linear function. Why?

### Investigation — Equation of a vertical line

Fig. 2.4 shows a vertical line.

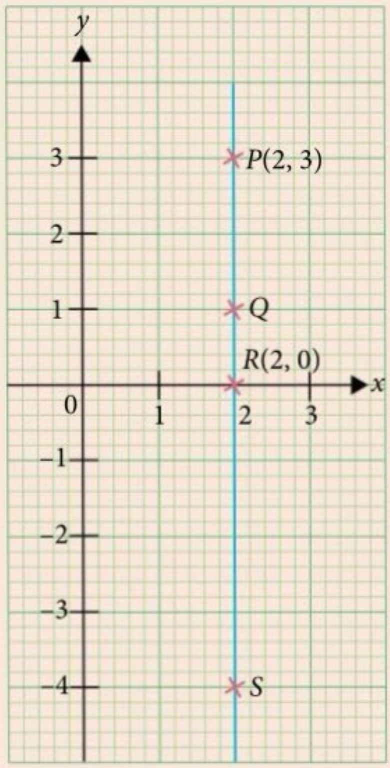



Fig. 2.4

1. What is the gradient of the vertical line?
2. $P$, $Q$, $R$ and $S$ lie on the line. Write down the coordinates of $Q$ and of $S$.
3. What do you notice about the $x$-coordinates of all the four points on the vertical line?
4. What can you say about a point $(2, k)$, where $k$ is a real number?
5. What do you think the equation of the vertical line is?

From the above Investigation, since the gradient $m$ of a vertical line is undefined, we cannot write the equation of a vertical line in the form $y = mx + c$. As the $x$-coordinates of all the points on a vertical line are equal to the same constant value $a$,

the equation of a vertical line is

$x = a$.

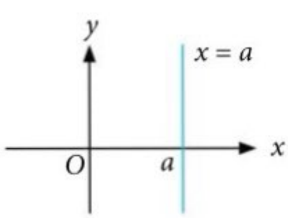


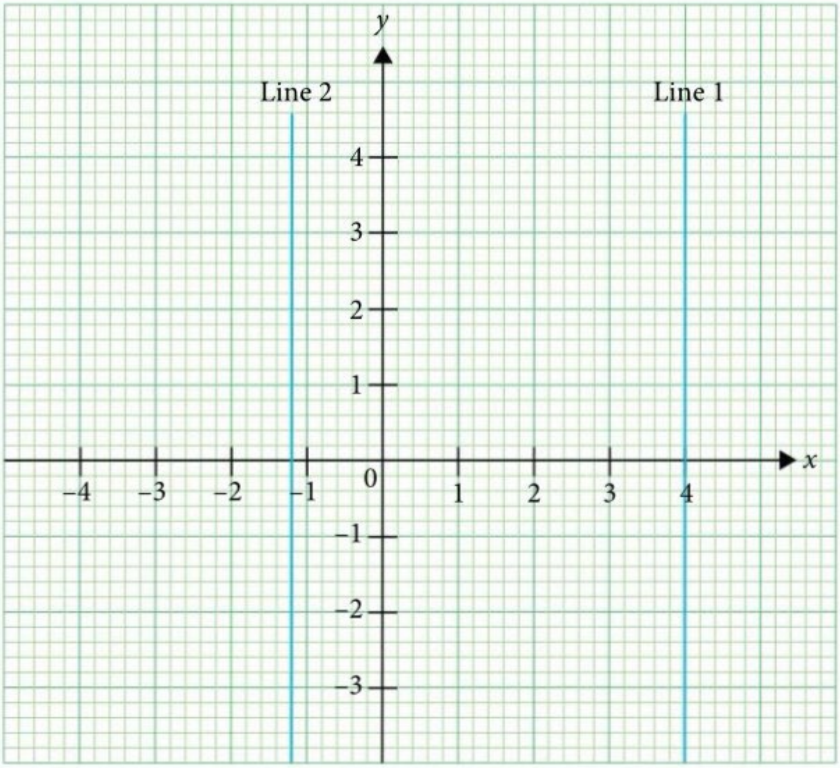

Fig. 2.5

**Information**

$x = a$ is not a linear function because there are many values of $y$ for one value of $x$.

**Practise Now 1B**

Similar and Further Questions
**Exercise 2A**
Question 2(a), (b)




**(a)** Write down the equation of each of the given vertical lines.

**(b)** On the graph, draw each of the lines with the following equations.

**(i)** $x = -3.5$

**(ii)** $x = 0$

Describe the lines.

### Reflection

What is a quick way to remember whether $x = a$ is horizontal or vertical?

## 2.2 Graphs of linear equations in the form $ax + by = k$

### A. Plotting graphs of $ax + by = k$

We have learnt how to draw graphs of linear equations in the form $y = mx + c$, where $m$ and $c$ are constants. In this section, we shall learn to draw the graphs of linear equations in the form $ax + by = k$, where $a$, $b$ and $k$ are constants.

## Investigation — Graphs of $ax + by = k$

1. Consider the equation $2x + y = 3$.
   - **(i)** Using a graphing software, draw the graph of $2x + y = 3$.
   - **(ii)** Do the points $A(2, -1)$ and $B(-2, 5)$ lie on the graph in part **(i)**?
     Do the coordinates of each of the points satisfy the equation $2x + y = 3$?
     Explain your answers.
   - **(iii)** The point $(1, p)$ lies on the graph in part **(i)**. Determine the value of $p$.
   - **(iv)** The point $(q, -7)$ lies on the graph in part **(i)**. Determine the value of $q$.
   - **(v)** On the same axes in part **(i)**, draw the graph of $y = -2x + 3$. What do you notice?
     Hence, show algebraically that $y = -2x + 3$ can be obtained from $2x + y = 3$.
2. Consider the equation $3x - 4y = 6$.
   - **(i)** Using a graphing software, draw the graph of $3x - 4y = 6$.
   - **(ii)** The point $(2, r)$ lies on the graph in part **(i)**. Determine the value of $r$.
   - **(iii)** The point $(s, -1.5)$ lies on the graph in part **(i)**. Determine the value of $s$.
   - **(iv)** State the coordinates of two other points that satisfy the equation $3x - 4y = 6$.
   - **(v)** On the same axes in part **(i)**, draw the graph of $y = \frac{3}{4}x - \frac{3}{2}$. What do you notice?
     Hence, show algebraically that $y = \frac{3}{4}x - \frac{3}{2}$ can be obtained from $3x - 4y = 6$.

**Big Idea**

**Equivalence**
Equations of the form $ax + by = k$ can be rewritten in the form $y = mx + c$, where $m = -\frac{a}{b}$ and $c = \frac{k}{b}$. We say that these two equations are equivalent. Equivalence of equations allows us to rewrite one equation into another form that is easier for us to work with.

### Worked Example 1

**Drawing the graph of $ax + by = k$**

The variables $x$ and $y$ are connected by the equation $2x - 3y = 2$. Some values of $x$ and the corresponding values of $y$ are given in the table.

| $x$ | $-2$ | $-0.5$ | $4$ |
|---|---|---|---|
| $y$ | $-2$ | $p$ | $2$ |

**(a)** Calculate the value of $p$.
**(b)** On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the $x$-axis and 2 cm to represent 1 unit on the $y$-axis, draw the graph of $2x - 3y = 2$ for $-2 \leqslant x \leqslant 4$.
**(c)** The point $(1, q)$ lies on the graph in part **(b)**. Find the value of $q$.
**(d)** **(i)** On the same axes in part **(b)**, draw the graph of $y = 1$.
**(ii)** State the $x$-coordinate of the point on the graph of $2x - 3y = 2$ that has a $y$-coordinate of 1.

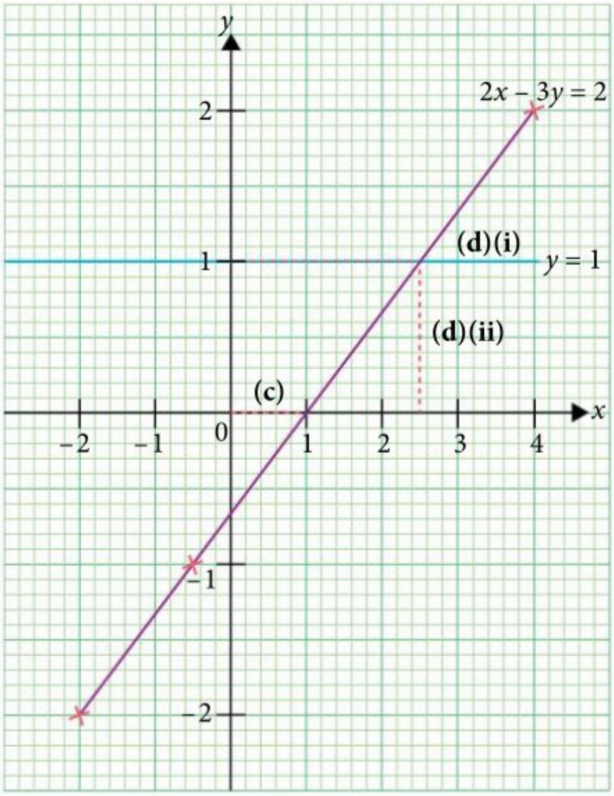
**Attention**
$-2 \leqslant x \leqslant 4$ represents values of $x$ that are more than or equal to $-2$ but less than or equal to 4.

**Solution**

**(a)** When $x = -0.5$, $y = p$,

$$2(-0.5) - 3p = 2$$
$$3p = -3$$
$$\therefore p = -1$$

**(b)**




**(c)** From the graph in part **(b)**,
when $x = 1$,
$q = y = 0$

**(d)** **(ii)** $x$-coordinate of point = 2.5

**Practise Now 1C**

Similar and Further Questions
**Exercise 2A**
Questions 4, 6

The variables $x$ and $y$ are connected by the equation $3x + y = 1$. Some values of $x$ and the corresponding values of $y$ are given in the table.

| $x$ | –2 | 0 | 2 |
|---|---|---|---|
| $y$ | $p$ | 1 | –5 |

**(a)** Find the value of $p$.

**(b)** On a sheet of graph paper, using a scale of 4 cm to represent 1 unit on the $x$-axis and 1 cm to represent 1 unit on the $y$-axis, draw the graph of $3x + y = 1$ for $-2 \leqslant x \leqslant 2$.

**(c)** The point $(-1, q)$ lies on the graph in part **(b)**. Find the value of $q$.

**(d)** **(i)** On the same axes in part **(b)**, draw the graph of $x = -0.5$.

**(ii)** State the $y$-coordinate of the point on the graph of $3x + y = 1$ that has an $x$-coordinate of $-0.5$.

## B. Sketching graphs of $ax + by = k$

In Worked Example 1 of Section 2.2A, we have learnt how to plot the graph of $2x - 3y = 2$ on a sheet of graph paper. Sketching a graph does not require a grid. Instead, we identify the points that provide information about the graph, such as where the graph cuts the $x$- and $y$-axes.

**Worked Example 2**

**Sketching the graph of $ax + by = k$**

Sketch the graph of $x + 3y = 6$.

**Solution**

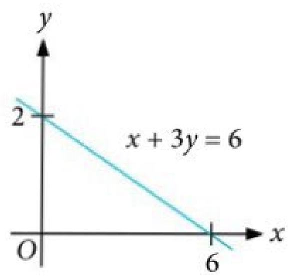
Substitute $x = 0$ into $x + 3y = 6$:

$$0 + 3y = 6$$
$$y = 2$$

Substitute $y = 0$ into $x + 3y = 6$:

$$x + 3(0) = 6$$
$$x = 6$$

$\therefore$ the graph cuts the $x$- and $y$- axes at $(6, 0)$ and $(0, 2)$ respectively.




**Problem-solving Tip**

For linear graphs, use a ruler to draw the line. Label the $x$- and $y$-values where the graph cuts the axes and write down the equation of the line.

**Attention**

For non-linear graphs, other points that are characteristic of the graphs must also be labelled in the sketch. Some examples are turning points and asymptotes.

We will learn more about sketching non-linear graphs in Book 3.

**Practise Now 2**

Similar and Further Questions
Exercise 2A
Questions 3(a)-(f), 5

Sketch the graph of each of the following functions.

**(a)** $3x - y = 1$

**(b)** $4y + \frac{1}{2}x = -2$

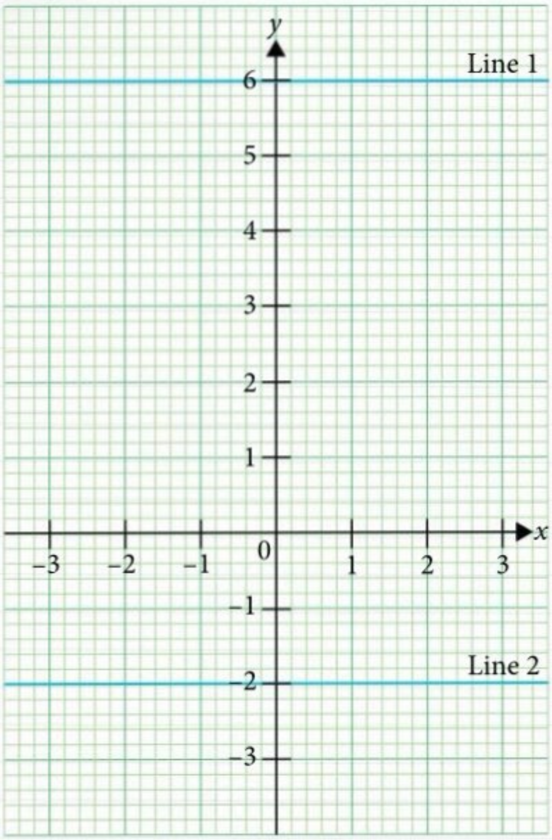
**(c)** $2y = 4x + 3$

**(d)** $x + y = 0$

Basic Intermediate Advanced

## Exercise 2A

**1.**




**(a)** Write down the equation of each of the given horizontal lines.

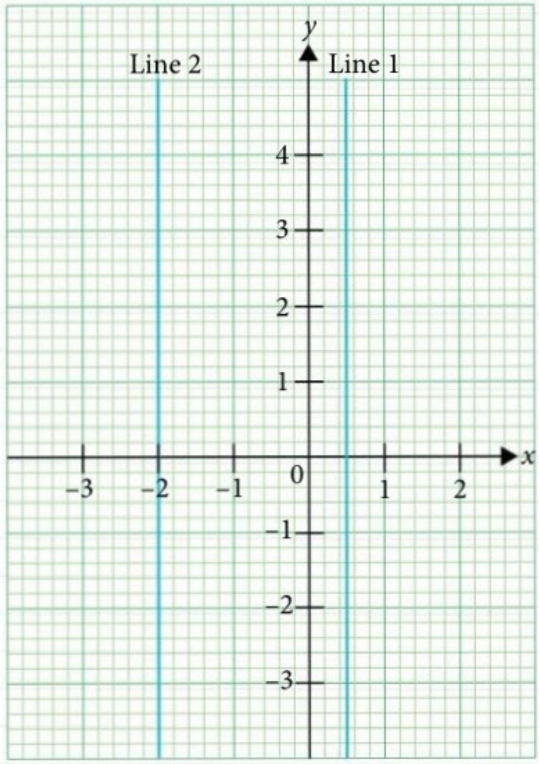
**(b)** On the graph, draw each of the lines with the following equations.

**(i)** $y = -3$

**(ii)** $y = 3\frac{1}{2}$

Describe the lines.

**2.**




**(a)** Write down the equation of each of the given vertical lines.

**(b)** On the graph, draw each of the lines with the following equations.

**(i)** $x = 1$

**(ii)** $x = -2\frac{1}{2}$

Describe the lines.

**3.** Sketch the graph of each of the following functions.

**(a)** $y - 2x = -1$

**(b)** $3x + 2y = 3$

**(c)** $2x - 5y = -10$

**(d)** $2.5x - 3y = -4$

**(e)** $\frac{1}{2}y + x = -2$

**(f)** $5x - 4y = 0$

## Exercise 2A

**4.** The variables $x$ and $y$ are connected by the equation $-x + 2y = 4$. Some values of $x$ and the corresponding values of $y$ are given in the table.

| $x$ | $-5$ | $0$ | $5$ |
|---|---|---|---|
| $y$ | $p$ | $2$ | $q$ |

**(i)** Find the value of $p$ and of $q$.

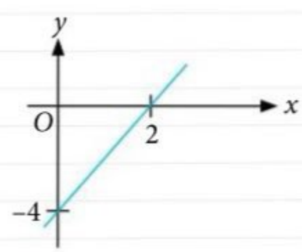
**(ii)** On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the $x$-axis and 2 cm to represent 1 unit on the $y$-axis, draw the graph of $-x + 2y = 4$ for $-5 \leqslant x \leqslant 5$.

**(iii)** The point $(r, 0.5)$ lies on the graph in part **(ii)**. Find the value of $r$.

**(iv)** **(a)** On the same axes in part **(ii)**, draw the graph of $x = 3$.

**(b)** State the $y$-coordinate of the point on the graph of $-x + 2y = 4$ that has an $x$-coordinate of 3.

**5.** The following shows the graph of a function.




**(i)** Determine the equation of the function.

**(ii)** The points $P$ and $Q$ lie on the graph and have the coordinates $(1, p)$ and $(q, 3)$ respectively. Find the values of $p$ and $q$.

OPEN **(iii)** Point $R$ has coordinates $(4, r)$. Give two examples of the value of $r$ such that $R$ does not lie on the graph.

**6.** Consider the equation $-2x + y = -3$.

**(i)** Copy and complete the table.

| $x$ | $-1$ | $0$ | $2$ |
|---|---|---|---|
| $y$ | | | |

**(ii)** On a sheet of graph paper, using a scale of 4 cm to represent 1 unit on the $x$-axis and 2 cm to represent 1 unit on the $y$-axis, draw the graph of $-2x + y = -3$ for $-1 \leqslant x \leqslant 2$.

**(iii)** **(a)** On the same axes in part **(ii)**, draw the graph of $y = -1$.

**(b)** Find the area of the trapezium bounded by the lines $-2x + y = -3$, $y = -1$, and the $x$- and $y$-axes.

# 2.3 Solving simultaneous linear equations using graphical method

We have learnt how to draw the graphs of $2x - 3y = 2$, $y = -3$, $x = -2\frac{1}{2}$, $-x + 2y = 4$, etc. Let us now apply these ideas to solve some problems.

**Investigation** Solving simultaneous linear equations graphically

1. Consider the linear equations $2x + 3y = 5$ and $3x - y = 2$.
   **(i)** Using a graphing software, draw the graphs of $2x + 3y = 5$ and $3x - y = 2$ on the same axes.
   **(ii)** What are the coordinates of the point of intersection of the two graphs?
   Five pairs of values of $x$ and $y$ are given in Table 2.2. Some of these points lie on the line $2x + 3y = 5$ and the rest lie on the line $3x - y = 2$.

| $x$ | −2 | 0 | 1 | 2 | 4 |
|---|---|---|---|---|---|
| $y$ | 3 | −2 | 1 | 4 | −1 |

Table 2.2

   **(iii)** Substitute the first pair of values of $x$ and $y$ in Table 2.2 (i.e. $x = -2$ and $y = 3$) into each of the two equations $2x + 3y = 5$ and $3x - y = 2$. What do you notice?
   **(iv)** Repeat part **(iii)** for the other pairs of values of $x$ and $y$ in Table 2.2. What do you notice?
2. Consider the linear equations $3x - 4y = 10$ and $5x + 7y = 3$.
   **(i)** Using a graphing software, draw the graphs of $3x - 4y = 10$ and $5x + 7y = 3$ on the same axes.
   **(ii)** What are the coordinates of the point of intersection of the two graphs?
   **(iii)** Hence, state the pair of values of $x$ and $y$ that satisfies both the equations $3x - 4y = 10$ and $5x + 7y = 3$.
3. What can we conclude about the coordinates of the point of intersection of the two graphs and the pair of values of $x$ and $y$ that satisfies both the equations? Explain your answer.

In the above Investigation, the graphs of $2x + 3y = 5$ and $3x - y = 2$ intersect at the point (1, 1). The set of values of $x = 1$ and $y = 1$ satisfies the two linear equations simultaneously. We say that $x = 1$ and $y = 1$ is the **solution** of the **simultaneous linear equations** $2x + 3y = 5$ and $3x - y = 2$.

Can we say the same for the linear equations $3x - 4y = 10$ and $5x + 7y = 3$?

## A. Choice of appropriate scales for graphs

If the scale for drawing a graph is not given, we have to choose a suitable scale before drawing the graph. The following guidelines may be useful:

- Use a convenient scale for both the $x$-axis and the $y$-axis. For example, we may use 1 cm to represent 1 unit, 2 units, 4 units, 5 units or 10 units. Avoid using awkward scales such as 1 cm to represent 3 units or 1 cm to represent 4.3 units.
- The scale used for the $x$-axis need not be the same as the scale used for the $y$-axis.
- Choose a suitable scale so that the graph will occupy more than half the size of the graph paper.
- Look at the largest and the smallest value of $x$ and estimate the scale to be used. Repeat the process for the values of $y$.

## Class Discussion

**Choice of appropriate scales for graphs and accuracy of graphs**

1. Using a suitable scale, draw the graph of $y = 3x - 1$ for $-3 \leqslant x \leqslant 3$. Compare your graph with that of your classmate. Do the graphs look different?
2. Use your graph in Question 1 to find
   (i) the value of $y$ when $x = 1.3$, (ii) the value of $x$ when $y = -2.8$.
3. How do you check for the accuracy of your answers in Question 2?
4. If your answers in Question 2 are inaccurate, how can you improve your graph?

## B. Solving simultaneous linear equations using graphical method

**Worked Example 3**

### Solving simultaneous linear equations using graphical method

Using the graphical method, solve the simultaneous equations

$$2x - 5y = 32,$$
$$2x + 3y = 0.$$

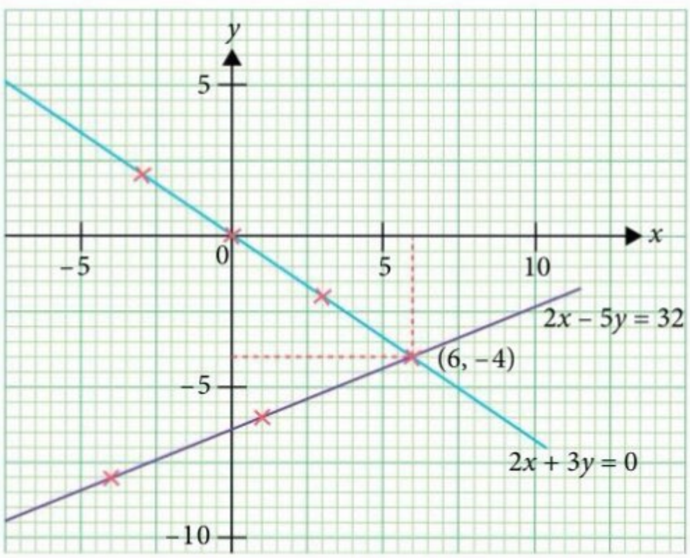
**Solution**

$2x - 5y = 32$

| $x$ | $-4$ | $1$ | $6$ |
|---|---|---|---|
| $y$ | $-8$ | $-6$ | $-4$ |

$2x + 3y = 0$

| $x$ | $-3$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $2$ | $0$ | $-2$ |




The graphs intersect at the point $(6, -4)$.

$\therefore$ the solution is $x = 6$ and $y = -4$.

**Recall**

We only need to plot 3 points to obtain the graph of a linear equation. In fact, a straight line can be determined by plotting 2 points. We use the 3rd point to check for mistakes in the graph.

**Problem-solving Tip**

An appropriate scale to use here would be 2 cm to 5 units for both axes.

**Big Idea**

**Diagrams**

The Cartesian coordinate system allows us to represent a pair of simultaneous equations graphically so that we can find the point of intersection which also gives the solution that satisfies both equations.

**Practise Now 3**

Similar and Further Questions
Exercise 2B
Questions 1(a)–(f), 2(a)–(d), 3

1. Using the graphical method, solve the simultaneous equations
$$x + y = 3,$$
$$3x + y = 5.$$
2. Using the graphical method, solve the simultaneous equations
$$7x - 2y + 11 = 0,$$
$$6x + y + 4 = 0.$$

## C. Coincident lines and parallel lines

### Class Discussion

**Finding number of points of intersection between coincident lines and between parallel lines**

Use a graphing software for this activity.

**1.** **(a)** For each of the following parts, draw the graphs of the pairs of simultaneous equations on the same axes.

**(i)** $x + y = 1$
$3x + 3y = 3$

**(ii)** $2x + 3y = -1$
$20x + 30y = -10$

**(iii)** $x - 2y = 5$
$5x - 10y = 25$

**(b)** What do you notice about the graphs of each pair of simultaneous equations?

**(c)** Does each pair of simultaneous equations have any solutions? If yes, what are the solutions?

**2.** **(a)** For each of the following parts, draw the graphs of the pairs of simultaneous equations on the same axes.

**(i)** $x + y = 1$
$3x + 3y = 15$

**(ii)** $2x + 3y = -1$
$20x + 30y = -40$

**(iii)** $x - 2y = 5$
$5x - 10y = 30$

**(b)** What do you notice about the graphs of each pair of simultaneous equations?

**(c)** Does each pair of simultaneous equations have any solutions? If yes, what are the solutions?

Similar and Further Questions
**Exercise 2B**
Questions 4(a)–(d), 5(a), (b)

From the above Class Discussion, we notice that the graphs of each pair of simultaneous equations in Question 1 are *identical*, i.e. the two lines coincide. Since every point on each line is a point of intersection of the graphs, the graphs have an infinite number of points of intersection. Hence, the pair of simultaneous equations has an *infinite number of solutions*.

The graphs of the simultaneous equations in Question 2 are *parallel lines*. Since the graphs do not intersect, they have no point of intersection. Hence, the simultaneous equations have *no solution*.

### Thinking Time

When given a pair of simultaneous equations, how can we determine, without solving, that the pair of simultaneous equations has

**(a)** only one solution?

**(b)** infinitely many solutions?

**(c)** no solution?

## Exercise 2B

**1.** Using the graphical method, solve each of the following pairs of simultaneous equations.

(a) $3x - y = 0$, $2x - y = 1$
(b) $x - y = -3$, $x - 2y = -1$
(c) $3x - 2y = 7$, $2x + 3y = 9$
(d) $3x + 2y = 4$, $5x + y = 2$
(e) $2x + 5y = 25$, $3x - 2y = 9$
(f) $3x - 4y = 25$, $4x - y = 16$

**2.** Using the graphical method, solve each of the following pairs of simultaneous equations.

(a) $x + 4y - 12 = 0$, $4x + y - 18 = 0$
(b) $3x + y - 2 = 0$, $2x - y - 3 = 0$
(c) $3x - 2y - 13 = 0$, $2x + 2y = 0$
(d) $2x + 4y + 5 = 0$, $-x + 5y + 1 = 0$

**3.** (a) Consider the equation $y = 2x + 9$.

(i) Copy and complete the table.

| $x$ | $-8$ | $0$ | $4$ |
|---|---|---|---|
| $y$ | | | |

(ii) On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the $x$-axis and 1 cm to represent 5 units on the $y$-axis, draw the graph of $y = 2x + 9$ for $-8 \leqslant x \leqslant 4$.

(b) Consider the equation $y = \frac{1}{4}x + 2$.

(i) Copy and complete the table.

| $x$ | $-8$ | $0$ | $4$ |
|---|---|---|---|
| $y$ | | | |

(ii) On the same axes in part **(a)(ii)**, draw the graph of $y = \frac{1}{4}x + 2$ for $-8 \leqslant x \leqslant 4$.

(c) Hence, solve the simultaneous equations $2x - y = -9$ and $x - 4y = -8$.

**4.** Using the graphical method, state if each of the following pairs of simultaneous equations has no solutions or infinitely many solutions and explain why.

(a) $x + 2y = 3$, $2x + 4y = 6$
(b) $4x + y = 2$, $4x + y = -3$
(c) $2y - x = 2$, $4y - 2x = 4$
(d) $2y + x = 4$, $2y + x = 6$

**5.** Using the graphical method, state if each of the following pairs of simultaneous equations has no solutions or infinitely many solutions and explain why.

(a) $y = 3 - 5x$, $5x + y - 1 = 0$
(b) $3y + x = 7$, $15y = 35 - 5x$

# 2.4 Solving simultaneous linear equations using algebraic methods

In Book 1, we have learnt how to solve linear equations in one variable such as $3x - 4 = 11$ and $4x - 10 = 5x + 7$.

In Section 2.3, we have learnt that from the graphs of two linear equations, the coordinates of the point(s) of intersection give the solution(s) to the pair of simultaneous linear equations. In this section, we shall take a look at two algebraic methods that can be used to solve a pair of simultaneous equations: the **elimination method** and the **substitution method**.

> **Recall**
> To solve a linear equation in one variable $x$ means to find the value of $x$ so that the values of the expressions on both sides of the equation are equal, i.e. $x$ satisfies the equation.

## A. Solving simultaneous linear equations using elimination method

Let us use the **elimination method** to solve the following pair of equations.

We shall label the equations as equation (1) and equation (2).

$$3x - y = 12 \quad \text{— (1)}$$
$$2x + y = 13 \quad \text{— (2)}$$

> **Recall**
> The absolute value of $-1$ is 1, and the absolute value of 1 is 1.

The elimination method is usually used when the absolute values of the coefficients of one variable in both the equations are the same. For example, the absolute values of the coefficients of $y$ in equations (1) and (2) are the same.

> **Big Idea**
> **Equivalence**
> We can add equations (1) and (2) based on the idea of equivalence. Do you know why?

What happens when we add the two equations?

(1) + (2): $\quad (3x - y) + (2x + y) = 12 + 13$

$$3x + 2x - y + y = 25$$

Notice that the terms in $y$ are eliminated. We now have a linear equation in one variable $x$.

$$5x = 25$$
$$x = 5$$

Substitute $x = 5$ into (1): $\quad 3(5) - y = 12$

$$y = 3$$

$\therefore$ the solution of the simultaneous equations is $x = 5$ and $y = 3$.

**Check:** Substitute $x = 5$ and $y = 3$ into (1) and (2):

In (1), LHS $= 3(5) - 3$
$= 12$
$=$ RHS

In (2), LHS $= 2(5) + 3$
$= 13$
$=$ RHS

Since $x = 5$ and $y = 3$ satisfies both the equations, it is the solution of the simultaneous equations.

> **Problem-solving Tip**
> It is a good practice to check your solution by substituting the values of the unknowns found into the original equations.

**Worked Example 4**

### Solving simultaneous linear equations using elimination method

Using the elimination method, solve the simultaneous equations

$3x + 7y = 17,$
$3x - 6y = 4.$

**Solution**

$3x + 7y = 17$ —— (1)
$3x - 6y = 4$ —— (2)

(1) – (2): $(3x + 7y) - (3x - 6y) = 17 - 4$
$3x + 7y - 3x + 6y = 13$
$13y = 13$
$y = 1$

Substitute $y = 1$ into (1): $3x + 7(1) = 17$
$3x = 10$
$x = \frac{10}{3}$

$\therefore$ the solution is $x = \frac{10}{3}$ and $y = 1$.

**Check:** Substitute $x = \frac{10}{3}$ and $y = 1$ into (1) and (2):

In (1), LHS $= 3\left(\frac{10}{3}\right) + 7(1)$
$= 17$
$=$ RHS

In (2), LHS $= 3\left(\frac{10}{3}\right) - 6(1)$
$= 4$
$=$ RHS

**Problem-solving Tip**

To eliminate a variable, the absolute values of the coefficient of the variable in both the equations must be the same.

**Attention**

The coefficient of $x$ in both the equations is 3. Hence, when we subtract equation (2) from equation (1), the terms in $x$ are eliminated.

**Attention**

It is acceptable to leave numerical answers of algebraic terms as improper fractions or mixed numbers.

**Practise Now 4**

Similar and Further Questions
Exercise 2C
Questions 1(a)–(l), 5(a), 6(a), (b)

1. Using the elimination method, solve each of the following pairs of simultaneous equations.
   (a) $x - y = 3$
   $4x + y = 17$
   (b) $7x + 2y = 19$
   $7x + 8y = 13$
   (c) $43x + 9y = 4$
   $17x - 9y = 26$
   (d) $4x - 5y = 17$
   $2x - 5y = 8$

2. Using the elimination method, solve the simultaneous equations
   $3x - y + 14 = 0,$
   $2x + y + 1 = 0.$

It is sometimes necessary to manipulate one or both of the equations before we can eliminate a variable by addition or subtraction.

**Worked Example 5**

### Solving simultaneous linear equations using elimination method

Using the elimination method, solve the following pairs of simultaneous equations.

**(a)** $3x + 2y = 8$
$4x - y = 7$

**(b)** $13x - 6y = 20$
$7x + 4y = 18$

**Solution**

**(a)** $3x + 2y = 8$ —— (1)
$4x - y = 7$ —— (2)
$2 \times (2)$: $8x - 2y = 14$ —— (3)
$(1) + (3)$: $(3x + 2y) + (8x - 2y) = 8 + 14$
$11x = 22$
$x = 2$
Substitute $x = 2$ into (2): $4(2) - y = 7$
$y = 1$
$\therefore$ the solution is $x = 2$ and $y = 1$.

**Attention**
- In this case, it is easier to eliminate $y$.
- We multiply equation (2) by 2 so that the absolute value of the coefficient of $y$ is the same as equation (1).

**Problem-solving Tip**
It is a good practice to check your solution by substituting the values of the unknowns found into the original equations.

**(b)** $13x - 6y = 20$ —— (1)
$7x + 4y = 18$ —— (2)
$2 \times (1)$: $26x - 12y = 40$ —— (3)
$3 \times (2)$: $21x + 12y = 54$ —— (4)
$(3) + (4)$: $(26x - 12y) + (21x + 12y) = 40 + 54$
$47x = 94$
$x = 2$
Substitute $x = 2$ into (1): $13(2) - 6y = 20$
$6y = 6$
$y = 1$
$\therefore$ the solution is $x = 2$ and $y = 1$.

**Attention**
- The LCM of 6 and 4 is 12.
- We multiply equation (1) by 2 and equation (2) by 3 so that the absolute values of the coefficients of $y$ in both the equations are the same.

**Practise Now 5**

Similar and Further Questions
**Exercise 2C**
Questions 2(a)–(f), 3(a)–(f), 5(b)–(d), 6(c)–(f)

1. Using the elimination method, solve each of the following pairs of simultaneous equations.
   **(a)** $2x + 3y = 18$
   $3x - y = 5$
   **(b)** $x + 4y = 11$
   $2x + 3y = 7$
2. Using the elimination method, solve each of the following pairs of simultaneous equations.
   **(a)** $9x + 2y = 5$
   $7x - 3y = 13$
   **(b)** $5x - 4y = 17$
   $2x - 3y = 11$

**Thinking Time**

In Worked Example 5(**b**), is it easier to eliminate $x$? Explain your answer by showing how $x$ can be eliminated.

**Worked Example 6**

### Solving simultaneous fractional equations using elimination method

Using the elimination method, solve the simultaneous equations

$$\frac{2}{3}x - \frac{y}{9} = 6,$$

$$x - \frac{y}{3} = 6.$$

**Solution**

**Method 1:**

$$\frac{2}{3}x - \frac{y}{9} = 6 \quad \text{—— (1)}$$

$$x - \frac{y}{3} = 6 \quad \text{—— (2)}$$

$\frac{3}{2} \times (1)$: $x - \frac{y}{6} = 9$ —— (3)

$(3) - (2)$: $\left(x - \frac{y}{6}\right) - \left(x - \frac{y}{3}\right) = 9 - 6$

$$-\frac{y}{6} + \frac{y}{3} = 3$$

$$\frac{y}{6} = 3$$

$$y = 18$$

Substitute $y = 18$ into (2): $x - \frac{18}{3} = 6$

$$x = 12$$

$\therefore$ the solution is $x = 12$ and $y = 18$.

**Method 2:**

$$\frac{2}{3}x - \frac{y}{9} = 6 \quad \text{—— (1)}$$

$$x - \frac{y}{3} = 6 \quad \text{—— (2)}$$

$9 \times (1)$: $6x - y = 54$ —— (3)

$3 \times (2)$: $3x - y = 18$ —— (4)

$(3) - (4)$: $(6x - y) - (3x - y) = 54 - 18$

$$3x = 36$$

$$x = 12$$

Substitute $x = 12$ into (4): $3(12) - y = 18$

$$y = 18$$

$\therefore$ the solution is $x = 12$ and $y = 18$.

**Practise Now 6**

Similar and Further Questions
**Exercise 2C**
Questions 7(a)–(d)

Using the elimination method, solve the simultaneous equations

$$\frac{x}{2} - \frac{y}{3} = 4,$$

$$\frac{2}{5}x - \frac{y}{6} = 3\frac{1}{2}.$$

## B. Solving simultaneous linear equations using substitution method

We can also solve a pair of simultaneous equations using the **substitution method**.

In this method, we first rearrange one equation to express one variable in terms of the other variable. Next, we substitute this expression into the other equation to obtain an equation in only one variable.

**Worked Example 7**

**Solving simultaneous linear equations using substitution method**

Using the substitution method, solve the following pairs of simultaneous equations.

**(a)** $7x - 2y = 21$
$4x + y = 57$

**(b)** $3x + 2y = 7$
$9x + 8y = 22$

**Solution**

**(a)** $7x - 2y = 21$ —— (1)
$4x + y = 57$ —— (2)

From (2), $y = 57 - 4x$ —— (3)    rearrange (2) to express $y$ in terms of $x$

Substitute (3) into (1): $7x - 2(57 - 4x) = 21$    replace $y$ in (1) with expression in $x$ to obtain an equation in $x$

$7x - 114 + 8x = 21$    solve the equation in $x$

$15x = 135$

$x = 9$

Substitute $x = 9$ into (3): $y = 57 - 4(9)$    substitute the solution of $x$ into the expression for $y$

$= 21$

$\therefore$ the solution is $x = 9$ and $y = 21$.

> **Problem-solving Tip**
> It is easier to obtain the value of $y$ by substituting the value of $x$ into equation (3) instead of equation (1) or (2).

**(b)** $3x + 2y = 7$ —— (1)
$9x + 8y = 22$ —— (2)

From (1), $2y = 7 - 3x$

$y = \dfrac{7-3x}{2}$ —— (3)

Substitute (3) into (2): $9x + 8\left(\dfrac{7-3x}{2}\right) = 22$

$9x + 4(7 - 3x) = 22$

$9x + 28 - 12x = 22$

$-3x = -6$

$3x = 6$

$x = 2$

Substitute $x = 2$ into (3): $y = \dfrac{7-3(2)}{2}$

$= \dfrac{1}{2}$

$\therefore$ the solution is $x = 2$ and $y = \dfrac{1}{2}$.

> **Reflection**
> 1. If we express $x$ in terms of $y$ using equation (1) or (2) in the worked example, will we get the same solution? Which way is easier?
> 2. Can we use the method of elimination to solve Worked Example 7? Which method do you prefer? Why?
> 3. How is the substitution method similar to or different from the elimination method?

**Practise Now 7**

Similar and Further Questions
**Exercise 2C**
Questions 4(a)–(h), 8(a)–(f), 10–12

1. Using the substitution method, solve the simultaneous equations
$3y - x = 7,$
$2x + 3y = 4.$

2. Using the substitution method, solve the simultaneous equations
$3x - 2y = 8,$
$4x + 3y = 5.$

## Thinking Time

Shaha was asked to solve the simultaneous equations

$$2x + y = 6,$$
$$x = 1 - \frac{1}{2}y,$$

using the substitution method.

She did it this way:

$2x + y = 6$ —— (1)

$x = 1 - \frac{1}{2}y$ —— (2)

Substitute (2) into (1): $2\left(1 - \frac{1}{2}y\right) + y = 6$

$$2 - y + y = 6$$
$$2 = 6$$

What happened here? Explain your answer.

**Worked Example 8**

### Solving simultaneous fractional equations using substitution method

Using the substitution method, solve the simultaneous equations

$$\frac{x+1}{y+2} = \frac{1}{2},$$
$$\frac{x-2}{y-1} = \frac{1}{3}.$$

**Solution**

$\frac{x+1}{y+2} = \frac{1}{2}$ —— (1)

$\frac{x-2}{y-1} = \frac{1}{3}$ —— (2)

From (1), $2(x + 1) = y + 2$
$2x + 2 = y + 2$
$y = 2x$ —— (3)

From (2), $3(x - 2) = y - 1$
$3x - 6 = y - 1$
$3x - y = 5$ —— (4)

Substitute (3) into (4): $3x - 2x = 5$
$x = 5$

Substitute $x = 5$ into (3): $y = 2(5)$
$= 10$

$\therefore$ the solution is $x = 5$ and $y = 10$.

**Recall**

Consider $\frac{a}{b} = \frac{c}{d}$, where $b, d \neq 0$.
Multiply by $bd$ on both sides,
$bd \times \frac{a}{b} = bd \times \frac{c}{d}$
$\therefore ad = bc$

**Practise Now 8**

Similar and Further Questions
Exercise 2C
Questions 9(a)–(d), 13(a)–(d)

Using the substitution method, solve each of the following pairs of simultaneous equations.

**(a)** $\dfrac{x-1}{y-3} = \dfrac{2}{3}$

$\dfrac{x-2}{y-1} = \dfrac{1}{2}$

**(b)** $3x + 2y = 3$

$\dfrac{1}{x+y} = \dfrac{3}{x+2y}$

You may have noticed by now that:

Both the elimination method and the substitution method use the same principle: reduce the two given simultaneous equations in two variables to an equation in one variable.

However, depending on the given simultaneous equations, sometimes it is easier to use one method over the other.

## Reflection

1. How can I decide when to use the elimination or the substitution method to solve a pair of simultaneous linear equations?
2. When will I use the graphical method to solve a pair of simultaneous linear equations?
3. Looking back at Worked Example 5(**a**) on page 60, the pair of equations (1) and (2) were manipulated to an equivalent pair of equations (1) and (3) to obtain the solution. Are equations (2) and (3) also an equivalent pair? Would they give the same solution? Do you see similar ideas about equivalent equations when using the substitution method?

Basic | Intermediate | Advanced

## Exercise 2C

**1.** Using the elimination method, solve each of the following pairs of simultaneous equations.

**(a)** $x + y = 16$
$x - y = 0$

**(b)** $x - y = 5$
$x + y = 19$

**(c)** $11x + 4y = 12$
$9x - 4y = 8$

**(d)** $4y + x = 11$
$3y - x = 3$

**(e)** $3x + y = 5$
$x + y = 3$

**(f)** $2x + 3y = 5$
$2x + 7y = 9$

**(g)** $7x - 3y = 15$
$11x - 3y = 21$

**(h)** $3y - 2x = 9$
$2y - 2x = 7$

**(i)** $3a - 2b = 5$
$2b - 5a = 9$

**(j)** $5c - 2d = 9$
$3c + 2d = 7$

**(k)** $3f + 4h = 1$
$5f - 4h = 7$

**(l)** $6j - k = 23$
$3k + 6j = 11$

**2.** Using the elimination method, solve each of the following pairs of simultaneous equations.

**(a)** $7x - 2y = 17$
$3x + 4y = 17$

**(b)** $16x + 5y = 39$
$4x - 3y = 31$

**(c)** $x + 2y = 3$
$3x + 5y = 7$

**(d)** $3x + y = -5$
$7x + 3y = 1$

**(e)** $7x - 3y = 13$
$2x - y = 3$

**(f)** $9x - 5y = 2$
$3x - 4y = 10$

## Exercise 2C

**3.** Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $7x - 3y = 18$
$6x + 7y = 25$

(b) $4x + 3y = -5$
$3x - 2y = 43$

(c) $2x + 3y = 8$
$5x + 2y = 9$

(d) $5x + 4y = 11$
$3x + 5y = 4$

(e) $4x - 3y = -1$
$5x - 2y = 4$

(f) $5x - 4y = 23$
$2x - 7y = 11$

**4.** Using the substitution method, solve each of the following pairs of simultaneous equations.

(a) $x + y = 7$
$x - y = 5$

(b) $3x - y = 0$
$2x + y = 5$

(c) $2x - 7y = 5$
$3x + y = -4$

(d) $5x - y = 5$
$3x + 2y = 29$

(e) $5x + 3y = 11$
$4x - y = 2$

(f) $3x + 5y = 10$
$x - 2y = 7$

(g) $x + y = 9$
$5x - 2y = 4$

(h) $5x + 2y = 3$
$x - 4y = -6$

**5.** Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $x + y = 0.5$
$x - y = 1$

(b) $2x + 0.4y = 8$
$5x - 1.2y = 9$

(c) $10x - 3y = 24.5$
$3x - 5y = 13.5$

(d) $6x + 5y = 10.5$
$5x - 3y = -2$

**6.** Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $4x - y - 7 = 0$
$4x + 3y - 11 = 0$

(b) $7x + 2y - 33 = 0$
$3y - 7x - 17 = 0$

(c) $5x - 3y - 2 = 0$
$x + 5y - 6 = 0$

(d) $5x - 3y - 13 = 0$
$7x - 6y - 20 = 0$

(e) $7x + 3y - 8 = 0$
$3x - 4y - 14 = 0$

(f) $3x + 5y + 8 = 0$
$4x + 13y - 2 = 0$

**7.** Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $\frac{x+1}{y+2} = \frac{3}{4}$
$\frac{x-2}{y-1} = \frac{3}{5}$

(b) $\frac{x}{3} - \frac{y}{2} = \frac{5}{6}$
$3x - \frac{2}{5}y = 3\frac{2}{5}$

(c) $\frac{x}{4} - \frac{3}{8}y = 3$
$\frac{5}{3}x - \frac{y}{2} = 12$

(d) $\frac{x-3}{5} = \frac{y-7}{2}$
$11x = 13y$

**8.** Using the substitution method, solve each of the following pairs of simultaneous equations.

(a) $2x + 5y = 12$
$4x + 3y = -4$

(b) $4x - 3y = 25$
$6x + 5y = 9$

(c) $3x + 7y = 2$
$6x - 5y = 4$

(d) $9x + 2y = 5$
$7x - 3y = 13$

(e) $2y - 5x = 25$
$4x + 3y = 3$

(f) $3x - 5y = 7$
$4x - 3y = 3$

**9.** Using the substitution method, solve each of the following pairs of simultaneous equations.

(a) $\frac{x}{5} + y + 2 = 0$
$\frac{x}{3} - y - 10 = 0$

(b) $\frac{x+y}{3} = 3$
$\frac{3x+y}{5} = 1$

(c) $3x - y = 23$
$\frac{x}{3} + \frac{y}{4} = 4$

(d) $\frac{x}{3} + \frac{y}{2} = 4$
$\frac{2}{3}x - \frac{y}{6} = 1$

**10.** If $x = 3$ and $y = -1$ is the solution of the simultaneous equations
$3px + qy = 11,$
$-qx + 5y = p,$
find the value of $p$ and of $q$.

**11.** If $x = -11$ and $y = 5$ is the solution of the simultaneous equations
$px + 5y = q,$
$qx + 7y = p,$
find the value of $p$ and of $q$.

Basic Intermediate Advanced

## Exercise 2C

12. A computer animation shows a cat moving in a straight line. Its height, $h$ metres, above the ground, is given by $8s - 3h = -9$, where $s$ is the time in seconds after it starts moving. In the same animation, a mouse starts to move at the same time as the cat and its movement is given by $-29s + 10h = 16$. Find the height above the ground and the time when the cat meets the mouse.

13. Using either the elimination or the substitution method, solve each of the following pairs of simultaneous equations.

(a) $\dfrac{2}{x+y} = \dfrac{1}{2x+y}$

$3x + 4y = 9$

(b) $\frac{1}{5}(x-2) = \frac{1}{4}(1-y)$

$\frac{1}{7}\left(x+2\frac{2}{3}\right) = \frac{1}{3}(3-y)$

(c) $\dfrac{5x+y}{9} = 2-\dfrac{x+y}{5}$

$\dfrac{7x-3}{2} = 1+\dfrac{y-x}{3}$

(d) $\dfrac{x+y}{3} = \dfrac{x-y}{5} = 2x - 3y + 5$

# 2.5 Applications of simultaneous equations in real-world contexts

In this section, we will learn how to apply the concept of simultaneous equations to solve mathematical and real-life problems.

Consider the following problem:

> 7 cups of coffee and 4 pieces of toast cost \$10.60. 5 cups of coffee and 4 pieces of toast cost \$8.60. Find the cost of each item.

We can solve the problem by drawing a diagram (see **Method 1**). This is essentially the same as using algebra to solve the problem (see **Method 2**).

**Method 1:**

 +  = \$10.60

Cost of 7 cups of coffee    Cost of 4 pieces of toast

 +  = \$8.60

Cost of 5 cups of coffee    Cost of 4 pieces of toast

Cost of 2 cups of coffee = \$10.60 – \$8.60
= \$2

∴ cost of 1 cup of coffee = \$1

Cost of 4 pieces of toast = \$8.60 – 5 × \$1
= \$3.60

∴ cost of 1 piece of toast = \$0.90

**Method 2:**

Let the cost of 1 cup of coffee be \$$x$ and the cost of 1 piece of toast be \$$y$.

$7x + 4y = 10.6$ —— (1)

$5x + 4y = 8.6$ —— (2)

(1) – (2): $2x = 10.6 - 8.6$
$2x = 2$
$x = 1$

∴ cost of 1 cup of coffee = \$1

Substitute $x = 1$ into (2): $5(1) + 4y = 8.6$
$4y = 8.6 - 5$
$4y = 3.6$
$y = 0.9$

∴ cost of 1 piece of toast = \$0.90

**Worked Example 9**

### Finding two numbers given sum and difference

The sum of two numbers is 67 and their difference is 3. Find the two numbers.

**Solution**

Let the smaller number be $x$ and the greater number be $y$.

$x + y = 67$ —— (1)

$y - x = 3$ —— (2)

(1) + (2): $2y = 70$
$y = 35$

Substitute $y = 35$ into (1): $x + 35 = 67$

$$x = 32$$

$\therefore$ the two numbers are 32 and 35.

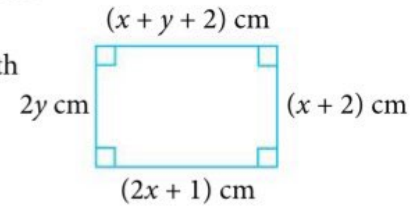
## Thinking Time

Can you solve Worked Example 9 using only one variable $x$?

**Practise Now 9**

Similar and Further Questions
**Exercise 2D**
Questions 1, 2, 3, 6–10, 18

1. The sum of two numbers is 36 and their difference is 9. Find the two numbers.
2. One third of the sum of two angles is 60° and one quarter of their difference is 28°. Find the two angles.
3. The figure shows a rectangle with its length and breadth as indicated. Find the perimeter of the rectangle.




**Worked Example 10**

### Finding ages

The sum of the present ages of Ken and his mother is 60. Two years ago, Ken's mother was three times as old as Ken. Calculate

**(i)** Ken's present age,

**(ii)** the age of Ken's mother when he was born.

### Solution

**(i)** Let the present age of Ken's mother be $x$ years and that of Ken be $y$ years.
Then two years ago, Ken's mother was $(x - 2)$ years old and Ken was $(y - 2)$ years old.

$x + y = 60$ —— (1)

$x - 2 = 3(y - 2)$ —— (2)

From (2), $x - 2 = 3y - 6$

$x = 3y - 4$ —— (3)

Substitute (3) into (1): $(3y - 4) + y = 60$

$$4y = 64$$

$$y = 16$$

$\therefore$ Ken's present age = 16 years

**(ii)** Substitute $y = 16$ into (3): $x = 3(16) - 4$

$$= 44$$

$\therefore$ age of Ken's mother when he was born $= 44 - 16$

$$= 28 \text{ years}$$

**Practise Now 10**

Similar and Further Questions
**Exercise 2D**
Questions 4, 5, 11–16, 19, 20

1. In five years' time, Li Ting's father will be three times as old as Li Ting. Four years ago, her father was six times as old as her. Find their present ages.
2. To visit the two conservatories at Gardens by the Bay in Singapore, 11 adults and 5 children pay \$280 whereas 14 adults and 9 children pay \$388. Find the total amount a family of 2 adults and 3 children pays to visit the two conservatories.

**Worked Example 11**

## Finding a fraction

If 1 is added to the numerator and 2 to the denominator of a fraction, the value obtained is $\frac{2}{3}$.

If 2 is subtracted from its numerator and 1 from its denominator, the resulting value is $\frac{1}{3}$.

Find the fraction.

***Solution**

Let the numerator of the fraction be $x$ and its denominator be $y$, i.e. the fraction is $\frac{x}{y}$.

$\frac{x+1}{y+2} = \frac{2}{3}$ —— (1)

$\frac{x-2}{y-1} = \frac{1}{3}$ —— (2)

From (1), $3(x + 1) = 2(y + 2)$

$3x + 3 = 2y + 4$

$3x - 2y = 1$ —— (3)

From (2), $3(x - 2) = y - 1$

$3x - 6 = y - 1$

$3x - y = 5$ —— (4)

(4) − (3): $y = 4$

Substitute $y = 4$ into (4): $3x - 4 = 5$

$3x = 9$

$x = 3$

$\therefore$ the fraction is $\frac{3}{4}$.

**Practise Now 11**

Similar and Further Questions
**Exercise 2D**
Question 17

If 1 is added to the numerator and to the denominator of a fraction, the value obtained is $\frac{4}{5}$.

If 5 is subtracted from its numerator and from its denominator, the resulting value is $\frac{1}{2}$.

Find the fraction.

Worked Example 12

### Finding a two-digit number

The sum of the digits of a two-digit number is 8. When the digits of the number are reversed and the number is subtracted from the original number, the result obtained is 18. Find the original number.

**Solution**

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

**Stage 1: Understand the problem**

*What does it mean to reverse the digits of a two-digit number?*
*Which number is subtracted from which number?*

**Stage 2: Think of a plan**

*Can we solve this problem using algebra?*
*If we let the tens digit of the original number be x and its ones digit be y, how do we represent the number using x and y?*
*Is it xy or 10x + y? What is the difference?*

**Stage 3: Carry out the plan**

Let the tens digit of the original number be $x$ and its ones digit be $y$.
Then the original number is $10x + y$, the number obtained when the digits of the original number are reversed is $10y + x$.

$$x + y = 8 \quad \text{—— (1)}$$
$$10x + y - (10y + x) = 18 \quad \text{—— (2)}$$

From (2), $10x + y - 10y - x = 18$
$$9x - 9y = 18$$
$$x - y = 2 \quad \text{—— (3)} \qquad \text{divide by 9 throughout}$$

(1) + (3): $2x = 10$
$$x = 5$$

Substitute $x = 5$ into (1): $5 + y = 8$
$$y = 3$$

$\therefore$ the original number is 53.

**Stage 4: Look back**

*How can we check that the answer is correct?*
By performing the operation as stated in the question: $53 - 35 = 18$

**Practise Now 12**

Similar and Further Questions
Exercise 2D
Question 21

A two-digit number is such that the sum of its digits is 11. When the digits of the number are reversed and the number is subtracted from the original number, the result obtained is 9. Find the original number.

## Introductory Problem Revisited

**(a)** Can you now write two linear equations involving two variables $c$, the money Cheryl has, and $d$, the money Albert has, to represent the information given in the Introductory Problem? If so, can you draw the graphs of these two equations? What do you notice?

**(b)** Now that you have learnt how to solve simultaneous equations using algebraic methods, solve the Introductory Problem using the techniques you have learnt. Which method is easier to use?

## Reflection

1. What do I need to consider when assigning variables to the unknown to solve a problem?
2. What are the basic steps that I should take to solve word problems involving simultaneous equations?
3. How can I tell that my answers are correct without referring to the given answers?

Advanced | Intermediate | Basic

## Exercise 2D

1. The sum of two numbers is 138 and their difference is 88. Find the two numbers.

2. The difference between two numbers is 10 and their sum is four times the smaller number. Find the two numbers.

3. The sum of two numbers is 48. If the smaller number is one fifth of the larger number, find the two numbers.

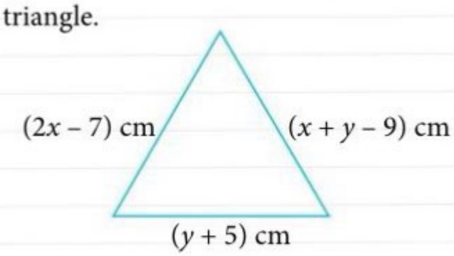
4. A belt and a wallet cost \$42. Seven belts and four wallets cost \$213. Find the cost of each item.

5. 8 kg of potatoes and 5 kg of carrots cost \$28 whereas 2 kg of potatoes and 3 kg of carrots cost \$11.20. Find the cost of 1 kg of each item.

6. Two numbers are such that if 7 is added to the first number, a number twice the second number is obtained. If 20 is added to the second number, the number obtained is four times the first number. Find the two numbers.

7. One fifth of the sum of two angles is 24° and half their difference is 14°. Find the two angles.

8. The figure shows an equilateral triangle with its sides as indicated. Find the length of each side of the triangle.

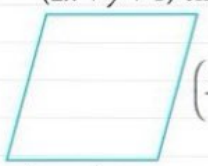


Basic Intermediate Advanced

## Exercise 2D

**9.** The figure shows a rectangle with its length and breadth as indicated. Given that the perimeter of the rectangle is 120 cm, find the area of the rectangle.


**10.** The figure shows a rhombus with its sides as indicated. Find the perimeter of the figure.




**11.** A polar bear and a giant panda reside in a zoo. The sum of their ages in 2013 was 11 years. In 2022, polar bear was three times as old as the giant panda was in 2013. Find their ages in 2014.

**12.** 6 adults and 4 senior citizens pay \$228 while 13 adults and 7 senior citizens pay \$459 to visit an exhibition at the museum. Find the total amount 2 adults and a senior citizen pay to visit the exhibition.

**13.** Nadia intends to buy either Gift A, which costs \$10, or Gift B, which costs \$8, as Christmas gifts for each of her parents, 2 siblings, 13 relatives and 10 friends. Given that she intends to spend \$230, find the number of each gift she should buy.

**14.** There are some chickens and goats on a farm. Given that the animals have a total of 50 heads and 140 legs, how many more chickens than goats are there?

**15.** \$80 is divided between Bernard and Cheryl such that one quarter of Bernard's share is equal to one sixth of Cheryl's share. How much does each of them receive?

**16.** Waseem deposited a total of \$25 000 in Bank A and Bank B at the beginning of 2020. Bank A and Bank B paid simple interest at rates of 0.6% and 0.65% per annum respectively. He withdrew all his money from the two banks at the end of 2020. If he earned the same amount of interest from each bank, find the amount of money he deposited in each bank.

**17.** If 1 is subtracted from the numerator and from the denominator of a fraction, the value obtained is $\frac{1}{2}$. If 1 is added to its numerator and to its denominator, the resulting value is $\frac{2}{3}$. Find the fraction.

**18.** Two numbers are such that when the larger number is divided by the smaller number, both the quotient and the remainder are equal to 2. If five times the smaller number is divided by the larger number, both the quotient and the remainder are also equal to 2. Find the two numbers.

**19.** Raju has \$10. If he buys 8 pears and 5 mangoes, he will be short of \$1.10. If he buys 5 pears and 4 mangoes, he will receive \$1.75 in change. Find the price of each fruit.

## Exercise 2D

20. Joyce's mother buys some shares of Company A on Day 0. On Day 7, the share price of Company A is \$4.60. If she sells all her shares of Company A and buys 2000 shares of Company B on Day 7, she would receive \$7400. On Day 12, the share price of Company A is \$4.80 and the share price of Company B is \$0.50 less than that on Day 7. If she sells all her shares of Company A and buys 5000 shares of Company B on Day 12, she would have to pay \$5800. Find
    **(i)** the number of shares of Company A Joyce's mother has,
    **(ii)** the share price of Company B on Day 12.

21. A two-digit number is such that the sum of its digit is $\frac{1}{8}$ of the number. When the digits of the number are reversed and the number is subtracted from the original number, the result obtained is 45. Find the original number.

## Looking Back

In this chapter, we have learnt how to **model** real-world context problems using pairs of linear functions, and to interpret the intersection point, if any, of the graphs of two linear functions as a solution to the problem. Solving a pair of simultaneous equations in two variables involves finding the values of two variables that will satisfy both equations at the same time. There are two important ideas behind solving simultaneous linear equations. Firstly, the use of the Cartesian coordinates provides a way to visualise the meaning behind solving simultaneous equations, with the use of graphs—a form of mathematical **diagrams**.

Secondly, the ability to write equations in another **equivalent** form helps to simplify the solving process. We say that two sets of equations are equivalent if they have the same solution set. In particular, we are able to solve simultaneous linear equations by means of elimination and substitution, which gives us the same solution set even though we have rewritten the equations in a different form. Although there are many other real-world situations that may be modelled by more complicated simultaneous equations, the ideas that we have learnt here will form the basis for us to solve such problems in the future.

## Summary

1. **(a)** The equation of a horizontal line is $y = c$. Its gradient is 0.
   **(b)** The equation of a vertical line is $x = a$. Its gradient is undefined.
   - Give an example of the equation of a horizontal line and an example of the equation of a vertical line.

2. Equations in the form $ax + by = k$, where $a$, $b$ and $k$ are constants, are linear equations because they can be rewritten in the form $y = mx + c$.
   - Give an example of a linear equation in the form $ax + by = k$ and then rewrite it in the form $y = mx + c$.

3. When sketching graphs of equations in the form $ax + by = k$, where $a$, $b$ and $k$ are constants, identify and label the $x$- and $y$-intercepts.
   - Give an example of the equation of a linear function and sketch it.

4. To **solve** a pair of **simultaneous linear equations** in two variables $x$ and $y$ means to find a value of $x$ and a value of $y$ that *satisfy* both equations at the same time.
   Graphically, it means finding the $x$-coordinate and the $y$-coordinate of the *point of intersection* of the graphs of the two linear equations.

5. A pair of simultaneous linear equations in two variables can be solved by
   - the **graphical method**,
   - the **elimination method**,
   - the **substitution method**.

   Both the elimination and substitution methods work by reducing the two given equations in two variables to an equation in one variable.
   - Which method do you prefer? Does it depend on the given pair of equations?

6. **(a)** A pair of simultaneous linear equations in $x$ and $y$ has an *infinite number of solutions* if the graphs of the two equations are *identical*.
   - Give an example of such a pair of simultaneous linear equations.

   **(b)** A pair of simultaneous linear equations in $x$ and $y$ has *no solution* if the graphs of the two equations are *parallel*.
   - Give an example of such a pair of simultaneous linear equations.

7. For mathematical and real-life problems that involve simultaneous equations, we formulate a pair of linear equations in two variables before solving for the unknowns in the problems.

# CHAPTER 3

## Linear Inequalities

Postage rates to send postcards, letters and parcels vary depending on the destination and the mass of the item. Often, an exact cost is not assigned to a specific mass. Instead, the price is determined by the range that the mass of the parcel falls into. In situations that describe a range of values rather than exact values, inequalities serve as more appropriate **models** than equations.

In this chapter, we will learn more about inequalities and how we use them to solve problems in real-world contexts.

### Learning Outcomes

What will we learn in this chapter?

- What linear inequalities are
- How to solve linear inequalities and simultaneous linear inequalities in one variable, and represent the solution on a number line
- How to represent linear inequalities in two variables graphically
- How to apply linear inequalities and simultaneous linear inequalities to solve real-world problems

## Introductory Problem

The overall mark for a design project is calculated based on the weightage shown in Table 3.1:

| Component | Proposal | Reflections | Written Report | Presentation |
|---|---|---|---|---|
| Weightage | 15% | 10% | 35% | 40% |

Table 3.1

Yasir obtained 65 out of 100 marks for his proposal, 8 out of 10 marks for the reflections and 75 out of 100 marks for the written report.

The maximum mark for the presentation is 100, and an overall mark of 70 will qualify a student for an award.

**(a)** Find the range of marks Yasir must score for his presentation to qualify for an award.

**(b)** A student will receive funding for the project if the overall mark obtained is at least 85.
Explain if Yasir can receive funding for this project.

In this chapter, we will learn how to solve such problems by forming linear inequalities.

# 3.1 Simple inequalities

In Book 1, we have learnt how to represent numbers on a number line. A number on the number line is *more than* any number on its *left* and *less than* any number on its *right* (see Fig. 3.1).

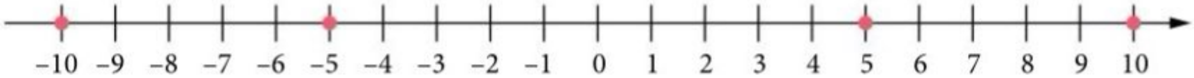



Fig. 3.1

Since the number 10 is to the right of the number 5, 10 is more than 5 (we write $10 > 5$). Similarly, –5 is to the right of –10, so –5 is more than –10 (we write $-5 > -10$). Alternatively, we can say that –10 is less than –5 (we write $-10 < -5$). $10 > 5$ and $-10 < -5$ are known as **inequalities**.

The different notations and their meanings are summarised in Table 3.2.

| Notation | Meaning |
|---|---|
| $a > b$ | $a$ is more than $b$. |
| $a < b$ | $a$ is less than $b$. |
| $a \geqslant b$ | $a$ is more than or equal to $b$. |
| $a \leqslant b$ | $a$ is less than or equal to $b$. |

Table 3.2

**Information**

- $a > b$ and $a < b$ are called strict inequalities as $a \neq b$.
- The **Law of Trichotomy** states that for any two numbers $x$ and $y$, exactly one of the relations $x > y$, $x = y$ or $x < y$ holds.

Inequalities are used in our everyday life to relate two quantities. Learning to interpret the language of inequalities is important for us to solve associated problems in our daily contexts.

## Class Discussion — Real-life examples of inequalities


(a) (b)

Fig. 3.2

1. Fig. 3.2(**a**) shows a common road sign seen along roads in New South Wales, Australia. It is located along school zones, in which motorists are required to reduce their speeds to 40 km/h or less during the stated hours.
   A motorist is travelling in a school zone at a speed of $x$ km/h during the stated hours.
   Write down an inequality which $x$ must satisfy.
2. There are six ratings in film classification in a particular country and Fig. 3.2(**b**) shows one of the advisory logos, NC16 (No Children under 16).
   Imran, who is $y$ years old, is watching a movie that is classified as NC16.
   Write down an inequality which $y$ must satisfy.
   Shaha, who is $z$ years old, is not allowed to watch this movie.
   Write down an inequality which $z$ must satisfy.
3. Can you find other examples in our daily life in which two quantities are related by an inequality?

## Investigation — Properties of inequalities

In this Investigation, we shall explore some properties of inequalities.
Fill in each blank with '>' or '<'.

**Part 1:** Adding/subtracting a real number on/from both sides of an inequality

1. Consider the inequality $6 < 12$.
   (a) (i) $6 + 2 \;\square\; 12 + 2$ (ii) $6 - 4 \;\square\; 12 - 4$
   (b) If $6 < 12$ and $a$ is a real number, then $6 + a \;\square\; 12 + a$ and $6 - a \;\square\; 12 - a$.
   (c) If $12 > 6$ and $a$ is a real number, then $12 + a \;\square\; 6 + a$ and $12 - a \;\square\; 6 - a$.
2. Consider the inequality $-6 < 12$.
   (a) (i) $-6 + 2 \;\square\; 12 + 2$ (ii) $-6 - 4 \;\square\; 12 - 4$
   (b) If $-6 < 12$ and $a$ is a real number, then $-6 + a \;\square\; 12 + a$ and $-6 - a \;\square\; 12 - a$.
   (c) If $12 > -6$ and $a$ is a real number, then $12 + a \;\square\; -6 + a$ and $12 - a \;\square\; -6 - a$.
3. Consider the inequality $6 > -12$.
   (a) (i) $6 + 2 \;\square\; -12 + 2$ (ii) $6 - 4 \;\square\; -12 - 4$
   (b) What conclusions can you draw from your answers in part (**a**)?

4. From Questions 1 to 3, can you generalise the conclusions for $x < y$ and $x > y$? What about $x \leqslant y$ and $x \geqslant y$?

**Part 2:** Multiplying/dividing by a real number on both sides of an inequality

5. Consider the inequality $6 < 12$.
   - **(a)** **(i)** $6 \times 2$ ☐ $12 \times 2$ **(ii)** $6 \times (-4)$ ☐ $12 \times (-4)$
   - **(b)** If $6 < 12$ and $a$ is a *positive* real number, then $6 \times a$ ☐ $12 \times a$.
     If $6 < 12$ and $a$ is a *negative* real number, then $6 \times a$ ☐ $12 \times a$.
   - **(c)** If $12 > 6$ and $a$ is a *positive* real number, then $12 \times a$ ☐ $6 \times a$.
     If $12 > 6$ and $a$ is a *negative* real number, then $12 \times a$ ☐ $6 \times a$.
6. Consider the inequality $-6 < 12$.
   - **(a)** **(i)** $-6 \times 2$ ☐ $12 \times 2$ **(ii)** $-6 \times (-4)$ ☐ $12 \times (-4)$
   - **(b)** If $-6 < 12$ and $a$ is a *positive* real number, then $-6 \times a$ ☐ $12 \times a$.
     If $-6 < 12$ and $a$ is a *negative* real number, then $-6 \times a$ ☐ $12 \times a$.
   - **(c)** If $12 > -6$ and $a$ is a *positive* real number, then $12 \times a$ ☐ $-6 \times a$.
     If $12 > -6$ and $a$ is a *negative* real number, then $12 \times a$ ☐ $-6 \times a$.
7. Consider the inequality $6 > -12$.
   - **(a)** **(i)** $6 \times 2$ ☐ $-12 \times 2$ **(ii)** $6 \times (-4)$ ☐ $-12 \times (-4)$
   - **(b)** What conclusions can you draw from your answers in part **(a)**?
8. Consider each of the inequalities in Questions 5 to 7.
   - **(a)** Divide by a *positive* real number on both sides of the inequality. Is the inequality sign reversed in each case?
   - **(b)** Divide by a *negative* real number on both sides of the inequality. Is the inequality sign reversed in each case?
9. From Questions 5 to 8, can you generalise the conclusions for $x < y$ and $x > y$? What about $x \leqslant y$ and $x \geqslant y$?

From the above Investigation, we observe the following:

- When we **add** or **subtract** a positive or a negative number on both sides of an inequality, the inequality sign does *not* change.
- When we multiply or divide by a **positive** number on both sides of an inequality, the inequality sign does *not* change.
- When we multiply or divide by a **negative** number on both sides of an inequality, the inequality sign is *reversed*.

Similar and Further Questions
**Exercise 3A**
Questions 1(a)–(f), 9(a)–(f)

For example, if $x < 2$ and $a$ is a real number, then $x + a < 2 + a$ and $x - a < 2 - a$;

if $x < 2$ and $a > 0$, then $ax < 2a$ and $\frac{x}{a} < \frac{2}{a}$;

if $x < 2$ and $a < 0$, then $ax > 2a$ and $\frac{x}{a} > \frac{2}{a}$.

## Reflection

1. Recall the properties of equations which were covered in Book 1. How are they similar to or different from the properties of inequalities?
2. Why does the inequality symbol reverse when both sides of the inequality are multiplied or divided by a negative number?

# 3.2 Solving simple linear inequalities

For an inequality with an unknown $x$, all values of $x$ that satisfy the inequality are called the **solutions** of the inequality.

Consider the inequality $x > -2$.

If $x = -1$, the inequality becomes $-1 > -2$, which is true.

Therefore, $x = -1$ is a *solution* of this inequality.

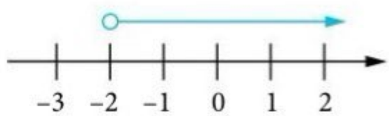
Similarly, $x = 0, 2, 3\frac{1}{2}$ and 5.6 are also solutions of this inequality. Can you think of another solution of this inequality? How many solutions does this inequality have?

If $x = -3$, the inequality becomes $-3 > -2$, which is not true.

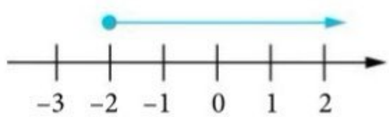
Therefore, $x = -3$ is not a solution of this inequality. Can you think of another value of $x$ which is not a solution of this inequality?

In fact, the solutions of the inequality $x > -2$ are all the real numbers on the right of $-2$ on the number line. We can represent the solutions using the number line. For example,

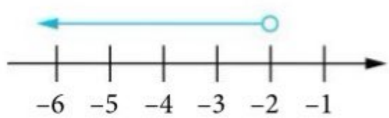
- the solutions of the inequality $x > -2$ are represented as




- the solutions of the inequality $x \geqslant -2$ are represented as




- the solutions of the inequality $x < -2$ are represented as



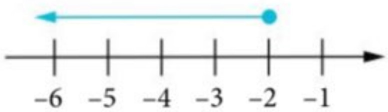

**Attention**

On a number line, a circle ○ is used to indicate that $x$ cannot take on a particular value whereas a dot ● is used to indicate that $x$ can take on the particular value.

**Big Idea**

**Diagrams**
Using a number line helps us to visualise the solution, or the range of possible values, to a given inequality. We follow conventions to construct these diagrams so that the information can be correctly represented and interpreted.

• the solutions of the inequality $x \leqslant -2$ are represented as

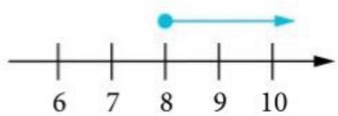



'To **solve** an inequality' means to find all the solutions that **satisfy** the inequality.
In this section, we will learn how to solve linear inequalities in one variable and represent the solution on a number line.

For example, if $4x \geqslant 32$,

then $\frac{4x}{4} \geqslant \frac{32}{4}$, divide both sides by 4; no change in the inequality sign since $4 > 0$

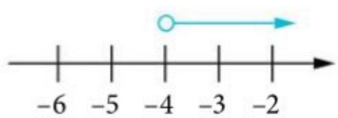
$x \geqslant 8$.




If $-5y < 20$,

then $\frac{-5y}{-5} > \frac{20}{-5}$, divide both sides by −5; reverse the inequality sign since $-5 < 0$

$y > -4$.




**Attention**

If we solve the equation $4x = 32$, then the solution is just $x = 8$, i.e. the equation has only one solution.
But the inequality $4x \geqslant 32$ has infinitely many solutions.

**Worked Example 1**

## Solving linear inequalities

Solve each of the following inequalities and represent the solutions on a number line.

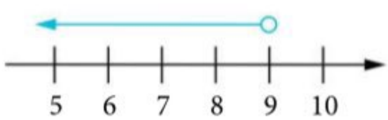
**(a)** $3x < 27$ **(b)** $-2x \geqslant 4$

### Solution

**(a)** $3x < 27$

$\frac{3x}{3} < \frac{27}{3}$ divide both sides by 3; no change in the inequality sign since $3 > 0$

$\therefore x < 9$



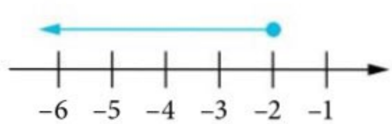

**(b)** $-2x \geqslant 4$

$-1 \times (-2x) \leqslant -1 \times 4$ multiply both sides by −1; reverse the inequality sign since $-1 < 0$

$2x \leqslant -4$

$\frac{2x}{2} \leqslant \frac{-4}{2}$ divide both sides by 2; no change in the inequality sign since $2 > 0$

$\therefore x \leqslant -2$




**Big Idea**

**Equivalence**

Similar to solving equations, we convert inequalities into equivalent forms to solve them. The inequality $3x < 27$ is equivalent to the inequality $x < \frac{27}{3}$. Likewise for **(b)**, all subsequent inequalities are equivalent to the original inequality $-2x \geqslant 4$.

**Big Idea**

**Notations**

Some problems have multiple solutions within a range of values, or even infinitely many solutions. Using an inequality symbol, we can write an inequality such as $x \leqslant -2$ to express these ideas in a clear and concise manner.

**Practise Now 1**

Similar and Further Questions
**Exercise 3A**
Questions 2(a)–(d)

Solve each of the following inequalities and represent the solutions on a number line.

(a) $5x > 30$ (b) $-4x \geqslant 20$
(c) $15x \leqslant 45$ (d) $-6x < 15$

**Worked Example 2**

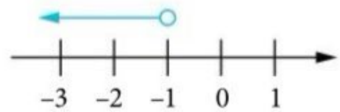
### Solving linear inequalities

Solve each of the following inequalities, representing each solution on a number line.

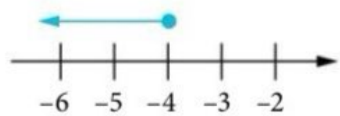
(a) $x + 4 < 3$ (b) $-4y - 5 \geqslant 11$

**Solution**

(a)
$$x + 4 < 3$$
$$x + 4 - 4 < 3 - 4 \quad \text{subtract 4 from both sides}$$
$$x < -1$$




(b)
$$-4y - 5 \geqslant 11$$
$$-4y - 5 + 5 \geqslant 11 + 5 \quad \text{add 5 to both sides}$$
$$-4y \geqslant 16$$
$$-1 \times (-4y) \leqslant -1 \times 16 \quad \text{multiply both sides by } -1\text{; reverse the inequality sign since } -1 < 0$$
$$4y \leqslant -16$$
$$\frac{4y}{4} \leqslant \frac{-16}{4} \quad \text{divide both sides by 4; no change in the inequality sign since } 4 > 0$$
$$y \leqslant -4$$




**Practise Now 2**

Similar and Further Questions
**Exercise 3A**
Questions 2(e)–(h), 4(a)–(d)

Solve each of the following inequalities, representing each solution on a number line.

(a) $x - 3 \geqslant 7$ (b) $-2y + 4 > 3$

## Thinking Time

1. Given an equation in the form $ax + b = c$, where $a$, $b$ and $c$ are constants and $a > 0$, list the steps you would take to find the value of $x$. Would the steps change if $a < 0$?
2. Given an inequality in the form $ax + b > c$, where $a$, $b$ and $c$ are constants and $a > 0$, list the steps you would take to find the range of values of $x$. How would the steps change if $a < 0$?
3. The inequalities $ax + b < c$, $ax + b \leqslant c$, $ax + b > c$ and $ax + b \geqslant c$ correspond to the linear equation $ax + b = c$. How is the solution of each inequality related to that of the corresponding linear equation?

**Worked Example 3**

### Solving linear inequalities

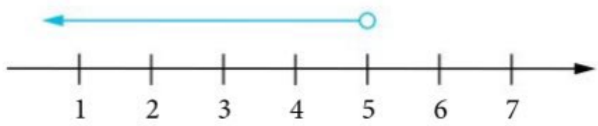
Solve the inequality $8 - x > 3$ and represent the solution on a number line.

**(i)** If $x$ is a prime number, write down the largest possible value of $x$ that satisfies the inequality.

**(ii)** Write down the positive integer values of $x$ that satisfy the inequality.

**Solution**

$$\begin{aligned} 8 - x &> 3 \\ 8 - 8 - x &> 3 - 8 && \text{subtract 8 from both sides} \\ -x &> -5 \\ -1 \times (-x) &< -1 \times (-5) && \text{multiply both sides by } -1\text{; reverse the inequality sign since } -1 < 0 \\ x &< 5 \end{aligned}$$



**(i)** The largest prime value of $x$ is 3.

**(ii)** The positive integer values of $x$ are 1, 2, 3 and 4.

> **Recall**
> A prime number is a positive integer that has exactly 2 different factors, 1 and itself.

**Practise Now 3**

Similar and Further Questions
**Exercise 3A**
Questions 3(a)–(c), 5

Solve the inequality $5 - x < -9$ and represent the solution on a number line.

**(i)** If $x$ is a prime number, write down the smallest possible value of $x$ that satisfies the inequality.

**(ii)** Given that $x$ is a perfect cube, find the smallest possible value of $x$.

**Worked Example 4**

### Solving more complicated linear inequalities

Solve each of the following inequalities.

**(a)** $-3 > 2(1 - x)$ **(b)** $\frac{1}{4} \leqslant \frac{y+1}{7}$

**Solution**

**(a)**
$$\begin{aligned} -3 &> 2(1 - x) \\ -3 &> 2 - 2x && \text{expand the RHS} \\ -3 - 2 &> 2 - 2x - 2 && \text{subtract 2 from both sides} \\ -5 &> -2x \\ -2x &< -5 && \text{change sides} \\ \frac{-2x}{-2} &> \frac{-5}{-2} && \text{divide both sides by } -2\text{; reverse the inequality sign since } -2 < 0 \\ x &> \frac{5}{2} \end{aligned}$$

> **Attention**
> You can leave your answer as $x > \frac{5}{2}$ or $x > 2\frac{1}{2}$.

**(b)**
$$\begin{aligned} \frac{1}{4} &\leqslant \frac{y+1}{7} \\ 4 \times 7 \times \frac{1}{4} &\leqslant 4 \times 7 \times \frac{y+1}{7} && \text{multiply both sides by } 4 \times 7\text{; no change in the inequality sign since } 4 \times 7 > 0 \\ 7 &\leqslant 4(y + 1) \\ 7 &\leqslant 4y + 4 && \text{expand the RHS} \\ 7 - 4 &\leqslant 4y + 4 - 4 && \text{subtract 4 from both sides} \\ 3 &\leqslant 4y \\ 4y &\geqslant 3 && \text{change sides} \end{aligned}$$

> **Problem-solving Tip**
> The LCM of 4 and 7 is $4 \times 7$.

$$\frac{4y}{4} \geqslant \frac{3}{4}$$ divide both sides by 4; no change in the inequality sign since $4 > 0$

$$y \geqslant \frac{3}{4}$$

**Practise Now 4**

Similar and Further Questions
**Exercise 3A**
Questions 6(a)–(f), 7(a)–(d)

1. Solve each of the following inequalities.
   (a) $5(3 + x) \geqslant 9$ (b) $\frac{1}{3} > \frac{y+1}{2}$ (c) $\frac{1}{3}(z+1) + 2 \leqslant \frac{1}{2}$
2. Given that $p$ satisfies the inequality $\frac{1}{2} - \frac{3}{4}(p-3) > -1$, find the largest possible value of $p$ if $p$ is a perfect square.

**Worked Example 5**

## Solving more complicated linear inequalities

Solve each of the following inequalities.

**(a)** $4x < 3x + 6$ **(b)** $3x \geqslant 8x - 5$

### Solution

**(a)**
$$4x < 3x + 6$$
$$4x - 3x < 3x + 6 - 3x$$ subtract $3x$ from both sides
$$x < 6$$

**(b)**
$$3x \geqslant 8x - 5$$
$$3x - 8x \geqslant 8x - 5 - 8x$$ subtract $8x$ from both sides
$$-5x \geqslant -5$$
$$\frac{-5x}{-5} \leqslant \frac{-5}{-5}$$ divide both sides by $-5$; reverse the inequality sign since $-5 < 0$
$$x \leqslant 1$$

**Recall**

To solve linear inequalities, we convert them into equivalent forms and isolate the variable e.g. $x$ on the LHS.

**Practise Now 5**

Similar and Further Questions
**Exercise 3A**
Question 8(a)–(d)

Solve each of the following inequalities.

(a) $4x < 2x + 3$ (b) $3x > 7x + 10$
(c) $5x \geqslant 2(2x + 3)$ (d) $2x + 5 \leqslant 6x - 13$

## Reflection

1. What have I learnt about the properties of inequalities in Section 3.1 that could help me to solve the problems in this section?
2. How is solving a linear inequality in one variable similar to or different from solving a linear equation in one variable?

## Exercise 3A

**1.** Fill in each box with '$<$', '$>$', '$\leqslant$' or '$\geqslant$'.

(a) $5 + h \;\square\; 7 + h$, where $h$ is a real number

(b) $5 - k \;\square\; 7 - k$, where $k$ is a real number

(c) If $x > y$, then $5x \;\square\; 5y$.

(d) If $x < y$, then $\frac{x}{-20} \;\square\; \frac{y}{-20}$.

(e) If $x \geqslant y$, then $-3x \;\square\; -3y$.

(f) If $x \leqslant y$, then $\frac{x}{10} \;\square\; \frac{y}{10}$.

**2.** Solve each of the following inequalities, representing each solution on a number line.

(a) $4x < 28$ (b) $-12x \geqslant 126$

(c) $-y \leqslant -5$ (d) $\frac{1}{2}y > -2$

(e) $a + 2 < 3$ (f) $b - 3 \geqslant 4$

(g) $-c + 3 > 5$ (h) $4 - d \leqslant 4$

**3.** (a) Find the smallest rational value of $y$ that satisfies the inequality $8 \leqslant 7y$.

(b) Given that $x$ satisfies the inequality $20x > 33$, find the smallest value of $x$ if $x$ is a prime number.

(c) Find the greatest odd integer value of $x$ that satisfies the inequality $-3x > -105$.

**4.** Solve each of the following inequalities, representing each solution on a number line.

(a) $2a - 5 > 2$ (b) $2 + 5b < 0$

(c) $-2c - 1 \leqslant 2$ (d) $1 - 4d \geqslant 1$

**5.** Solve the inequality $7 + 2x \leqslant 16$ and represent the solution on a number line.

(i) If $x$ is an integer, write down the largest possible value of $x$ that satisfies the inequality.

(ii) Given that $x$ is a perfect square, find the largest possible value of $x$.

**6.** Solve each of the following inequalities, representing each solution on a number line.

(a) $2(e - 5) \leqslant 2$ (b) $4(f + 1) > 5$

(c) $8 < 2(7 - g)$ (d) $-5 \geqslant 2(-2h)$

(e) $\frac{4a}{3} \geqslant 2$ (f) $\frac{5b}{7} < \frac{1}{4}$

**7.** Solve each of the following inequalities, representing each solution on a number line.

(a) $\frac{2p-1}{3} > \frac{7}{11}$ (b) $\frac{10q+9}{2} \leqslant \frac{3}{8}$

(c) $\frac{3(9-2r)}{4} \geqslant \frac{2}{5}$ (d) $\frac{5(-5s-1)}{6} > -\frac{9}{5}$

**8.** Solve each of the following inequalities.

(a) $3x < x - 7$ (b) $2x + 8 > 9x$

(c) $5x \geqslant 3(x - 4)$ (d) $4x - 2 \leqslant 7x - 5$

**9.** State whether each of the following statements is always, never or sometimes true. If a statement is never true, then give a counterexample. If a statement is sometimes true, then state when it is true. If $a < b$ and $c < d$, then

(a) $a + c < b + c$. (b) $b + c < b + d$.

(c) $a + c < b + d$. (d) $a + d < b + c$.

(e) $a - c < b - d$. (f) $c - a < c - b$.

# 3.3 Solving problems involving linear inequalities

In this section, we shall use inequalities to solve problems.

**Worked Example 6**

**Solving problem involving inequalities**

A croissant costs \$1.60. By forming an inequality, find the maximum number of croissants that can be bought with \$20.

**Solution**

Let the number of croissants that can be bought with \$20 be $x$.

Then $160x \leqslant 2000$ — \$1.60 = 160 cents, \$20 = 2000 cents

$\frac{160x}{160} \leqslant \frac{2000}{160}$ — divide both sides by 160; no change in the inequality sign since $160 > 0$

$x \leqslant 12\frac{1}{2}$

$\therefore$ the maximum number of croissants that can be bought with \$20 is 12.

**Attention**

In such cases, it is better to leave the answer in mixed numbers so that the largest integer value can be easily determined.

**Practise Now 6**

Similar and Further Questions
**Exercise 3B**
Questions 1, 2

A bus can ferry a maximum of 45 students. By forming an inequality, find the minimum number of buses that are needed to ferry 520 students.

**Worked Example 7**

**Solving problem involving inequalities**

Ali's car consumes petrol at an average rate of 8 litres daily. Before Ali begins his journey, he tops up the petrol in his car to 100 litres. Given that there must be at least 15 litres of petrol in the tank at all times, form an inequality and solve it to find the maximum number of days he can travel before he has to top up the petrol in his car.

**Solution**

Let $x$ be the number of days he can travel before he tops up the petrol in his car.

$100 - 8x \geqslant 15$

$100 - 8x - 100 \geqslant 15 - 100$ — subtract 100 from both sides

$-8x \geqslant -85$

$\frac{-8x}{-8} \leqslant \frac{-85}{-8}$ — divide both sides by $-8$; reverse the inequality sign since $-8 < 0$

$x \leqslant 10\frac{5}{8}$

$\therefore$ Ali can travel a maximum of 10 days before he has to top up the petrol in his car.

**Practise Now 7**

Similar and Further Questions
**Exercise 3B**
Questions 3, 4

The minimum mark to obtain a Grade A is 75. Joyce managed to achieve an average of Grade A for three of her Science quizzes. What is the minimum mark she scored in her first quiz if she scored 76 and 89 marks in her second and third quizzes respectively?

## Introductory Problem Revisited

In the Introductory Problem, you had to find the range of marks Yasir should score for the presentation in order to qualify for an award and explain if he could receive funding for his design project. After learning how to solve linear inequalities in one variable, by letting the mark Yasir scores for the presentation be $x$, can you form a suitable inequality to solve the problem? Discuss and work out the solution with your classmates.

**Worked Example 8**

### Solving more complicated problem involving inequalities

Waseem has 12 \$10-notes and \$5-notes in his wallet. If the total value of all the notes is less than \$95, what is the maximum number of \$10-notes that he has?

**Solution**

Let the number of \$10-notes be $x$.
Then the number of \$5-notes is $12 - x$.

$10x + 5(12 - x) < 95$

$10x + 60 - 5x < 95$ expand the LHS

$5x + 60 < 95$

$5x + 60 - 60 < 95 - 60$ subtract 60 from both sides

$5x < 35$

$\frac{5x}{5} < \frac{35}{5}$ divide both sides by 5; no change in the inequality sign since $5 > 0$

$x < 7$

$\therefore$ Waseem has at most 6 \$10-notes.

**Practise Now 8**

Similar and Further Questions
**Exercise 3B**
Questions 5–7

An IQ test consists of 20 multiple choice questions. 3 points are awarded for a correct answer and 1 point is deducted for a wrong answer. No points are awarded or deducted for an unanswered question. Raju attempted a total of 19 questions and his total score for the IQ test was above 32. Find the minimum number of correct answers he obtained.

## Exercise 3B

1. On weekends, a movie ticket costs \$13.50. Form an inequality and solve it to find the maximum number of tickets Vasi can buy with \$265.

2. Cheryl signs up for a savings account that guarantees an increase in her savings at a fixed rate of \$15 per month. She plans to withdraw the money when it is more than twice the original amount of \$200. Find the least number of complete months that Cheryl has to keep the money in the account for.

3. David scored 66 marks for his first class test and 72 marks for his second class test. What is the minimum mark he must score for his third class test to meet his target of obtaining an average of 75 marks or more for the three tests?

4. If the sum of three consecutive integers is less than 75, find the cube of the largest possible integer.

5. In a Math Olympiad quiz, 5 points are awarded for a correct answer and 2 points are deducted for a wrong answer or if a question is left unanswered. Bernard attempted all 30 questions and his total score for the quiz was not more than 66. Find the maximum number of correct answers he obtained.

6. Sara opened her money box to find 50 \$5-notes and \$2-notes. If the total value of all the notes is more than \$132, find the minimum number of \$5-notes she has.

7. David wants to rent a car from either Company A or B. Company A quotes a rental rate of \$45 per day. Company B quotes a rate of \$38 per day, together with a fixed charge of \$75. Help David decide which company he should rent a car from.

# 3.4 Simultaneous linear inequalities

In Sections 3.2 and 3.3, we learnt how to solve linear inequalities such as $2x + 3 > 4 - 3x$.
How do we determine the solution if the variable is defined by two inequality statements, such as $4x + 14 \leqslant x + 5 < 3x - 1$?

## A. Compound inequalities

A compound inequality contains two inequality statements joined by the word "or" or "and".

If the compound inequality is joined by the word "or", the entire compound sentence is true as long as either one of the inequality statements is true.

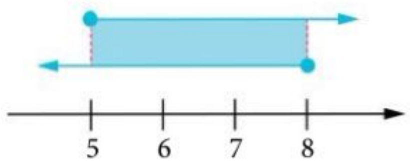
If the compound inequality is joined by the word "and", both inequality statements must be true at the same time. In this section, we shall learn how to solve compound linear inequalities in one variable joined by the word "and", which are also known as **simultaneous linear inequalities**.

## B. Simultaneous linear inequalities

To solve linear inequalities simultaneously, we first find the solution(s) to each inequality separately. Then we consider only the *common solutions* of the inequalities.

For example, given that $x \geqslant 5$ and $x \leqslant 8$, then the range of values of $x$ which satisfies both inequalities is $5 \leqslant x \leqslant 8$.

We can also represent $x \geqslant 5$ and $x \leqslant 8$ on a number line, and then identify the overlapping region that represents the range of values of $x$ which satisfies both inequalities.




Does $x = 1$ satisfy *both* $3x \leqslant x + 6$ *and* $2x + 4 < 3x + 6$?

What about $x = -3$?

**Big Idea**

**Diagrams**

The number line is a diagram that can be used to represent the solution sets of the individual inequality. It facilitates the identification of common solutions of the simultaneous linear inequalities.

Can you recall the features and conventions of a number line?

**Worked Example 9**

### Solving simultaneous linear inequalities

Find the range of values of $x$ for which $3x \leqslant x + 6$ and $2x + 4 < 3x + 6$.

**Solution**

Solving the two linear inequalities separately,

$$3x \leqslant x + 6 \quad \text{and} \quad 2x + 4 < 3x + 6$$

$$3x - x \leqslant x + 6 - x \qquad 2x + 4 - 3x < 3x + 6 - 3x$$

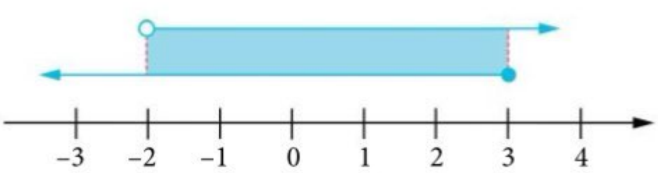
$$2x \leqslant 6 \qquad -x + 4 < 6$$

$$x \leqslant \frac{6}{2} \qquad -x + 4 - 4 < 6 - 4$$

$$x \leqslant 3 \qquad -x < 2$$

$$x > -2$$

Representing $x \leqslant 3$ and $x > -2$ on a number line,




$\therefore$ the solutions satisfying both inequalities lie in the overlapping shaded region, i.e. $-2 < x \leqslant 3$.

**Big Idea**

**Notations**

A compound inequality written without "and" / "or" is understood to be "and". Thus, "$-2 < x \leqslant 3$" represents the compound inequality "$x \leqslant 3$ and $x > -2$". We read it as "$x$ is greater than $-2$ *and* is less than or equal to 3".

**Practise Now 9**

Similar and Further Questions

**Exercise 3C**

Questions 1(a), (b), 2(a), (b), 8, 15(a)-(c)

Find the range of values of $x$ for which $2x - 3 \leqslant 7$ and $2x + 1 \geqslant -3x - 4$.

**Worked Example 10**

### Solving simultaneous linear inequalities

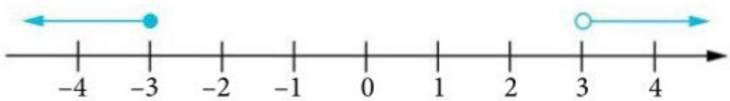
Solve the inequalities $4x + 14 \leqslant x + 5 < 3x - 1$.

**Solution**

Solving the two linear inequalities separately,

$$\begin{aligned} 4x + 14 &\leqslant x + 5 \\ 4x + 14 - x &\leqslant x + 5 - x \\ 3x + 14 &\leqslant 5 \\ 3x + 14 - 14 &\leqslant 5 - 14 \\ 3x &\leqslant -9 \\ x &\leqslant -3 \end{aligned} \quad \text{and} \quad \begin{aligned} x + 5 &< 3x - 1 \\ x + 5 - 3x &< 3x - 1 - 3x \\ -2x + 5 &< -1 \\ -2x + 5 - 5 &< -1 - 5 \\ -2x &< -6 \\ 2x &> 6 \\ x &> 3 \end{aligned}$$

Representing $x \leqslant -3$ and $x > 3$ on a number line,




$\therefore$ these two simultaneous linear inequalities have *no solution*.

**Attention**

Since this compound inequality has no connecting word, it is understood to be "and". It is translated into the following compound sentence:
$4x + 14 \leqslant x + 5$ and $x + 5 < 3x - 1$.

**Attention**

Since there is no overlapping region on the number line, there is no solution that satisfies both inequalities.

**Practise Now 10**

1. Solve the inequalities $8x + 13 \leqslant 4x - 3 < 5x - 11$.
2. Solve the inequalities $\dfrac{y-2}{3} < \dfrac{2y+1}{5} \leqslant 3$.

Similar and Further Questions
**Exercise 3C**
Questions 3(a)–(c), 9, 10(a)–(d), 11(a)–(d)

## Reflection

1. How is solving a linear inequality in one variable similar to or different from solving simultaneous linear inequalities in one variable?
2. How can I check that my solutions for a pair of simultaneous linear inequalities are correct?

# 3.5 Solving problems involving simultaneous linear inequalities

In this section, we shall formulate a pair of simultaneous linear inequalities and use them to solve problems.

**Worked Example 11**

### Solving problems involving simultaneous linear inequalities

There are 35 \$2-notes and \$10-notes in an envelope. If there are more \$2-notes than \$10-notes and the total value of all the notes is at least \$190, find the possible numbers of \$2-notes in the envelope.

**Solution**

Let the number of \$2-notes be $x$.

Then the number of \$10-notes is $(35 - x)$.

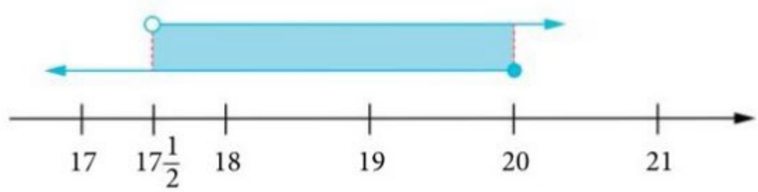
Since there are more \$2-notes than \$10-notes,

$$x > 35 - x$$
$$2x > 35$$
$$x > 17\frac{1}{2}$$

Since the total value of all the notes is at least \$190,

$$2x + 10(35 - x) \geqslant 190$$
$$2x + 350 - 10x \geqslant 190$$
$$-8x \geqslant -160$$
$$8x \leqslant 160$$
$$x \leqslant 20$$

Representing $x > 17\frac{1}{2}$ and $x \leqslant 20$ on a number line,




the solutions satisfying both inequalities is $17\frac{1}{2} < x \leqslant 20$.

$\therefore$ the possible numbers of \$2-notes in the envelope are 18, 19 or 20.

**Problem-solving Tip**

**Step 1:** Read the problem and identify the variables.
**Step 2:** Look for information in the problem that you can use to write an inequality.
**Step 3:** Look for more information that you can use to write a second inequality.
**Step 4:** Solve each inequality separately.
**Step 5:** Represent the solutions of the two inequalities on a number line, and identify the intersection of the two inequalities.

**Practise Now 11**

OPEN There are 48 20-cent coins and 50-cent coins in a box. The total value of all the coins is more than \$12, and there are at least twice as many 20-cent coins as 50-cent coins. Find how many 20-cent coins and 50-cent coins there could be in the box.

Similar and Further Questions
Exercise 3C
Questions 4, 12, 16

# 3.6 Linear inequalities in two variables

We have learnt in Chapter 2 that the graph of a linear equation in the form $ax + by = k$ is a straight line. This line represents all the possible pairs of $(x, y)$ values that satisfy the equation.

For example, Fig. 3.3(**a**) shows the graph of $y = x + 1$. The $x$- and $y$-coordinates of the points (1, 2) and (−1, 0) located on the graph will satisfy the equation $y = x + 1$.

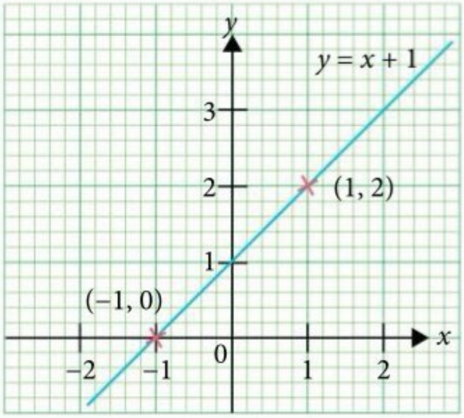
Now consider $y \geqslant x + 1$.

If we substitute $x = 1$ into the inequality, we get $y \geqslant 2$. For an $x$ value, there is a *range* of $y$ values. When $x = 1$, some possible values of $y$ are 2, 2.3, 3 and $3\frac{1}{5}$. On a graph, this corresponds to a series of coordinate pairs as shown in Fig. 3.3(**b**). Any of these coordinate pairs will *satisfy* the inequality $y \geqslant x + 1$.

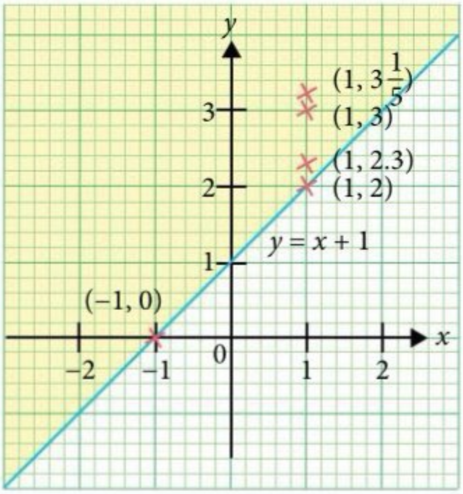
What are the values of $y$ when $x = 2$? Do these values include 2, 2.3, 3 and $3\frac{1}{5}$?

If we substitute various values of $x$ into $y \geqslant x + 1$, we will see that all coordinate pairs that satisfy the inequality $y \geqslant x + 1$ lie in the shaded region bounded by the line $y = x + 1$ as shown in Fig. 3.3(**b**).

Worked Example 12 shows how to represent some other inequalities in two variables graphically.




(**a**)




(**b**)

Fig. 3.3

Worked Example 12

## Representing linear inequalities in two variables graphically

Represent each of the following inequalities graphically by shading the region which contains all values of $x$ and $y$ that satisfy the inequality.

**(a)** $y \leqslant 2x - 1$

**(b)** $y < -x + 3.2$

**(c)** $2y + 4x > -1$

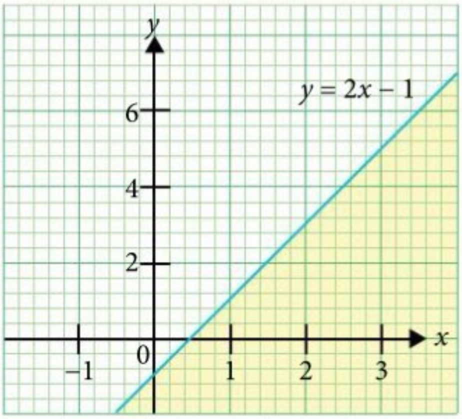
### Solution

**(a)** The point (2, 2) is in the shaded region. Substituting $x = 2$ and $y = 2$ into $y \leqslant 2x - 1$, we have

$2 \leqslant 2(2) - 1$

$2 \leqslant 3.$

The shaded region represents $y \leqslant 2x - 1$.

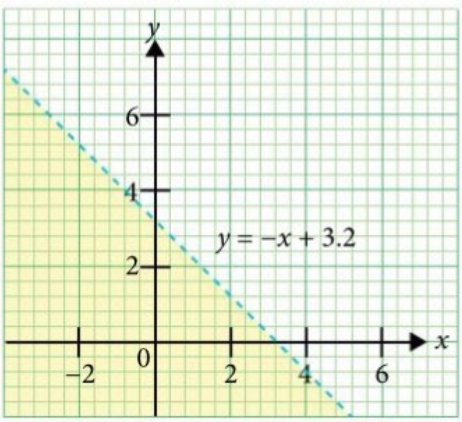



**Attention**

Sometimes, the region *not satisfying* the inequality is shaded.

**(b)** The point (0, 0) is in the shaded region. Substituting $x = 0$ and $y = 0$ into $y < -x + 3.2$, we have $0 < 3.2$.

The shaded region represents $y < -x + 3.2$.




**Attention**

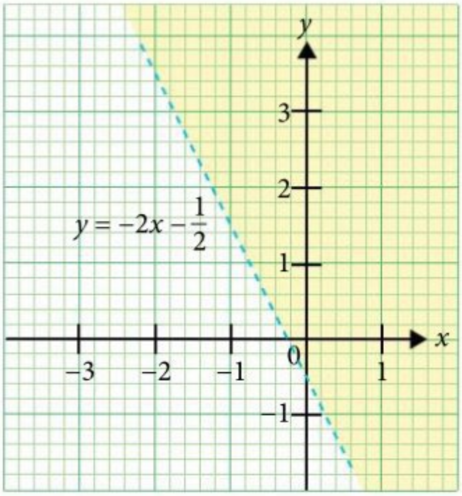
For **(b)**, if we substitute $x = 0$ into $y < -x + 3.2$, we get $y < 3.2$. This includes all values of $y$ that are less than, but *not equal* to, 3.2.

We use a *dashed line* for $y < -x + 3.2$ to show that the shaded region does not include this line.

**(c)** $$2y + 4x > -1$$
$$2y + 4x - 4x > -1 - 4x \quad \text{subtract } 4x \text{ from both sides}$$
$$\frac{2y}{2} > \frac{-1 - 4x}{2} \quad \text{divide both sides by 2; no change in the inequality sign since } 2 > 0$$
$$y > -2x - \frac{1}{2}$$

The point (0, 0) is in the shaded region. Substituting $x = 0$ and $y = 0$ into $y > -2x - \frac{1}{2}$, we have $0 > -\frac{1}{2}$.

The shaded region represents $y > -2x - \frac{1}{2}$.




**Practise Now 12**

Similar and Further Questions
Exercise 3C
Question 5(a)-(c)

Represent each of the following inequalities graphically by shading the region which contains all values of $x$ and $y$ that satisfy the inequality.

**(a)** $y < 3x - 2.4$

**(b)** $\frac{1}{2}x + 2y \geqslant 1$

**(c)** $\frac{3}{2} - 2x \geqslant \frac{1}{2}y$

In Section 3.4, we have used a number line to locate solutions which satisfy two inequalities in one variable. Similarly, by representing each inequality graphically, we can locate common solutions to two or more inequalities involving two variables. Worked Example 13 illustrates this.

**Worked Example 13**

### Representing two or more inequalities in two variables graphically

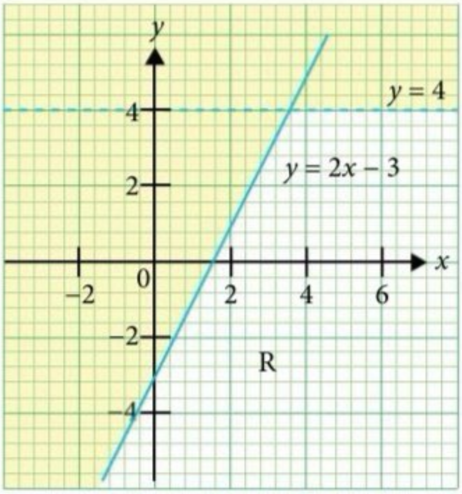
Using a sheet of graph paper, draw the graphs and identify the region which represents the inequalities in each of the following.

**(a)** $y \leqslant 2x - 3$ and $y < 4$

**(b)** $2y < x - 1$, $y \geqslant -x$, $x < 6$ and $y \geqslant -3$

*Solution*

**(a)** The unshaded region labelled R represents $y \leqslant 2x - 3$ and $y < 4$.

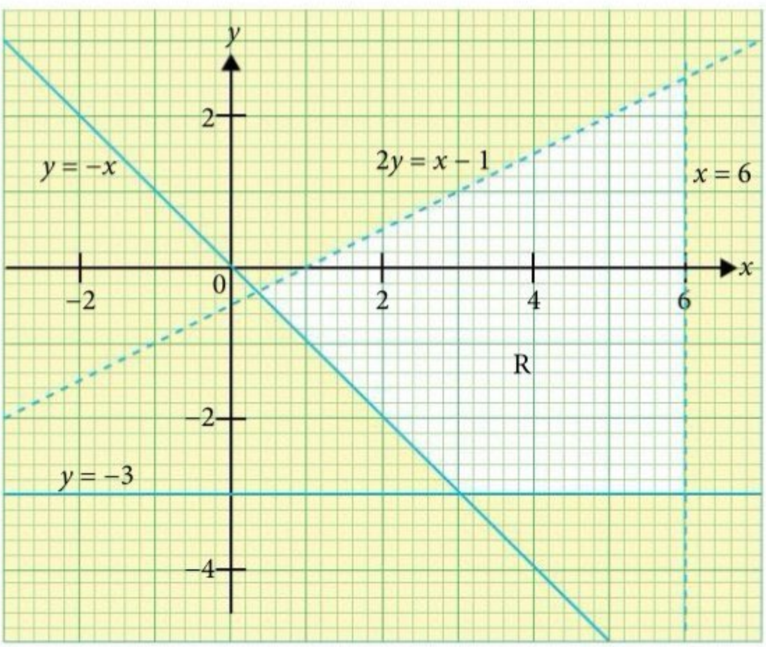



**Attention**

Generally, when representing more than two inequalities, the region *not satisfying* each inequality is shaded. This leaves the region satisfying all the inequalities unshaded.

To locate this region, represent each inequality on the graph by shading the region *not satisfying* the inequality.

**(b)** The unshaded region labelled R represents $2y < x - 1$, $y \geqslant -x$, $x < 6$ and $y \geqslant -3$.




**Practise Now 13**

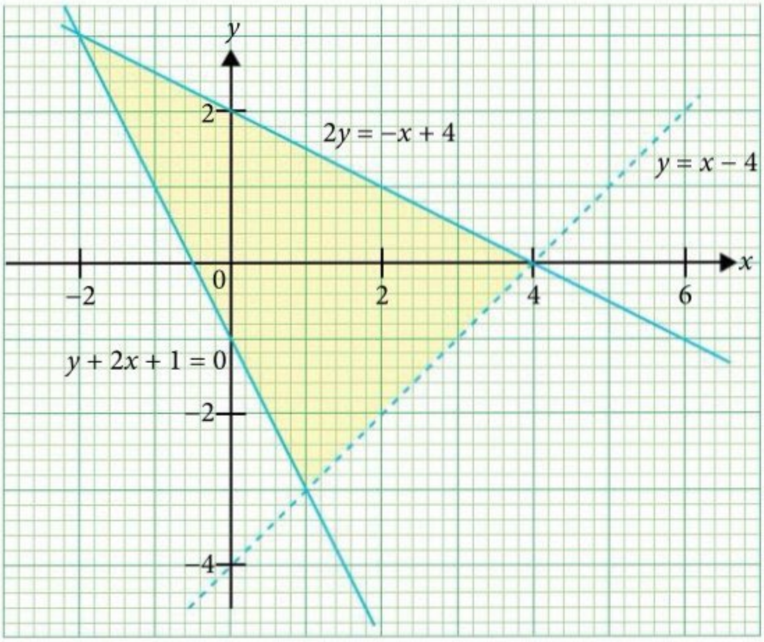
Similar and Further Questions
Exercise 3C
Questions 6(a), (b), 7(a), (b), 13(a), (b), 14

1. Using a sheet of graph paper, plot and identify the region which represents all the inequalities in each of the following:

   **(a)** $y < 2x + 1$, $y < 5$, $y \geqslant 0$ and $x \leqslant 4$

   **(b)** $2y + x < 3$, $y \geqslant x - 2$ and $x > -1$

2. Write down the inequalities that are represented by the shaded region shown in the figure.




3. Two numbers $p$ and $q$ are selected such that $p$ is more than twice of $q$, and the sum of $p$ and $q$ is greater than 5.
   **(i)** If $p$ is less than 8, write down the inequalities that describe $p$ and $q$.
   **(ii)** On a sheet of graph paper, using the vertical axis to represent $p$ and the horizontal axis to represent $q$, plot and identify the region which represents the inequalities in part **(i)**.
   OPEN **(iii)** Give an example of a pair of values of $p$ and $q$.

## Reflection

1. How does what I have learnt about graphs of equations in the form $ax + by = k$ help me in representing linear inequalities of the form $ax + by > k$, $ax + by < k$, $ax + by \geqslant k$ or $ax + by \leqslant k$ graphically?
2. How is finding common solutions to two or more linear inequalities in two variables similar to or different from finding common solutions to two or more linear inequalities in one variable?

Advanced | Intermediate | Basic

## Exercise 3C

1. Find the range of values of $x$ which satisfy each of the following pairs of inequalities.
   **(a)** $x - 4 \leqslant 3$ and $3x \geqslant -6$
   **(b)** $2x + 5 < 15$ and $3x - 2 > -6$

2. Find the integer values of $x$ which satisfy each of the following pairs of inequalities.
   **(a)** $5x - 1 < 4$ and $3x + 5 \geqslant x + 1$
   **(b)** $2x - 5 \geqslant 1$ and $3x - 1 < 26$

Basic Intermediate Advanced

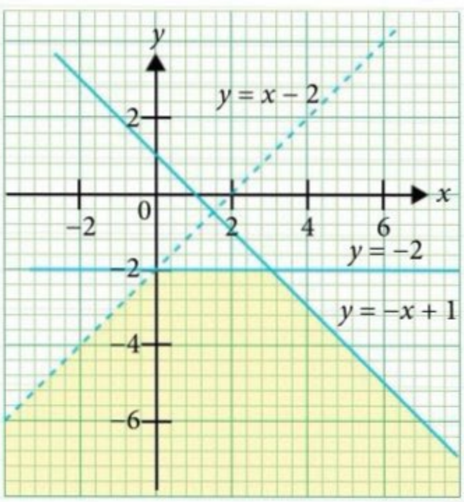
## Exercise 3C

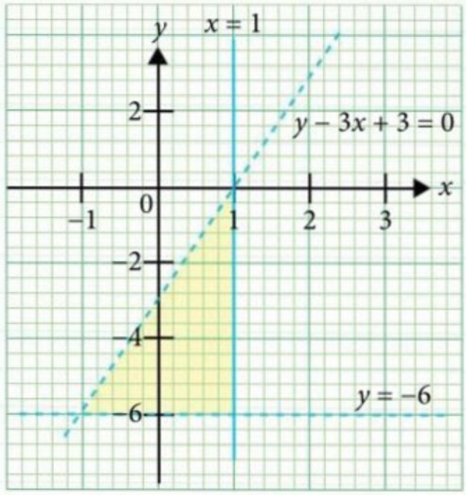
3. Solve each of the following inequalities, illustrating the solution on a number line.
   (a) $-4 \leqslant 2x \leqslant 3x - 2$
   (b) $1 - x < -2 \leqslant 3 - x$
   (c) $3x - 3 < x - 9 < 2x$

4. A fruit seller bought a crate of apples for \$66.50. After selling each apple for 55 cents, he was able to cover his cost but did not make a profit of more than \$20. How many apples could he have sold?

5. Represent each of the following inequalities graphically by shading the region which contains all values of $x$ and $y$ that satisfy the inequality.
   (a) $y > 4 - x$
   (b) $y + 2x \leqslant \frac{2}{5}$
   (c) $3x - 4y \geqslant 8$

6. On a sheet of graph paper, plot and identify the region which represents all the inequalities in each of the following.
   (a) $\frac{1}{2}y < x + \frac{3}{4}$, $x \leqslant 2$ and $y > 4$
   (b) $y > x - 2.2$, $2y + x \geqslant 0$ and $y < x$

7. For each of the following, write down the inequalities represented by the shaded region.
   (a)



   (b)



8. Given that $x$ is a prime number, find the values of $x$ for which $\frac{1}{2}x - 4 > \frac{1}{3}x$ and $\frac{1}{6}x + 1 < \frac{1}{8}x + 3$.

9. An integer $x$ is such that $x + 2 < 5\sqrt{17} < x + 3$. State the value of $x$.

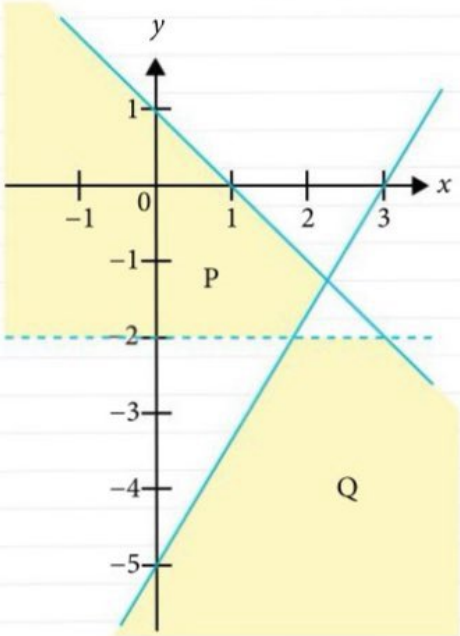
10. Solve each of the following inequalities.
    (a) $3 - a \leqslant a - 4 \leqslant 9 - 2a$
    (b) $1 - b < b - 1 < 11 - 2b$
    (c) $3 - c < 2c - 1 < 5 + c$
    (d) $3d - 5 < d + 1 \leqslant 2d + 1$

11. Solve each of the following inequalities.
    (a) $\frac{a}{4} + 3 \leqslant 4 \leqslant \frac{a}{4} + 6$
    (b) $\frac{b}{3} \geqslant \frac{b}{2} + 1 \geqslant b - 1$
    (c) $2(1 - c) > c - 1 \geqslant \frac{c - 2}{7}$
    (d) $d - 5 < \frac{2d}{5} \leqslant \frac{d}{2} + \frac{1}{5}$

12. Waseem and Joyce intend to buy a present for Sara. Waseem agrees to pay at least twice as much as but at most \$150 more than Joyce. Given that the present costs no more than \$210, what is the greatest amount Waseem has to pay?

## Exercise 3C

**13.** Determine the inequalities represented by the region
**(a)** P, **(b)** Q.




**14.** Two integers $a$ and $b$ are selected such that the following are satisfied:

- the sum of $a$ and two times of $b$ is less than 10,
- the sum of $a$ and $b$ is more than 2, and
- $b$ is greater than $a$.

By representing the above graphically, write down two possible pairs of the integers $a$ and $b$.

**15.** Find a possible set of integer values for $a$, $b$ and $c$ such that the solution to the inequalities $ax + 3 > 2x + b$ and $cx - 1 > 5x - 4$ is
**(a)** $x < -2$,
**(b)** $-3 < x < 4$,
**(c)** not possible.

**16.** Cheryl wants to give 12 good friends a treat on her birthday. Her friends can either have a muffin or an ice-cream cone. A muffin costs \$3.20 and an ice-cream cone costs \$2.40. Cheryl intends to spend no more than \$30 on the treat. Will she be able to do so if more of her friends want to have muffin than ice-cream cones? Explain your answer with your working clearly shown.

## Looking Back

In this chapter, we learnt about linear inequalities and how to formulate and solve linear inequalities in one variable. Inequalities can be used to **model** real-world problems to show a range of values instead of a single value. For example, we use inequalities to describe the speed limit within which vehicles can travel. Sometimes, inequalities are used when we are not certain of the exact value but we are aware of the range of values that it is part of. Therefore, it is important that we learn how to interpret the notation of inequalities so that we can express these ideas in a concise and precise manner. Besides the algebraic notation, the number line is another useful **diagram** to represent the range of values that a variable can take.

To solve problems in real-world contexts, we must identify how the quantities relate to each other and write a statement using an appropriate inequality symbol to represent the relationship. As we solve the inequality, we observe that the method of solving is similar to that of solving equations — that each subsequent inequality obtained from our algebraic manipulation is equivalent to the preceding one. As seen from the examples of equations and inequalities in this chapter, **equivalence** is the basis for solving linear equations as well as linear inequalities. This is an important idea that will carry us through the solving of other equations and inequalities.

Summary

1. **Linear inequality**
   - An inequality is a statement that relates two quantities.
   - An example of a linear inequality in one variable is $x \geqslant 75$.
     - Give two other examples of a linear inequality in one variable.

2. **Properties of a linear inequality**
   - The inequality sign is *reversed* only when we multiply or divide by a *negative* number on both sides of an inequality.

   Fill in the blanks in the table below.

| Case | Adding a number | Subtracting a number | Multiplying or dividing by a number | |
|---|---|---|---|---|
| | | | $c > 0$ | $d < 0$ |
| $x > y$ | $x + a > y + a$ | $x - b > y - b$ | $cx > cy$<br>$\frac{x}{c} > \frac{y}{c}$ | $dx < dy$<br>$\frac{x}{d} \square \frac{y}{d}$ |
| $x \geqslant y$ | $x + a \geqslant y + a$ | $x - b \square y - b$ | $cx \geqslant cy$<br>$\frac{x}{c} \square \frac{y}{c}$ | $dx \square dy$<br>$\frac{x}{d} \leqslant \frac{y}{d}$ |
| $x < y$ | $x + a < y + a$ | $x - b \square y - b$ | $cx \square cy$<br>$\frac{x}{c} \square \frac{y}{c}$ | $dx > dy$<br>$\frac{x}{d} > \frac{y}{d}$ |
| $x \leqslant y$ | $x + a \leqslant y + a$ | $x - b \square y - b$ | $cx \leqslant cy$<br>$\frac{x}{c} \leqslant \frac{y}{c}$ | $dx \square dy$<br>$\frac{x}{d} \square \frac{y}{d}$ |

3. **Solving a linear inequality**
   - To solve a linear inequality in the form $ax + b \leqslant c$ or $ax + b < c$, we isolate the variable $x$ on the left-hand side. This gives us the *solution* of the inequality.

   For example, $3x - 8 \leqslant 4$

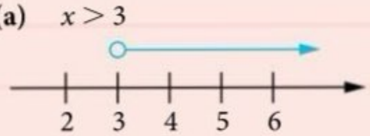
$$3x \leqslant 12$$
$$\frac{3x}{3} \leqslant \frac{12}{3} \quad \text{divide both sides by 3; no change in the inequality sign since } 3 > 0$$
$$x \leqslant 4$$

$$-2x + 10 > 5$$
$$-2x > -5$$
$$\frac{-2x}{-2} < \frac{-5}{-2} \quad \text{divide both sides by } -2\text{; reverse the inequality sign since } -2 < 0$$
$$x < \frac{5}{2}$$

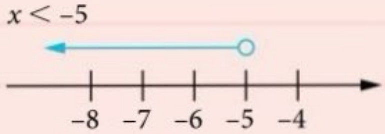
4. **Representing solution on a number line**
   - We can represent the *solutions* of inequalities using a number line.

   For example:

   (a) $x > 3$

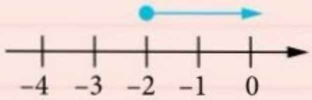



   (b) $x < -5$

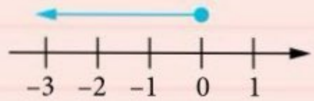


Summary

(c) $x \geqslant -2$



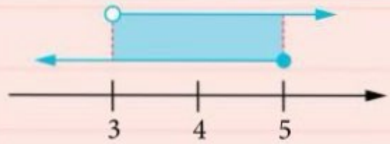
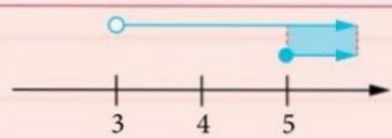
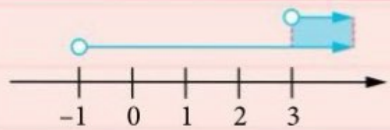
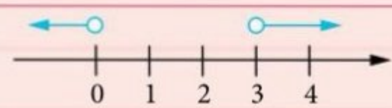

(d) $x \leqslant 0$




**5. Simultaneous linear inequalities**

- To solve a pair of simultaneous linear inequalities, we consider the range of values that *satisfies* both inequalities.
- To find the range of values that satisfies both inequalities, we can represent the solution of each linear inequality on a number line and identify the overlapping region.

  For example,

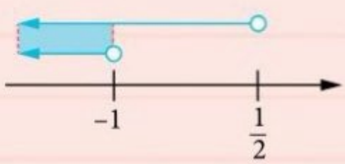
| | Simultaneous linear inequalities | Representation on a number line | Range of values of $x$ that satisfies both inequalities |
|---|---|---|---|
| (a) | $x > 3$ and $x \leqslant 5$ |   | $3 < x \leqslant 5$ |
| (b) | $x > 3$ and $x \geqslant 5$ |   | $x \geqslant 5$ |
| (c) | $x > 3$ and $x > -1$ |   | $x > 3$ |
| (d) | $x > 3$ and $x < 0$ |   | No solution |

Thus, to solve $2x + 1 < x < 1 - x$, we have

$2x + 1 < x$ and $x < 1 - x$

$x < -1$ $\quad$ $2x < 1$

$$x < \frac{1}{2}$$




The common solution is $x < -1$.

Summary

6. **Linear inequality in two variables**
   - An example of a linear inequality in two variables is $y > 2x - 1$.
   - A linear inequality in two variables can be represented graphically:
     1. Replace the inequality sign with an equal sign and draw the graph of the equation. A solid line is used if the inequality sign is $\leqslant$ or $\geqslant$, and a dashed line is used if the inequality sign is $<$ or $>$.
     2. Substitute a value of $x$ into the inequality to find the corresponding range of $y$ values.
     3. List some $y$ values in the range found in step 2. The selected value of $x$ and the list of $y$-values satisfy the given inequality.
     4. Shade the region in the graph which contains the coordinate pairs in step 3.
   - To find the values which satisfy two or more linear inequalities involving two variables, we can represent each inequality graphically and identify the overlapping region. This region contains common solutions to all the inequalities.
     - Give two other examples of a linear inequality in two variables and represent these inequalities on the same graph. Shade the region containing values which satisfy both inequalities.

# CHAPTER 4

# Expansion and Factorisation of Algebraic Expressions

Antique Egyptian papyrus and hieroglyph

Much of our knowledge about how mathematics was developed in the ancient times comes from our understanding of the solutions recorded on papyri, tablets and other primitive forms of writing.

For example, the famous Berlin Papyrus 6619 from ancient Egypt contains one interesting problem:

> The area of a square of 100 is equal to that of two smaller squares.
> The side of one is $\frac{1}{2} + \frac{1}{4}$ the side of the other.

If we try to solve this problem using modern algebraic **notations**, we realise that we need to work with algebraic expressions with more than one variable, as well as expressions that involve products of linear algebraic expressions.

In this chapter, we will further explore ideas in algebra, and extend our knowledge of algebraic manipulation to include more complicated expressions. In particular, we will learn how to expand products of linear expressions, and factorise quadratic expressions.

Berlin Papyrus 6619

## Learning Outcomes

What will we learn in this chapter?

- What quadratic expressions are
- How to add, subtract, expand, factorise and simplify quadratic expressions
- How to simplify algebraic expressions in two or more variables involving squares and cubes
- How to expand algebraic expressions
- How to factorise algebraic expressions of the form $ac + ad + bc + bd$
- What the three special algebraic identities are
- How to apply the three special algebraic identities to expand and factorise algebraic expressions
- Why the three special algebraic identities have useful applications in mathematics

**Introductory Problem**

The area of a rectangle is $(x^2 + 11x + 24)$ cm$^2$. If the breadth of the rectangle is $(x + 3)$ cm, what is the length of the rectangle?

In this chapter, we will learn how to expand and factorise algebraic expressions (including quadratic expressions).

# 4.1 Addition and subtraction of quadratic expressions

## A. Algebraic and linear expressions (Recap)

In Book 1, we have learnt that an **algebraic expression** is an expression containing letters, numbers and/or operations. Examples of algebraic expressions are:
$x + 8$, $\frac{3}{2}y - 7$, $\frac{6-x}{2}$ and $x - 3xy + \frac{4y}{3} - 7$.

> **Recall**
> We usually write $1 \times x$ as $x$, *not* $1x$.

**Class Discussion** — Recap of algebraic expressions

Let us consider the algebraic expression $x - 3xy + \frac{4y}{3} - 7$.

1. How many *variables* does the above expression contain? What are the variables?
2. How many *terms* does the above expression contain? What are the terms?
3. What is the *coefficient* of the variable(s) in each term of the above expression?
4. What do you call the last term of the above expression?
5. Can an algebraic expression consist of only one term? If yes, give an example of such an expression.

A **linear expression** in *one variable* $x$ is an algebraic expression that contains only one term in $x$, with or without a constant term. Examples of linear expressions are: $x + 8$, $5x$, $\frac{3}{2}y - 7$ and $\frac{6-x}{2}$.

**Class Discussion** — Recap of linear expressions

Which of the following are linear expressions? Give your reasons.

**(a)** $7 - 4x$ **(b)** $\frac{5}{3}y + 8$ **(c)** $\frac{2-x}{9}$ **(d)** $4x + y - 8$

**(e)** $4$ **(f)** $2x - 3xy + 7$ **(g)** $x^2 - 5x + 6$

## B. Quadratic expressions

In the Class Discussion on the previous page, we encountered the term $xy$.
What about $x \times x$? We can write it as $x^2$ (read as '$x$ squared').
This is similar to how we would write $3 \times 3 = 3^2$ and $4 \times 4 = 4^2$.

We call algebraic expressions such as $x^2 - 5x + 6$, $3x^2 - 9x$ and $-4x^2 + \frac{3}{2}$ **quadratic expressions**.

> **Quadratic expression**
> A quadratic expression in one variable $x$ is of the form $ax^2 + bx + c$, where $a$, $b$ and $c$ are constants and $a \neq 0$.
> If $a = 0$, then it becomes a linear expression.

> **Attention**
> $xy$ means $x \times y$ (*not* $x + y$).

> **Big Idea**
> **Notations**
> There are certain conventions to writing algebraic notations in a concise and precise manner to communicate mathematical ideas clearly. For example:
> - $x^2$ means $x \times x$ (*not* $2 \times x$);
> - $-4x^2$ means $-4 \times x \times x$.

## C. Addition and subtraction of quadratic expressions

In Book 1, we have learnt how to add or subtract real numbers, and **like terms** in $x$, as shown in Table 4.1. Similarly, to add or subtract like terms in $x^2$, we add or subtract their coefficients, as shown in the last column of Table 4.1.

| Real numbers | Terms in $x$ | Terms in $x^2$ |
|---|---|---|
| $(-2) + (-3) = -5$ | $(-2x) + (-3x) = -5x$ | $(-2x^2) + (-3x^2) = -5x^2$ |
| $5 + (-2) = 5 - 2 = 3$<br>$-2 + 5 = 5 - 2 = 3$ | $5x + (-2x) = 5x - 2x = 3x$<br>$-2x + 5x = 5x - 2x = 3x$ | $5x^2 + (-2x^2) = 5x^2 - 2x^2 = 3x^2$<br>$-2x^2 + 5x^2 = 5x^2 - 2x^2 = 3x^2$ |
| $2 - 5 = -3$ | $2x - 5x = -3x$ | $2x^2 - 5x^2 = -3x^2$ |
| $-5 - 2 = -7$ | $-5x - 2x = -7x$ | $-5x^2 - 2x^2 = -7x^2$ |
| $5 - (-2) = 5 + 2 = 7$ | $5x - (-2x) = 5x + 2x = 7x$ | $5x^2 - (-2x^2) = 5x^2 + 2x^2 = 7x^2$ |

Table 4.1

We have learnt that we *cannot* add or subtract *unlike terms*: e.g. $2x + 3$ cannot be simplified further because $2x$ and 3 are unlike terms. Similarly, $x^2 - 5x + 6$ cannot be simplified further by addition or subtraction because $x^2$, $-5x$ and 6 are unlike terms. Although $x^2$ and $-5x$ both contain the same variable $x$, they are unlike terms because the variables $x$ are of different powers.

**Worked Example 1**

**Adding and subtracting algebraic terms**

Without using a calculator, simplify the following.

**(a)** $5x^2 + (-11x^2)$ **(b)** $-8x^2 + 4x^2$
**(c)** $-2y^2 - 8y^2$ **(d)** $-3y^2 - (-9y^2)$

**Solution**

**(a)** $5x^2 + (-11x^2) = 5x^2 - 11x^2$
$= -6x^2$

**(b)** $-8x^2 + 4x^2 = 4x^2 - 8x^2$
$= -4x^2$

**(c)** $-2y^2 - 8y^2 = -10y^2$

**(d)** $-3y^2 - (-9y^2) = -3y^2 + 9y^2$
$= 9y^2 - 3y^2$
$= 6y^2$

**Practise Now 1**

Similar and Further Questions
**Exercise 4A**
Questions 1(a)–(h)

Without using a calculator, simplify the following.

**(a)** $x^2 + (-6x^2)$ **(b)** $10x^2 + (-19x^2)$

**(c)** $-13y^2 + 3y^2$ **(d)** $-28y^2 + 15y^2$

**(e)** $-8w^2 - 4w^2$ **(f)** $-11w^2 - 12w^2$

**(g)** $-x^2 - (-30x^2)$ **(h)** $-25x^2 - (-15x^2)$

We can add or subtract like terms only. Hence, we cannot simplify expressions that consist of unlike terms only, e.g. $x^2 + x$ or $x^2 - 7$, any further. However, $xy$ and $yx$ are like terms because $yx = xy$, so $xy + yx = xy + xy = 2xy$.

**Worked Example 2**

### Adding and subtracting algebraic terms

Without using a calculator, simplify each of the following expressions.

**(a)** $-3x^2 + (-4x^2) + 2 - 8$ **(b)** $-y^2 - 3xy - 5y^2 - (-4yx)$

**Solution**

**(a)** $-3x^2 + (-4x^2) + 2 - 8 = -7x^2 + (-6)$
$= -7x^2 - 6$

**(b)** $-y^2 - 3xy - 5y^2 - (-4yx) = -y^2 - 5y^2 - 3xy - (-4xy)$ group like terms and change $yx$ to $xy$
$= -6y^2 - 3xy + 4xy$
$= -6y^2 + xy$

**Practise Now 2**

Similar and Further Questions
**Exercise 4A**
Questions 2(a)–(f)

Without using a calculator, simplify each of the following expressions.

**(a)** $-5x^2 + (-2x^2) + 3 - 7$

**(b)** $8x^2 + (-6x^2) + 4x - 9x$

**(c)** $-4y^2 - yx + 3y^2 - (-5xy)$

**(d)** $-7y^2 - 9y - 5y^2 - (-3xy)$

**(e)** $10a^2 + (-12b^2) - 9 - (-3b^2) + 5 + (-6a^2)$

**(f)** $16h^2 - (-5k^2) - 18hk + 3h^2 + (-k^2) + 15kh$

## Reflection

1. What do I already know about the addition and subtraction of real numbers and like terms in $x$ that could help me learn the addition and subtraction of other like terms, for example, in $x^2$ or $xy$?
2. What have I learnt in this section that I am still unclear of?

# 4.2 Expansion of algebraic expressions of the form $(a + b)(c + d)$

## A. Expansion of $a(b + c)$ (Recap)

In Book 1, we have learnt how to **expand** an algebraic expression of the form $a(b + c)$ using the Distributive Law:

**Recall**

**Expansion** is the process of expressing an algebraic expression as the *sum* and/or *difference* of two or more terms.

**The Distributive Law**

$a(b + c) = ab + ac$

This is called the Distributive Law because the first factor $a$ is *distributed*, or multiplied separately, to each of the two terms, $b$ and $c$, in the second factor $(b + c)$. In particular,

**Negative of $(x + y)$**

$-(x + y) = -x - y$

**Negative of $(x - y)$**

$-(x - y) = -x + y$

**Worked Example 3**

**Expanding expressions using Distributive Law**

Expand each of the following expressions.

**(a)** $3(x + 2)$ **(b)** $-4(5x - y)$ **(c)** $8 - a(-3b + 2c)$

**Solution**

**(a)** $3(x + 2) = 3x + 6$

**(b)** $-4(5x - y) = -20x + 4y$ Distributive Law: $-4 \times (-y) = +4y$

**(c)** $8 - a(-3b + 2c) = 8 + 3ab - 2ac$

**Practise Now 3**

Similar and Further Questions
Exercise 4A
Questions 3(a)–(h)

Expand each of the following expressions.

**(a)** $2(x + 5)$ **(b)** $-3(6x - y)$

**(c)** $5 - a(-2b + 3c)$ **(d)** $-4 - 2a(-7x - 6y)$

**Worked Example 4**

### Simplifying algebraic expressions using Distributive Law

Simplify $-4x(y + 2z) - 3x(5y - z)$.

**Solution**

$-4x(y + 2z) - 3x(5y - z)$

$= -4xy - 8xz - 15xy + 3xz$ Distributive Law

$= -4xy - 15xy - 8xz + 3xz$ group like terms

$= -19xy - 5xz$

> **Big Idea**
>
> **Equivalence**
> Using the Distributive Law to write $-4x(y + 2z) - 3x(5y - z)$ in its equivalent form helps us to simplify the expression here.

**Practise Now 4**

Similar and Further Questions
**Exercise 4A**
Questions 4(a)–(d), 7(a)–(d)

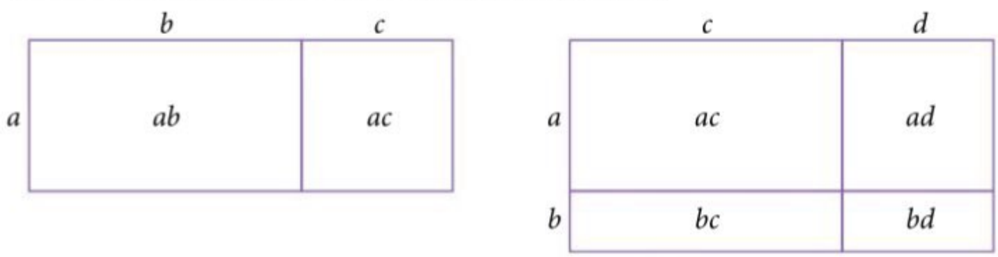
Simplify each of the following expressions.

**(a)** $-3x(y + 4z) - 5x(2y - z)$

**(b)** $2p(-4q - 3r) - 6q(3p + 2r)$

## B. Expansion of $(a + b)(c + d)$

Recall that in Book 1, we visualised the expansion of $a(b + c)$ using the rectangle in Fig. 4.1**(a)**.
In this section, we will learn how to expand an algebraic expression in the form $(a + b)(c + d)$. We can likewise visualise the expansion using the rectangle in Fig. 4.1**(b)**.




$a(b + c) = ab + ac$

**(a)**

$(a + b)(c + d) = ac + ad + bc + bd$

**(b)**

Fig. 4.1

> **Attention**
>
> If $a$, $b$, $c$ and $d$ are positive, we can think of this expansion as finding the area of a rectangle of sides $(a + b)$ and $(c + d)$. However, the result is the same for all values of $a$, $b$, $c$ and $d$.

**The Distributive Law for $(a + b)(c + d)$**

$$(a + b)(c + d) = ac + ad + bc + bd$$

> **Attention**
>
> Multiply each term in the first factor $(a + b)$ by each term in the second factor $(c + d)$.

**Worked Example 5**

### Expanding expressions using Distributive Law

Expand each of the following expressions.

**(a)** $(a + b)(7x + 5y)$  **(b)** $(3a + 1)(x - 4y)$

**(c)** $(x - 6y)(3c + 2d)$  **(d)** $(8p - 3q)(2r - 5s)$

**Solution**

**(a)** $(a + b)(7x + 5y) = 7ax + 5ay + 7bx + 5by$

**(b)** $(3a + 1)(x - 4y) = 3ax - 12ay + x - 4y$

**(c)** $(x - 6y)(3c + 2d) = 3cx + 2dx - 18cy - 12dy$

**(d)** $(8p - 3q)(2r - 5s) = 16pr - 40ps - 6qr + 15qs$

**Problem-solving Tip**

Draw the first arrow and write down the answer for the first term, before drawing the second arrow and so on.

**Recall**

**(c)** $-(x + y) = -x - y$

**(d)** $-(x - y) = -x + y$

**Practise Now 5**

Similar and Further Questions
**Exercise 4A**
Questions 5(a)–(f), 8(a)–(d)

Expand each of the following expressions.

**(a)** $(a + b)(8x + 7y)$  **(b)** $(2c + d)(5x + 9y)$

**(c)** $(5a + 2)(x - 2y)$  **(d)** $(6a + 5b)(3c - d)$

**(e)** $(x - 4y)(2c + 3d)$  **(f)** $(7x - 1)(3a + 2b)$

**(g)** $(6p - 5q)(3r - 4s)$  **(h)** $(2p - 9q)(7x - 3y)$

**(i)** $(-3a - 5b)(-7c + 3d)$  **(j)** $(-4r - 3s)(3 - 2t - 5u)$

**Problem-solving Tip**

**(j)** We have learnt how to expand $(a + b)(c + d)$. What do you think the expansion of $(a + b)(c + d + e)$ is?

**Worked Example 6**

### Simplifying algebraic expressions using Distributive Law

Simplify the expression $2ac - (3a - b)(c + 4b)$.

**Solution**

$2ac - (3a - b)(c + 4b)$

$= 2ac - (3ac + 12ab - bc - 4b^2)$  Distributive Law

$= 2ac - 3ac - 12ab + bc + 4b^2$  $-(x + y) = -x - y$ and $-(x - y) = -x + y$

$= 4b^2 - ac - 12ab + bc$

**Practise Now 6**

Similar and Further Questions
**Exercise 4A**
Questions 6(a)–(d), 9(a)–(d), 10, 11

Simplify each of the following expressions.

**(a)** $2ac - (3a + b)(c - 4d)$

**(b)** $2x(3y - 4z) - (3x + y)(y - 3z)$

**(c)** $(3p - q)(2r + s) - (p - 2q)(5r - 4s)$

**(d)** $(h + 6k)(2m - h) + (3h - 2m)(2k + h)$

Basic | Intermediate | Advanced

## Exercise 4A

1. Without using a calculator, simplify each of the following.
   (a) $2x^2 + (-11x^2)$
   (b) $5x^2 - 17x^2$
   (c) $-6y^2 + 15y^2$
   (d) $-30y^2 + 14y^2$
   (e) $-3e^2 - 10e^2$
   (f) $-12f^2 - 19f^2$
   (g) $-20g^2 - (-21g^2)$
   (h) $-18h^2 - (-5h^2)$

2. Without using a calculator, simplify each of the following expressions.
   (a) $-3x^2 + (-7x^2) + 9 - 18$
   (b) $14x^2 - 15x^2 + 8x - 10x$
   (c) $6y^2 + 19z + 9y^2 - 8yz$
   (d) $5y^2 - xy - y^2 - (-10yx)$
   (e) $w^2 + 2w - 7 - (-11w^2) - 5w - 1$
   (f) $-4h^2 - 9k^2 - (-2hk) + 3h^2 - 7k^2 + 2kh$

3. Expand each of the following expressions.
   (a) $10(x + 1)$ (b) $-4(3x - 2y)$
   (c) $8x(y - 1)$ (d) $-9x(3y - 2z)$
   (e) $2 + 3a(5 - 11b)$ (f) $-5 - 3c(2d + 3e)$
   (g) $7 - 6p(7q - 3r)$ (h) $11 - 8s(-12t - 7u)$

4. Simplify each of the following expressions.
   (a) $5x(y + 6z) - 2x(2y + 10z)$
   (b) $4a(b - 5c) + 2a(3b - 7c)$
   (c) $7d(3e - 4f) - 4d(3e - 2f)$
   (d) $-3h(3k - 4m) - 8h(2k + 3m)$

5. Expand each of the following expressions.
   (a) $(a + b)(4x + 9y)$ (b) $(5c + d)(5e + 2f)$
   (c) $(7m + 3)(n - 3p)$ (d) $(3t - 7u)(7v + 4w)$
   (e) $(2a - b)(x - 6y)$ (f) $(3h - 5k)(-q - 7r)$

6. Simplify each of the following expressions.
   (a) $3ac - (2a + b)(c + 3d)$
   (b) $2xy + (x - 5a)(6y + 7b)$
   (c) $9ps + (2p - 3r)(4q - 5s)$
   (d) $10hk - (-3m - h)(8k - 3n)$

7. Simplify each of the following expressions.
   (a) $3a(5b + c) - 2b(3c + a)$
   (b) $-2d(4f - 5h) - 8f(3d + 7h)$
   (c) $4k(13m - 5n) - 13m(4k - 5n)$
   (d) $-6w(7x - 12y) - 4y(11w - 9x)$

8. Expand each of the following expressions.
   (a) $(x + 9y)(a + 3b + 1)$
   (b) $(2p + 5q)(7 - r + 5s)$
   (c) $(11m - 12n)(4t - 3u - 10)$
   (d) $(-5w - 14y)(-2v - 9x - 6z)$

9. Simplify each of the following expressions.
   (a) $2a(5b + 4c) - (2a + c)(3b - 5c)$
   (b) $(7x - 3y)(w - 4z) + (z - 2w)(5x - 9y)$
   (c) $(10p + q)(3r + 2q) - (5p - 4q)(-r - 6q)$
   (d) $(4h - 11k)(2m - 13h) + (-13h - 12m)(8k + 9h)$

10. Subtract $(11a - 2b)(4c + 6d + 15)$ from $(-13b - 3c)(20 - 11a - 7d)$.

11. A farmer plants carrots and tomatoes on two separate rectangular plots of land. The plot used to plant carrots measures $(2x + 3y)$ m by $(5w - 8z)$ m and that used for tomatoes measures $(3z - 10w)$ m by $(-4x - 9y)$ m. Albert says that the total area used to plant the crops is $(10xw - 24yz - 12xz + 90wy)$ m$^2$. Showing full working, explain if you agree with him.

# 4.3 Expansion of quadratic and complex expressions

## A. Expansion of quadratic expressions of the form $px(qx + r)$

We have revisited in Section 4.2A that expressions like $2(x + 3)$ can be expanded using the *Distributive Law*:
$2(x + 3) = 2x + 6$.

We can also represent $2(x + 3)$ in another way, as shown in Fig. 4.2. This method will be useful for us to factorise quadratic expressions in Section 4.4 later.

First, we draw a **multiplication frame** as shown in Fig. 4.2(**a**).

The expression $2(x + 3)$ has two factors, namely 2 and $(x + 3)$.
To represent the factor 2, we place 2 1 discs in the left column of the multiplication frame as shown in Fig. 4.2(**b**).
To represent the factor $(x + 3)$, we place 1 $x$ disc and 3 1 discs in the top row of the multiplication frame as shown in Fig. 4.2(**b**).

Multiplying the first term in the left column, 1, by the first term in the top row, $x$, we get $x$, so we put 1 $x$ disc in the corresponding position as shown in Fig. 4.2(**c**).

Continuing the multiplication process for each disc of the two factors, we get $2x + 6$, as shown in Fig. 4.2(**d**).
Finally, we write the final answer $2(x + 3) = 2x + 6$ at the bottom of the final arrangement of the discs.

From Fig. 4.2(**d**), we see pictorially how we can use the Distributive Law for $2(x + 3)$ to obtain $2x + 6$.

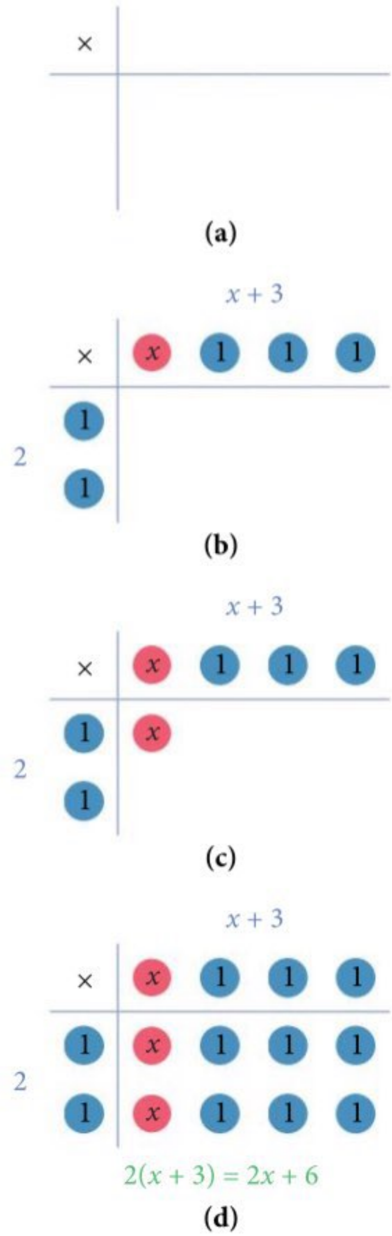
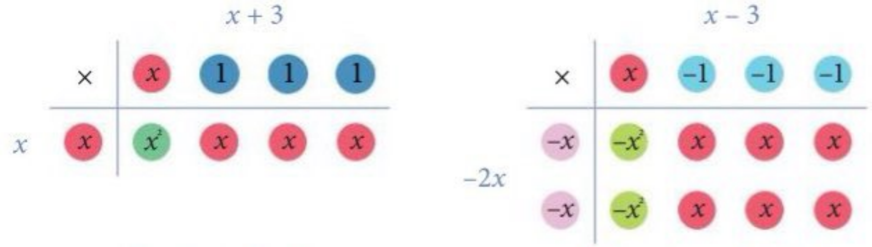


Fig. 4.2

Now, how do we use algebra discs and a multiplication frame to show the expansion of $x(x + 3)$ and $-2x(x - 3)$? The results are shown in Fig. 4.3**(a)** and **(b)** respectively.

The algebra discs $x^2$ and $-x^2$ are used to represent $x^2$ and $-x^2$ respectively.




$x(x + 3) = x^2 + 3x$ **(a)**

$-2x(x - 3) = -2x^2 + 6x$ **(b)**

Fig. 4.3

**Attention**

When a factor has both $x$ (or $-x$) discs and 1 (or −1) discs, the $x$ (or $-x$) discs are usually placed first.

From Fig. 4.3, we see pictorially how we can use the Distributive Law for:

**(a)** $x(x + 3)$ to obtain $x^2 + 3x$,

**(b)** $-2x(x - 3)$ to obtain $-2x^2 + 6x$.

## Investigation

### Expansion of expressions of the form $p(qx + r)$ and $px(qx + r)$

Expand each of the following expressions using algebra discs and a multiplication frame.

**(a)** $3(x + 4)$ **(b)** $-2(3x - 1)$ **(c)** $-4(3 - 2x)$

**(d)** $2x(x + 3)$ **(e)** $-3x(2x + 5)$ **(f)** $-4x(1 - 3x)$

**Worked Example 7**

### Expanding expressions of the form $px(qx + r)$ using Distributive Law

Without using algebra discs or multiplication frames, expand each of the following expressions.

**(a)** $-7x(4x + 3)$ **(b)** $-3y(10 - 9y)$

**Solution**

**(a)** $-7x(4x + 3) = -28x^2 - 21x$

**(b)** $-3y(10 - 9y) = -30y + 27y^2$
$= 27y^2 - 30y$

**Practise Now 7**

Similar and Further Questions
Exercise 4B
Questions 1(a)–(d)

Without using algebra discs or multiplication frames, expand each of the following expressions.

**(a)** $4x(2x + 3)$ **(b)** $11a(4 - a)$

**(c)** $-5x(3x + 4)$ **(d)** $-n(12n - 29)$

**Worked Example 8**

### Simplifying expressions of the form $px(qx + r)$

Simplify $x(2x - 1) - 2(x + 5)$.

**Solution**

$x(2x - 1) - 2(x + 5) = 2x^2 - x - 2x - 10$ Distributive Law
$= 2x^2 - 3x - 10$

**Big Idea**

**Equivalence**

Writing $x(2x - 1)$ and $-2(x + 5)$ in their equivalent expanded forms $2x^2 - x$ and $-2x - 10$ helps us to simplify the expression.

**Practise Now 8**

Similar and Further Questions
**Exercise 4B**
Questions 2(a)–(d), 8(a)–(f)

Simplify each of the following expressions.

**(a)** $x(7x - 4) - 3(x + 2)$ **(b)** $-2x(x - 8) - 5(3x - 4)$

**(c)** $-(5y + 8) - 3y(4 - 9y)$ **(d)** $-6k(7 - k) + 5k(-2k - 3)$

**Worked Example 9**

### Expanding and simplifying algebraic expressions involving squares and cubes

**(a)** Expand $ab(ac + b^2)$.

**(b)** Simplify $x(xy + y) - y(xy + x)$.

**Solution**

**(a)** $ab(ac + b^2) = a^2bc + ab^3$ Distributive Law

**(b)**
$$x(xy + y) - y(xy + x) = x^2y + xy - xy^2 - yx \quad \text{Distributive Law}$$
$$= x^2y - xy^2$$

**Problem-solving Tip**

**(a)** $a \times a = a^2$
$b \times b^2 = b \times b \times b$
$= b^3$

**(b)** $yx = xy$
$x^2y \neq xy^2$

**Practise Now 9**

Similar and Further Questions
**Exercise 4B**
Questions 3(a)–(d), 4(a)–(d), 9(a)–(d), 10(a), (b)

**(a)** Expand $xy(yz + x^2 - xy)$.

**(b)** Simplify $h^2(km + m) - m(h^2m - h^2k)$.

## B. Expansion of quadratic expressions of the form $(px + q)(rx + s)$

Let us begin with the expression $(x + 2)(x + 3)$.
We have learnt in Section 4.2B how to use the Distributive Law to expand $(a + b)(c + d)$ and obtain $ac + ad + bc + bd$.

Therefore, $(x + 2)(x + 3) = x^2 + 3x + 2x + 6$
$= x^2 + 5x + 6$

Let us learn how algebra discs and a multiplication frame can be used to get the same result, as shown in Fig. 4.4**(a)**.

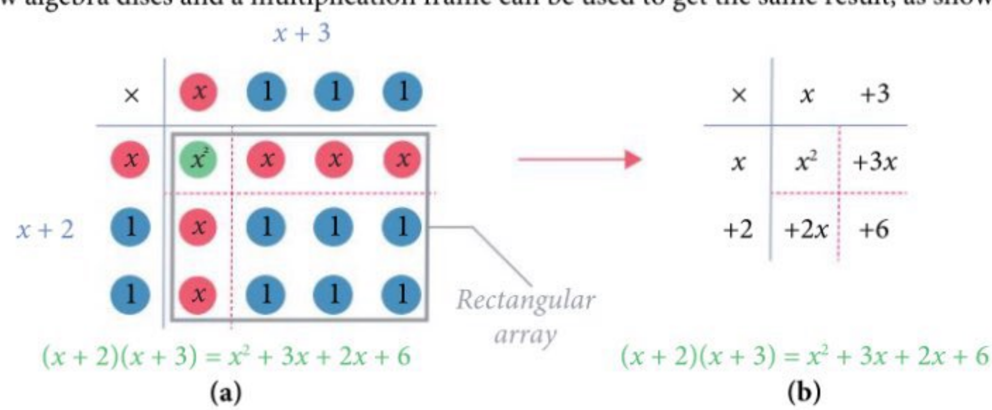



Fig. 4.4

From Fig. 4.4(**a**), we see that the discs representing $x^2 + 3x + 2x + 6$ form a **rectangular array**. This rectangular array can be divided into *four distinct regions* (by the two red dotted lines):

- the region corresponding to $x \times x$ contains the $x^2$ disc;
- the region corresponding to $2 \times 3$ contains the $1$ discs;
- the remaining two regions contain the $x$ discs.

Without using algebra discs, we can simply write the terms in the **multiplication frame** as shown in Fig. 4.4(**b**).

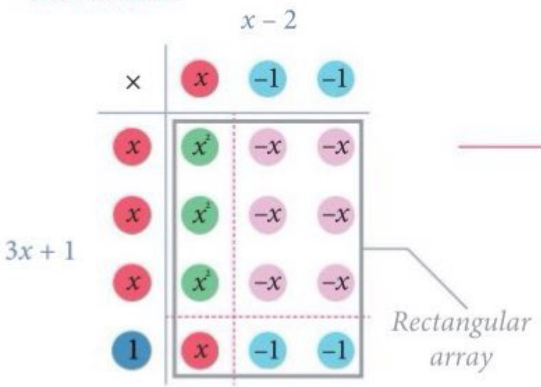
Fig. 4.5 shows how algebra discs and a multiplication frame can be used to expand $(3x + 1)(x - 2)$.

In this case, we end up with $-x$ and $-1$ discs as well. The discs representing $3x^2 - 6x + x - 2$ still form a **rectangular array**, which contains *four distinct regions*:

- the region corresponding to $x \times x$ contains the $x^2$ disc;
- the region corresponding to $1 \times (-2)$ now contains the $-1$ discs;
- the remaining two regions contain either the $x$ discs or the $-x$ discs, *but not both*.

**Attention**

In general, in a rectangular array,
- the top left region contains either the $x^2$ discs or the $-x^2$ discs (but not both);
- the bottom right region contains either the $1$ discs or the $-1$ discs (but not both);
- the remaining two regions contain either the $x$ discs or the $-x$ discs (but not both in the same region).




$$(3x + 1)(x - 2) = 3x^2 - 6x + x - 2$$
$$= 3x^2 - 5x - 2$$

(**a**)

| × | $x$ | $-2$ |
|---|---|---|
| $3x$ | $3x^2$ | $-6x$ |
| $+1$ | $+x$ | $-2$ |

$$(3x + 1)(x - 2) = 3x^2 - 6x + x - 2$$
$$= 3x^2 - 5x - 2$$

(**b**)

Fig. 4.5

## Investigation — Expansion of expressions of the form $(px + q)(rx + s)$

Expand each of the following quadratic expressions using algebra discs and a multiplication frame.

**(a)** $(x + 3)(x + 4)$ **(b)** $(2x + 1)(x - 3)$ **(c)** $(3y - 4)(2y - 1)$

**(d)** $(3 - 2x)(3x + 1)$ **(e)** $(3 - 2p)(4 - 3p)$ **(f)** $(-3x - 2)(2x + 3)$

Worked Example 10

## Expanding quadratic expressions using multiplication frame

Without using algebra discs, expand $(3x + 4)(7 - 8x)$ using a multiplication frame.

**Solution**

**Method 1:**

| $\times$ | $-8x$ | $+7$ |
|---|---|---|
| $3x$ | $-24x^2$ | $+21x$ |
| $+4$ | $-32x$ | $+28$ |

$$\begin{aligned}(3x + 4)(7 - 8x) &= (3x + 4)(-8x + 7)\\ &= -24x^2 + 21x - 32x + 28\\ &= -24x^2 - 11x + 28\end{aligned}$$

**Method 2:**

| $\times$ | $7$ | $-8x$ |
|---|---|---|
| $3x$ | $21x$ | $-24x^2$ |
| $+4$ | $+28$ | $-32x$ |

$$\begin{aligned}(3x + 4)(7 - 8x) &= 21x - 24x^2 + 28 - 32x\\ &= -24x^2 - 11x + 28\end{aligned}$$

**Reflection**

The region containing $x^2$ in the rectangular array is different for both methods. Does this affect the answer? Why or why not?

**Practise Now 10**

Similar and Further Questions
Exercise 4B
Questions 5(a)–(d), 11(a)–(f)

Without using algebra discs, expand each of the following expressions using a multiplication frame.

(a) $(4x + 7)(3x + 5)$
(b) $(9x - 4)(x + 6)$
(c) $(3y + 11)(2 - 7y)$
(d) $(5 - 3k)(4 - 7k)$

Worked Example 11

## Simplifying quadratic expressions

Simplify $(2x - 3)(x + 5) - 2(4x + 1)(5 - 3x)$.

**Solution**

| $\times$ | $x$ | $+5$ |
|---|---|---|
| $2x$ | $2x^2$ | $+10x$ |
| $-3$ | $-3x$ | $-15$ |

| $\times$ | $5$ | $-3x$ |
|---|---|---|
| $4x$ | $20x$ | $-12x^2$ |
| $+1$ | $+5$ | $-3x$ |

**Big Idea**

**Equivalence**

Writing $(2x - 3)(x + 5)$ and $(4x + 1)(5 - 3x)$ in their equivalent expanded forms $2x^2 + 7x - 15$ and $-12x^2 + 17x + 5$ helps us to simplify the expression.

$$\begin{aligned}(2x - 3)(x + 5) - 2(4x + 1)(5 - 3x) &= 2x^2 + 10x - 3x - 15 - 2(-12x^2 + 20x - 3x + 5)\\ &= 2x^2 + 7x - 15 - 2(-12x^2 + 17x + 5)\\ &= 2x^2 + 7x - 15 + 24x^2 - 34x - 10\\ &= 26x^2 - 27x - 25\end{aligned}$$

$-(x - y) = -x + y$;
$-(x + y) = -x - y$

**Practise Now 11**

Similar and Further Questions
Exercise 4B
Questions 12(a)–(d), 13(a)–(d), 17, 18

Simplify each of the following expressions.

(a) $(3x - 2)(x + 4) - 5x(x - 3)$
(b) $(5y - 1)(y + 6) + 3(4y - 5)(9 - 2y)$

**Problem-solving Tip**

(a) Use the Distributive Law to expand $-5x(x - 3)$.

Worked Example 12

### Expanding quadratic expressions in two variables

Expand $(x + 2y)(x - 3y)$.

**Solution**

**Method 1:**

$$(x + 2y)(x - 3y) = x^2 - 3xy + 2yx - 6y^2 \quad \text{Distributive Law}$$
$$= x^2 - xy - 6y^2$$

**Problem-solving Tip**

Recall that multiplication is commutative, therefore $yx = xy$.

**Method 2:**

| $\times$ | $x$ | $-3y$ |
|---|---|---|
| $x$ | $x^2$ | $-3xy$ |
| $+2y$ | $+2yx$ | $-6y^2$ |

$\therefore (x + 2y)(x - 3y) = x^2 - 3xy + 2yx - 6y^2$
$= x^2 - xy - 6y^2$

**Reflection**

Which method do you prefer? Why?

**Practise Now 12**

Similar and Further Questions
Exercise 4B
Questions 6(a)–(d), 14(a)–(d), 15(a)–(d)

1. Expand $(2x - 7y)(5x + y)$.
2. Simplify $(3w + 5v)(2v - 5w) - 6w(w - 2v)$.

## C. Expansion of expressions of the form $(px + q)(rx + s)(tx + u)$

In this section, we will learn to expand expressions of the form $(px + q)(rx + s)(tx + u)$.

Worked Example 13

### Expanding complex expressions

Expand $(x + 2)(x - 3)(2x - 5)$.

**Solution**

$(x + 2)(x - 3)(2x - 5)$
$= (x^2 - 3x + 2x - 6)(2x - 5)$ Distributive Law
$= (x^2 - x - 6)(2x - 5)$
$= 2x^3 - 5x^2 - 2x^2 + 5x - 12x + 30$ Distributive Law
$= 2x^3 - 7x^2 - 7x + 30$

**Problem-solving Tip**

Expand $(x + 2)(x - 3)$ first, before multiplying by $(2x - 5)$. Alternatively, we can start by expanding $(x - 3)(2x - 5)$ first.

**Practise Now 13**

Similar and Further Questions
Exercise 4B
Questions 7(a)–(d), 16(a)–(d)

Expand each of the following expressions.

**(a)** $(x + 1)(2 + 3x)(2 + x)$
**(b)** $(p + 2q)(p + q)(2p - 3q)$

## Reflection

1. What do I already know about using the Distributive Law to expand algebraic expressions that could help me expand quadratic expressions using a multiplication frame?
2. How is the expansion of quadratic expressions using a multiplication frame similar to using the Distributive Law?

Advanced | Intermediate | Basic

## Exercise 4B

**1.** Expand each of the following expressions.
(a) $5a(3a - 4)$ (b) $-8b(3b + 5)$
(c) $-5n(2 - 3n)$ (d) $-m(-m - 1)$

**2.** Simplify each of the following expressions.
(a) $4(2a + 3) + 5a(a + 3)$
(b) $9b(5 - 2b) + 3(6 - 5b)$
(c) $c(3c + 1) - 2c(c + 3)$
(d) $-6d(5d - 4) + 2d(3d - 2)$

**3.** Expand each of the following expressions.
(a) $-3a(2a + 3b^2)$ (b) $-4c(2c^2 - 5cd)$
(c) $-hk(7k - 3h)$ (d) $5xy(x - 4yz)$

**4.** Simplify each of the following expressions.
(a) $4k(3k + m) - 3k(2k - 5m)$
(b) $n(p - 2n) - 4n(n - 2p)$
(c) $3w(wt - 2t^2) + t(3wt - 4w^2)$
(d) $2x(-y - xy^2) - y(-2x + 3x^2y)$

**5.** Expand each of the following expressions.
(a) $(x + 3)(x + 7)$ (b) $(4y + 1)(3y + 5)$
(c) $(t + 1)(t - 8)$ (d) $(5 - v)(7 - v)$

**6.** Expand each of the following expressions.
(a) $(x + y)(x + 6y)$ (b) $(x + 3y)(x - 5y)$
(c) $(3c + 7d)(c - 2d)$ (d) $(3k - 5h)(-h - 7k)$

**7.** Expand each of the following expressions.
(a) $(a + 1)(a + 2)(a + 3)$
(b) $(1 + b)(b - 4)(5 + b)$
(c) $(m - n)(3m + 2n)(2m - 3n)$
(d) $(x - 6y)(4x - y)(3x - 4y)$

**8.** Simplify each of the following expressions.
(a) $7a(2a + 1) - 4(8a + 3)$
(b) $3(2b - 1) - 2b(5b - 3)$
(c) $3c(5 + c) - 2c(3c - 7)$
(d) $2d(3d - 5) - d(2 - d)$
(e) $-f(9 - 2f) + 4f(f - 8)$
(f) $-2h(3 + 4h) - 5h(h - 1)$

**9.** Expand each of the following expressions.
(a) $13x^2y(3xy - y)$
(b) $-8mn(-12m + nw - 7n^2)$
(c) $2p(3p + q^2p^2 + 7qr^3)$
(d) $-7s^2t(s - 4t^2 - 3su^3)$

**10.** Simplify each of the following expressions.
(a) $2x^2(z - yz - xz) + 3z(xz - x^2y + 2x^3)$
(b) $ab(ac + b^2 - c^2) - bc(a^2 - 2ac - 3ab^2)$

**11.** Expand each of the following expressions.
(a) $(2a + 1)(3a - 9)$
(b) $(5b - 2)(5b + 7)$
(c) $(4c - 5)(7c - 10)$
(d) $(3d + 14)(5 - 2d)$
(e) $(1 - f)(17f + 16)$
(f) $(19 - 3h)(10 - 9h)$

**12.** Simplify each of the following expressions.
(a) $5 + (x + 1)(x + 3)$
(b) $3y + (y + 7)(2y - 1)$
(c) $(3t + 2)(t - 9) + 2t(4t + 1)$
(d) $(w - 3)(w - 8) + (w - 4)(2w + 9)$

## Exercise 4B

**13.** Simplify each of the following expressions.
(a) $4a^2 - (3a - 4)(2a + 1)$
(b) $2b(b - 6) - (2b + 5)(7 - b)$
(c) $(4c - 3)(c + 2) - (3c - 5)(-c - 9)$
(d) $(2d + 3)(5d - 2) - 2(5d - 3)(d + 1)$

**14.** Expand each of the following expressions.
(a) $(x^2 + 2)(x + 5)$ (b) $(2x - 3y)(x + 5y - 2)$
(c) $(x + 2)(x^2 + x + 1)$ (d) $(3x^2 - 3x + 4)(3 - x)$

**15.** Simplify each of the following expressions.
(a) $5x(x - 6y) + (x + 3y)(3x - 4y)$
(b) $(7x - 3y)(x - 4y) + (5x - 9y)(y - 2x)$
(c) $(8x - y)(x + 3y) - (4x + y)(9y - 2x)$
(d) $(10x + y)(3x + 2y) - (5x - 4y)(-x - 6y)$

**16.** Simplify each of the following expressions.
(a) $-2x(x + 3) + (x + 2)(3x + 1)(x + 5)$
(b) $(-2x + 1)(x - 3)(4x + 1) - (2x + 5)(13x - 1)$
(c) $m(m + 2n)(-2m + n) + (4m + n)(m + n)(3m - n)$
(d) $(x - y)(2x + 3y)(4x - 6y) - 2x(x + y)(x - y)$

**17.** OPEN Q The total mass of a watermelon and a pack of lemons is $(7x - 5)(9 + 2x)$ kg. Write down two possible pairs of quadratic expressions for the mass of the watermelon and the corresponding mass of the pack of lemons.

**18.** A pair of shoes costs $\$(4x^2 - 9)$ and a bag costs $\$(2x^2 - 3x + 5)$. Find the total cost of five such pairs of shoes and four of the bags, in terms of $x$.

# 4.4 Factorisation of quadratic expressions

## A. Factorisation by extracting common factors

In Book 1, we have learnt how to **factorise** an algebraic expression of the form $ab + ac$ by extracting the common factor $a$ in both terms to obtain:

$$ab + ac = a(b + c).$$

Factorisation is the reverse of expansion:

$$a(b + c) \underset{\text{factorise}}{\overset{\text{expand}}{\rightleftarrows}} ab + ac$$

Fig. 4.6

> **Recall**
> **Factorisation** is the process of expressing an algebraic expression as the *product* of two or more factors.

> **Information**
> This diagram helps to depict the reverse process of expansion and factorisation.

When extracting negative common factors such as $-1$, we have to *change the sign* inside the brackets as shown below:

> **Extracting negative common factor**
> factor out $-1$
> $$-x - y = -(x + y)$$
> change sign

**Worked Example 14**

### Factorising linear expressions

Factorise each of the following expressions completely.

(a) $12x + 18$ (b) $-15x - 6$ (c) $-10ax + 25ay$

**Solution**

(a) $12x + 18 = 6(2x + 3)$ HCF of 12 and 18 = 6

factor out −3

(b) $-15x - 6 = -3(5x + 2)$ HCF of 6 and 15 = 3, so −3 is a common factor

change sign

(c) $-10ax + 25ay = 25ay - 10ax$ rearrange order of terms

$= 5a(5y - 2x)$ common factors are 5 and $a$

**Practise Now 14**

Similar and Further Questions
Exercise 4C
Questions 1(a)–(f), 12(a), (b)

Factorise each of the following expressions completely.

(a) $12x + 8$ (b) $21 + 35a$
(c) $-15x - 25$ (d) $-8 - 20p$
(e) $-27ax + 12ay$ (f) $-42xy - 12xz$
(g) $36p - 54pq + 18pr$ (h) $-9z - 24bz - 15cz$

**Worked Example 15**

### Factorising expressions involving squares and cubes

Factorise each of the following expressions completely.

(a) $6x^2 + 15x$ (b) $-a^3bc - a^2b^2$

**Solution**

(a) $6x^2 + 15x = 3x(2x + 5)$ HCF of 6 and 15 = 3;
HCF of $x^2$ and $x = x$

(b) $-a^3bc - a^2b^2 = -a^2b(ac + b)$ HCF of $a^3$ and $a^2 = a^2$;
HCF of $b$ and $b^2 = b$

**Problem-solving Tip**

(b) factor out −1

$-x - y = -(x + y)$

change sign

**Practise Now 15A**

Similar and Further Questions
Exercise 4C
Questions 2(a)–(d), 7(a)–(d)

Factorise each of the following expressions completely.

(a) $10x^2 + 8x$ (b) $10a^2 - 15a$
(c) $-49b - 28b^2$ (d) $2\pi r^2 + 2\pi rh$
(e) $x^2yz^3 - yz^2$ (f) $c^2d^3 + c^3d^2 - c^2d^2$

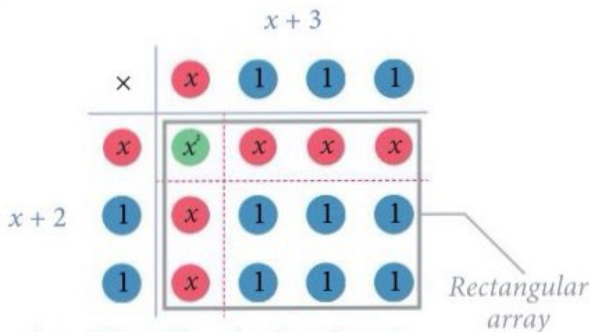
### Thinking Time

We have learnt how to factorise quadratic expressions of the form $ax^2 + bx$ in Worked Example 15(**a**).

1. Do you think we can factorise quadratic expressions of the form $ax^2 + c$? If yes, give an example of such an expression and factorise it.
2. By using the above method of extracting common factors, can we factorise quadratic expressions of the form $ax^2 + bx + c$, where $a$, $b$ and $c$ have no common factors other than 1 or −1? If not, how can we factorise such an expression?

## B. Factorisation of quadratic expressions of the form $x^2 + bx + c$, where $c > 0$ (and $b > 0$)

In Section 4.3B, we have learnt how to use a multiplication frame to help us expand quadratic expressions such as $(x + 2)(x + 3)$ to give $x^2 + 5x + 6$, which is shown again in Fig. 4.7.
We have also learnt that the discs representing $x^2 + 3x + 2x + 6$ form a **rectangular array**, which is divided into *four distinct regions*:




$$\begin{aligned}(x + 2)(x + 3) &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6\end{aligned}$$

**(a)**

| × | $x$ | $+3$ |
|---|---|---|
| $x$ | $x^2$ | $+3x$ |
| $+2$ | $+2x$ | $+6$ |

$$\begin{aligned}(x + 2)(x + 3) &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6\end{aligned}$$

**(b)**

Fig. 4.7

How can we do the reverse and factorise $x^2 + 5x + 6$ to obtain $(x + 2)(x + 3)$?
Unlike Section 4.4A, we cannot only extract the common factor $x$ from the first two terms as such:

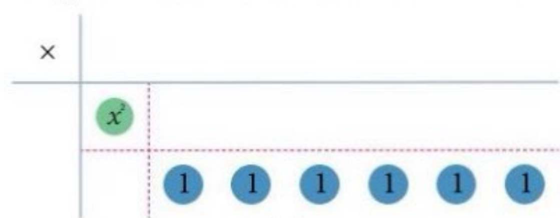
$$x^2 + 5x + 6 = x(x + 5) + 6.$$

To factorise an expression, we need to express it as a product of two or more factors. However, $x(x + 5) + 6$ is not the product of two factors. Therefore, we need to learn another method to express $x^2 + 5x + 6$ as $(x + 2)(x + 3)$.

## Using algebra discs and multiplication frame

To factorise $x^2 + 5x + 6$, we need to arrange the algebra discs in the multiplication frame.
First, notice that there is only one region to put the $x^2$ disc: the top left region of the rectangular array. The $x^2$ disc is the result of the product of $x$ in the factor $(x + 2)$ and $x$ in the factor $(x + 3)$, i.e. $x \times x = x^2$.
Second, recall that the 6 $1$ discs in the bottom right region in Fig. 4.7**(a)** is the result of the product of 2 in the factor $(x + 2)$ and 3 in the factor $(x + 3)$, i.e. $6 = 2 \times 3$.
But 6 is also equal to $1 \times 6$.
Thus, we have to consider all the **possible pairs of factors** whose product gives the **constant term** 6. Then we use *guess and check* as follows.

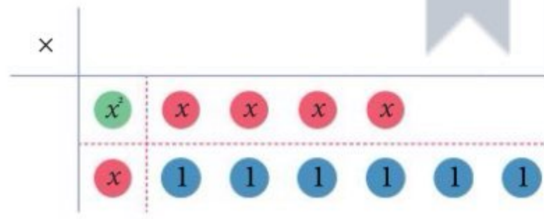
Suppose we start with $6 = 1 \times 6$. Then we arrange the 6 $1$ discs into 1 row in the *bottom right region* of the rectangular array to give $1 \times 6$ as shown in Fig. 4.8**(a)**.




**(a)**

Fig. 4.8

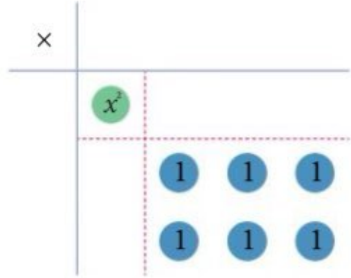
Next, we consider the **term in $x$**, which is $5x$. So we only have 5 $x$ discs to put in the *top right* and *bottom left* regions as shown in Fig. 4.8(**b**). But we are *short* of 2 $x$ discs to complete the rectangular array. Therefore, this arrangement of the 1 discs is wrong.



(**b**)

Fig. 4.8

Let us try the other option: $6 = 2 \times 3$. This means we have to arrange the 6 1 discs into 2 rows of 3 1 discs in the *bottom right region* of the rectangular array to give $2 \times 3$ as shown in Fig. 4.9(**a**).

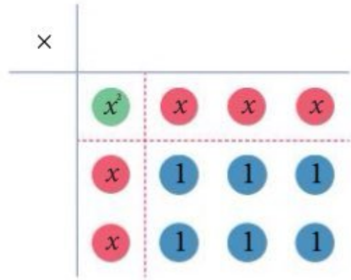


(**a**)

**Reflection**

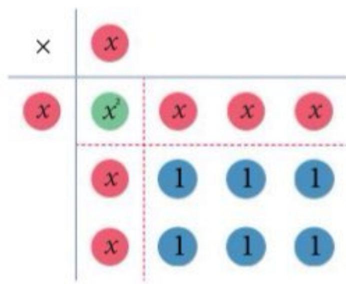
Does it matter if we arrange the 6 1 discs in Fig. 4.9(**a**) into 3 rows of 2 1 discs instead? Explain.

Next, we put the 5 $x$ discs in the *top right* and *bottom left* regions as shown in Fig. 4.9(**b**). Here, all the discs for $x^2 + 5x + 6$ are successfully arranged in a rectangular array. Notice that the term **$5x$ is split into $2x$ and $3x$** to complete the rectangular array, and the *coefficients* of $2x$ and $3x$ are the *pair of factors* of $6 = 2 \times 3$.



(**b**)

Thereafter, we need to find the 2 factors by filling in the leftmost column and the topmost row with the corresponding discs. We start by putting the 2 $x$ discs as shown in Fig. 4.9(**c**), because $x \times x = x^2$.

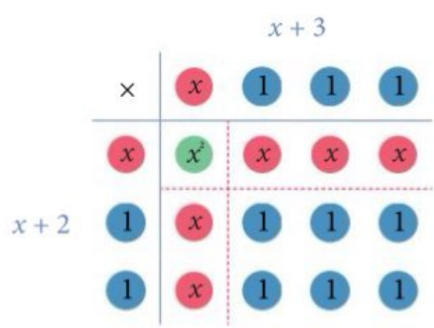


(**c**)

**Problem-solving Tip**

To fill the leftmost column and the topmost row in Fig. 4.9(**c**) and (**d**) with the correct discs, remember that the $x$ discs come first, followed by the 1 discs.

Then we put the 1 discs as shown in Fig. 4.9(**d**) because $1 \times x = x$, $x \times 1 = x$ and $1 \times 1 = 1$. We count the discs and write the first factor $x + 2$ on the left and the second factor $x + 3$ on top of the final arrangement as shown in Fig. 4.9(**d**).
Lastly, we write the final answer $x^2 + 5x + 6 = (x + 2)(x + 3)$.




$x^2 + 5x + 6 = (x + 2)(x + 3)$

(**d**)

Fig. 4.9

## Using multiplication frame only

The use of the algebra discs helps us learn how to fill in the multiplication frame. Let us factorise $x^2 + 5x + 6$ without using algebra discs.

First, we write the term in $x^2$ and the constant term 6 in the multiplication frame as shown in Fig. 4.10(**a**).
We know that the other two regions in the rectangular array must each contain a term in $x$ such that their sum is $5x$ (the $x$-term in the expression $x^2 + 5x + 6$).

| $\times$ | | |
|---|---|---|
| | $x^2$ | |
| | | $+6$ |

**(a)**

From the use of the algebra discs previously, we have to consider all the **possible pairs of factors** of the **constant term** 6 whose product is 6, i.e.

$$6 = 1 \times 6$$
$$= 2 \times 3.$$

**Method 1:** Use *guess and check* for each pair of factors of 6 using a multiplication frame to see which case will give the correct answer.

**Method 2:** Use some *reasoning* and choose the pair of factors of 6 with a sum of 5 (which is the coefficient of $5x$). Since $2x + 3x = 5x$ and $x + 6x \neq 5x$, we write $2x$ and $3x$ in the remaining two regions as shown in Fig. 4.10(**b**).

| $\times$ | | |
|---|---|---|
| | $x^2$ | $+3x$ |
| | $+2x$ | $+6$ |

**(b)**

Thereafter, we find the two factors by filling in the leftmost column and the topmost row with the corresponding term in $x$ and the constant term. We start by filling in the terms in $x$ as shown in Fig. 4.10(**c**) because $x \times x = x^2$.

| $\times$ | $x$ | |
|---|---|---|
| $x$ | $x^2$ | $+3x$ |
| | $+2x$ | $+6$ |

**(c)**

Then we fill in the constant terms as shown in Fig. 4.10(**d**) because $2 \times x = 2x$ and $x \times 3 = 3x$. Check that $2 \times 3 = 6$.
Lastly, we write the final answer $x^2 + 5x + 6 = (x + 2)(x + 3)$ at the bottom of the multiplication frame.

| $\times$ | $x$ | $+3$ |
|---|---|---|
| $x$ | $x^2$ | $+3x$ |
| $+2$ | $+2x$ | $+6$ |

$\therefore x^2 + 5x + 6 = (x + 2)(x + 3)$

**(d)**

Fig. 4.10

### Investigation

### Factorisation of quadratic expressions of the form $x^2 + bx + c$, where $c > 0$ (and $b > 0$)

Factorise each of the following quadratic expressions

**(i)** using algebra discs and a multiplication frame;

**(ii)** using only a multiplication frame.

**(a)** $x^2 + 7x + 6$ **(b)** $x^2 + 6x + 8$ **(c)** $x^2 + 9x + 8$

**(d)** $x^2 + 8x + 12$ **(e)** $x^2 + 7x + 12$ **(f)** $x^2 + 13x + 12$

## C. Factorisation of quadratic expressions of the form $x^2 + bx + c$, where $c > 0$ (and $b < 0$)

In Section 4.4B, we have learnt how to factorise quadratic expressions of the form $x^2 + bx + c$, where $c > 0$ (and $b > 0$).
What about expressions where $c > 0$ but $b < 0$? Will the technique of factorisation be similar?

Let us use the multiplication frame to factorise $x^2 - 5x + 6$. Because $b = -5$ is negative, we need to consider ***both positive and negative factors*** of the **constant term** 6, i.e.

$$\begin{aligned} 6 &= 1 \times 6 = (-1) \times (-6) \\ &= 2 \times 3 = (-2) \times (-3). \end{aligned}$$

**Method 1:** Use *guess and check* for each pair of factors of 6 using a multiplication frame to see which case will give the correct answer.

**Method 2:** Use some *reasoning* and reject the 2 pairs of positive factors of 6 since their sum must be –5 (which is the coefficient of $-5x$). Then check whether $(-x) + (-6x)$ or $(-2x) + (-3x)$ gives $-5x$.
Since the answer is the latter, we will fill in the multiplication frame accordingly as shown in Fig. 4.11.

| × | | |
|---|---|---|
| | $x^2$ | |
| | | $+6$ |

**(a)**

| × | | |
|---|---|---|
| | $x^2$ | $-3x$ |
| | $-2x$ | $+6$ |

**(b)**

| × | $x$ | $-3$ |
|---|---|---|
| $x$ | $x^2$ | $-3x$ |
| $-2$ | $-2x$ | $+6$ |

$x^2 - 5x + 6 = (x - 2)(x - 3)$

**(c)**

Fig. 4.11

### Investigation — Factorisation of quadratic expressions of the form $x^2 + bx + c$, where $c > 0$ (and $b < 0$)

Find the two factors of each of the following quadratic expressions using a multiplication frame without algebra discs.

**(a)** $x^2 - 7x + 6$ **(b)** $x^2 - 6x + 8$ **(c)** $x^2 - 9x + 8$
**(d)** $x^2 - 8x + 12$ **(e)** $x^2 - 7x + 12$ **(f)** $x^2 - 13x + 12$

## D. Factorisation of quadratic expressions of the form $x^2 + bx + c$, where $c < 0$

In Sections 4.4B and 4.4C, we have learnt how to factorise quadratic expressions of the form $x^2 + bx + c$, where $c > 0$ and $b$ is either positive or negative.

What happens if $c < 0$ and $b$ is either positive or negative? Will the technique of factorisation be different?

Let us use the multiplication frame to factorise $x^2 + 5x - 6$ and $x^2 - 5x - 6$.

Since the **constant term** $-6$ is negative, then $-6 = 1 \times (-6) = (-1) \times 6$
$= 2 \times (-3) = (-2) \times 3$.

Consider $x^2 + 5x - 6$ first.

We can use *guess and check*, or we can use some *reasoning* to check which pair of factors of $-6$ will add up to 5 (which is the coefficient of $5x$).

Since $(-x) + 6x = 5x$, we will fill in the multiplication frame accordingly as shown in Fig. 4.12.

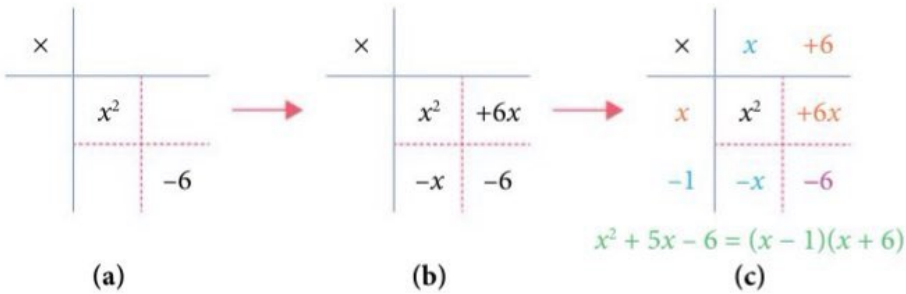



Fig. 4.12

Fig. 4.13 shows the factorisation of $x^2 - 5x - 6$. Since $(-6x) + x = -5x$, we fill in the multiplication frame as shown.

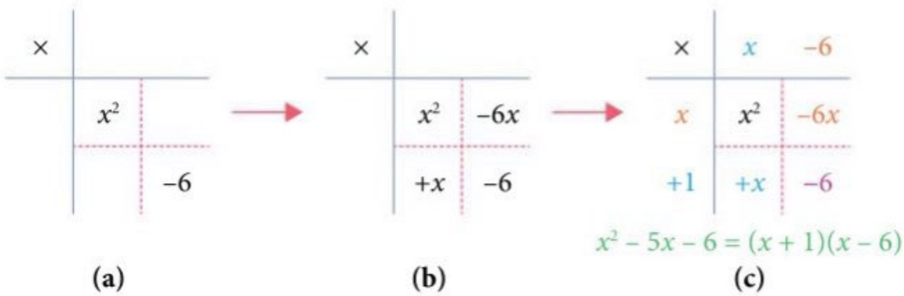



Fig. 4.13

### Investigation — Factorisation of quadratic expressions of the form $x^2 + bx + c$, where $c < 0$

Factorise each of the following quadratic expressions using a multiplication frame without algebra discs.

**(a)** $x^2 + x - 6$ **(b)** $x^2 - x - 6$ **(c)** $x^2 + 2x - 8$

**(d)** $x^2 - 2x - 8$ **(e)** $x^2 - 7x - 8$ **(f)** $x^2 + 7x - 8$

## Class Discussion — Factorisation of quadratic expressions

Fig. 4.14 shows the factorisation of the four quadratic expressions which we have done in Sections 4.4B–D.

| $\times$ | $x$ | $+3$ |
|---|---|---|
| $x$ | $x^2$ | $+3x$ |
| $+2$ | $+2x$ | $+6$ |

$x^2 + 5x + 6 = (x + 2)(x + 3)$

**(a)**

| $\times$ | $x$ | $-3$ |
|---|---|---|
| $x$ | $x^2$ | $-3x$ |
| $-2$ | $-2x$ | $+6$ |

$x^2 - 5x + 6 = (x - 2)(x - 3)$

**(b)**

| $\times$ | $x$ | $+6$ |
|---|---|---|
| $x$ | $x^2$ | $+6x$ |
| $-1$ | $-x$ | $-6$ |

$x^2 + 5x - 6 = (x - 1)(x + 6)$

**(c)**

| $\times$ | $x$ | $-6$ |
|---|---|---|
| $x$ | $x^2$ | $-6x$ |
| $+1$ | $+x$ | $-6$ |

$x^2 - 5x - 6 = (x + 1)(x - 6)$

**(d)**

Fig. 4.14

Notice that Fig. 4.14**(a)** and **(b)** have similar structures:

- When the constant term is positive (in this case, +6), the two corresponding factors of 6 must either be both positive, i.e. $6 = 1 \times 6$ or $2 \times 3$, or both negative, i.e. $6 = (-1) \times (-6)$ or $(-2) \times (-3)$.
- From Fig. 4.14**(a)**, if the coefficient of $x$ is also positive (in this case, +5), then the two corresponding factors of 6 must also be positive and have a sum of 5. In this case, $2 + 3 = 5$, so $x^2 + 5x + 6 = (x + 2)(x + 3)$.
- From Fig. 4.14**(b)**, if the coefficient of $x$ is negative (in this case, −5), then the two factors of 6 must also be negative and have a sum of −5. In this case, $(-2) + (-3) = -5$, so $x^2 - 5x + 6 = (x - 2)(x - 3)$.

**1.** Fig. 4.14**(c)** and **(d)** also have similar structures. Describe the structures.

**2.** How does knowing the similarities of the above structures help us factorise $x^2 + bx + c$, where $b = +5$ or $-5$ and $c = +6$ or $-6$?

**Practise Now 15B**

Similar and Further Questions
Exercise 4C
Questions 3(a)–(h)

Factorise each of the following quadratic expressions.

**(a)** $x^2 + 6x + 5$
**(b)** $x^2 + 2x - 15$
**(c)** $x^2 - x - 12$
**(d)** $x^2 + 5x - 14$
**(e)** $y^2 - 8y + 12$
**(f)** $y^2 - 6y + 5$
**(g)** $z^2 + 8z + 12$
**(h)** $z^2 - 7z - 8$

## Introductory Problem Revisited

Now that you have learnt how to factorise quadratic expressions of the form $x^2 + bx + c$, solve the Introductory Problem and discuss your solution with your classmates.

## E. Factorisation of quadratic expressions of the form $ax^2 + bx + c$, where $a \neq 1$

If the coefficient of $x^2$ is not equal to 1, i.e. $a \neq 1$, we can still use guess and check to factorise $ax^2 + bx + c$. We can also use reasoning to find the factors of $ax^2 + bx + c$, but we have to first observe a pattern. Let us use the following example to illustrate both methods.

Factorise $2x^2 + 7x - 15$.

First, notice that $2x^2 = 2x \times x$.

**Method 1: Guess and Check**

$$\begin{aligned} -15 &= 1 \times (-15) \quad \text{or} \quad (-1) \times 15 \\ &= 3 \times (-5) \quad \text{or} \quad (-3) \times 5 \end{aligned}$$

Which pair of factors of −15 should we put in the two ? positions?

| × | $x$ | ? |
|---|---|---|
| $2x$ | $2x^2$ | |
| ? | | $-15$ |

guess

Let us try −3 and 5 in these positions.

| × | $x$ | $-3$ |
|---|---|---|
| $2x$ | $2x^2$ | |
| $+5$ | | $-15$ |

check

$5x - 6x = -x \neq 7x$, so this cannot work.

| × | $x$ | $-3$ |
|---|---|---|
| $2x$ | $2x^2$ | $-6x$ |
| $+5$ | $+5x$ | $-15$ |

guess

We can try −3 and 5 again but in different positions.

| × | $x$ | $+5$ |
|---|---|---|
| $2x$ | $2x^2$ | |
| $-3$ | | $-15$ |

check

Complete the frame. Since $10x - 3x = 7x$, this is correct.

| × | $x$ | $+5$ |
|---|---|---|
| $2x$ | $2x^2$ | $+10x$ |
| $-3$ | $-3x$ | $-15$ |

$$\therefore\ 2x^2 + 7x - 15 = (2x - 3)(x + 5)$$

Fig. 4.15

In Fig. 4.15, we got the answer after two trials. Otherwise, we would have to repeat the process for the other six cases.

What if we have to factorise an expression such as $-6x^2 + 23x - 15$?
The coefficient of $x^2$, −6, has four pairs of factors, so there are four cases for $-6x^2$ to try (see Fig. 4.16). This is on top of the four cases for −15, giving us a total of $4 \times 4 = 16$ cases to check. This is both tedious and time consuming.

| × | $-x$ | |
|---|---|---|
| $6x$ | $-6x^2$ | |
| | | $-15$ |

or

| × | $x$ | |
|---|---|---|
| $-6x$ | $-6x^2$ | |
| | | $-15$ |

or

| × | $-2x$ | |
|---|---|---|
| $3x$ | $-6x^2$ | |
| | | $-15$ |

or

| × | $2x$ | |
|---|---|---|
| $-3x$ | $-6x^2$ | |
| | | $-15$ |

Fig. 4.16

Therefore, we can consider the next method, which is more efficient.

**Method 2: Use some reasoning**

Before we begin, we need to observe a pattern.

Consider the example: $2x^2 + 7x - 15 = (2x - 3)(x + 5)$. Notice that the products of the two terms in each diagonal (circled in Fig. 4.17) are equal. This can help us factorise $2x^2 + 7x - 15$.

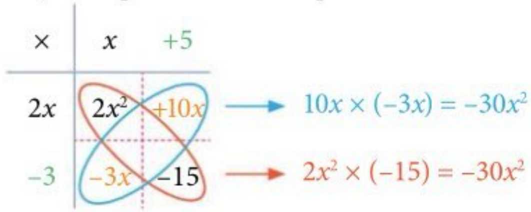
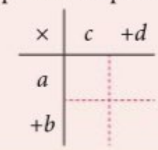



Fig. 4.17

> **Attention**
>
> Is the pattern in Fig. 4.17 always true for all quadratic expressions? Is there a reason for the pattern? Explain.
>
> **Hint:**
>
> 
> 

What numbers should we put in the two ? positions in Fig. 4.18(**a**)? The two numbers have a product of $-30$ ($2x^2 \times (-15) = -30x^2$) and a sum of 7 (coefficient of $x$ in $2x^2 + 7x - 15$ is 7).

$$\begin{aligned} -30 &= 1 \times (-30) \text{ or } (-1) \times 30 \\ &= 2 \times (-15) \text{ or } (-2) \times 15 \\ &= 3 \times (-10) \text{ or } (-3) \times 10 \\ &= 5 \times (-6) \text{ or } (-5) \times 6 \end{aligned}$$

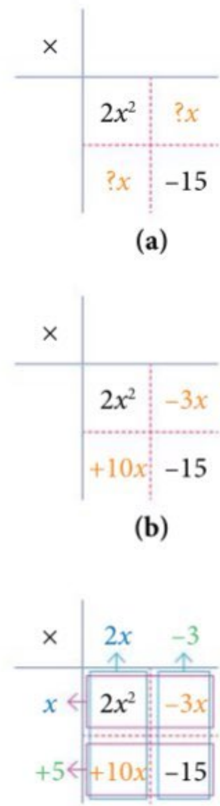
Notice that only $(-3) + 10 = 7$. So we put the numbers as shown in Fig. 4.18(**b**). Does it matter that the positions of $10x$ and $-3x$ are different from those in Fig. 4.17?

To complete the multiplication frame, we find the HCF of the terms in each row and column as shown in Fig. 4.18(**c**).

- For the first row containing $2x^2$ and $-3x$, the HCF is $x$, so we put $x$.
- For the second row containing $10x$ and $-15$, the HCF is 5, so we put 5.
- For the first column containing $2x^2$ and $10x$, the HCF is $2x$, so we put $2x$.
- For the second column containing $-3x$ and $-15$, the HCF is $-3$, so we put $-3$.

Fig. 4.18(**c**) looks different from Fig. 4.17 as the factors have switched places, but they are the same:

$$2x^2 + 7x - 15 = (2x - 3)(x + 5).$$




$\therefore 2x^2 + 7x - 15$
$= (2x - 3)(x + 5)$

(**c**)

Fig. 4.18

Now, how do we factorise an expression with a negative coefficient of $x^2$, e.g. $-2x^2 - 7x + 15$?

One method is to extract the negative sign as follows:

$$-2x^2 - 7x + 15 = -(2x^2 + 7x - 15).$$

We can factorise $2x^2 + 7x - 15$ using the above method to obtain $(2x - 3)(x + 5)$.

Then, $-2x^2 - 7x + 15 = -(2x^2 + 7x - 15)$
$= -(2x - 3)(x + 5)$.

We can also express the answer as $(3 - 2x)(x + 5)$ since $-(2x - 3) = -2x + 3 = 3 - 2x$.

**Practise Now 15C**

Similar and Further Questions
**Exercise 4C**
Questions 4(a)–(h), 8(a)–(h), 13(a), (b)

Factorise each of the following quadratic expressions completely.

**(a)** $2x^2 + 7x + 6$ **(b)** $3x^2 + 10x - 8$
**(c)** $6y^2 - 11y + 4$ **(d)** $7 - 13x - 2x^2$
**(e)** $-x^2 + 6x - 9$ **(f)** $-6x^2 + 23x - 15$
**(g)** $4x^2 - 6x - 4$ **(h)** $-5a^2 - 17a - 6$

**Problem-solving Tip**

For **(g)** and **(h)**, there is a common factor for all the 3 terms, so what should you do first?

**Worked Example 16**

### Checking whether two expressions are equivalent

Without expanding $(2x - 1)(x - 6)$ completely, and without factorising $x^2 + x - 6$, explain why the two expressions are not equivalent.

**Solution**

**Method 1:**

The coefficient of $x^2$ in $(2x - 1)(x - 6)$ is 2.
But the coefficient of $x^2$ in $x^2 + x - 6$ is 1.
$\therefore$ the two expressions are not equivalent.

**Method 2:**

The constant term in $(2x - 1)(x - 6)$ is 6.
But constant term in $x^2 + x - 6$ is $-6$.
$\therefore$ the two expressions are not equivalent.

**Big Idea**

**Equivalence**
In Book 1, we learnt that two expressions are **equivalent** if the value of both expressions is the same for *any* value we substitute into the variable of the expressions, e.g. $(x - 1)(x - 2)$ and its expanded form $x^2 - 3x + 2$ are equivalent.

**Practise Now 16**

Similar and Further Questions
**Exercise 4C**
Questions 9, 14

1. Without expanding $(2x + 3)(x - 5)$ completely, and without factorising $3x^2 - 18x + 15$, explain why the two expressions are not equivalent.
2. Bernard says that the two expressions $(y - 2)(2y + 1)$ and $4y^2 + 7y - 2$ are equivalent because the constant term of $(y - 2)(2y + 1)$ is also $-2$. Without expanding or factorising the expressions, explain if you agree with him.

**Worked Example 17**

### Factorising quadratic expressions in two variables using multiplication frame

Factorise $x^2 + 5xy - 6y^2$.

**Solution**

Let us compare the factorisation of $x^2 + 5x - 6$ and the factorisation of $x^2 + 5xy - 6y^2$.

**Problem-solving Tip**

If you cannot factorise $x^2 + 5xy - 6y^2$ directly, you can factorise $x^2 + 5x - 6$ first and then add in the appropriate $y$ and $y^2$ later.

**Factorisation of $x^2 + 5x - 6$:**

since $6x - x = 5x$

| × | | |
|---|---|---|
| | $x^2$ | |
| | | $-6$ |

→

| × | | |
|---|---|---|
| | $x^2$ | $+6x$ |
| | $-x$ | $-6$ |

→

| × | $x$ | $+6$ |
|---|---|---|
| $x$ | $x^2$ | $+6x$ |
| $-1$ | $-x$ | $-6$ |

$x^2 + 5x - 6 = (x - 1)(x + 6)$

**Factorisation of $x^2 + 5xy - 6y^2$:**

since $6xy - xy = 5xy$

| $\times$ | | |
|---|---|---|
| | $x^2$ | |
| | | $-6y^2$ |

$\rightarrow$

| $\times$ | | |
|---|---|---|
| | $x^2$ | $+6xy$ |
| | $-xy$ | $-6y^2$ |

$\rightarrow$

| $\times$ | $x$ | $+6y$ |
|---|---|---|
| $x$ | $x^2$ | $+6xy$ |
| $-y$ | $-xy$ | $-6y^2$ |

$x^2 + 5xy - 6y^2 = (x - y)(x + 6y)$

**Practise Now 17**

Similar and Further Questions
Exercise 4C
Questions 5(a)–(d), 10(a)–(e), 15

Factorise each of the following expressions completely.

**(a)** $x^2 + 2xy - 8y^2$
**(b)** $x^2 - 2xy - 15y^2$
**(c)** $6x^2 + 11xy + 5y^2$
**(d)** $6x^2 - 21xy + 18y^2$
**(e)** $-a^2 + 5ab - 6b^2$
**(f)** $-2c^2 + 12cd - 18d^2$
**(g)** $6pq^2 - 57pqr + 105pr^2$
**(h)** $3x^2y^2 - 2xy - 16$

## F. Factorisation of expressions of the form $ax^3 + bx^2 + cx$

Consider $2x^3 + 3x^2 - 2x$. While it is not a quadratic expression, it contains a common factor, $x$, and we can rewrite it as $2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2)$. Here, we see that $2x^2 + 3x - 2$ is a quadratic expression, which may be factorisable based on what we have learnt in Sections 4.4D and 4.4E. We will look at an example in Worked Example 18.

**Worked Example 18**

**Factorising expressions of the form $ax^3 + bx^2 + cx$**

Factorise $2x^3 + 3x^2 - 2x$.

**Solution**

$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2)$    extract common factor $x$

$= x(2x - 1)(x + 2)$

| $\times$ | $x$ | $+2$ |
|---|---|---|
| $2x$ | $2x^2$ | $+4x$ |
| $-1$ | $-x$ | $-2$ |

$2x^2 + 3x - 2 = (2x - 1)(x + 2)$

**Practise Now 18**

Similar and Further Questions
Exercise 4C
Questions 6(a)–(d), 11(a), (b)

Factorise each of the following expressions completely.

**(a)** $a^3 + 5a^2 + 6a$
**(b)** $b^3 + 4b^2 - 5b$
**(c)** $2c^3 + 7c^2 - 4c$
**(d)** $2d^3 - 9d^2 - 18d$
**(e)** $3e^3 - 5e^2 + 2e$
**(f)** $4f^3 - 17f^2 + 4f$
**(g)** $4g^3 - 4g^2 - 3g$
**(h)** $6h^3 - 5h^2 + h$

## Reflection

1. What do I already know about the expansion of quadratic expressions using a multiplication frame that could help me factorise quadratic expressions?
2. What do I already know about the factorisation of quadratic expressions in one variable that could help me factorise quadratic expressions in two variables?
3. What have I learnt in this section that I am still unclear of?

Basic | Intermediate | Advanced

## Exercise 4C

**1.** Factorise each of the following expressions fully.
(a) $8x + 64$ (b) $-12p - 27q$
(c) $16aw + 20av$ (d) $-36bc + 4bd$
(e) $14xy - 7x + 21xz$ (f) $-8tu - 4u - 11su$

**2.** Factorise each of the following expressions fully.
(a) $4x^2 + 16x$ (b) $18y^2 - 6y$
(c) $39xy - 15x^2z$ (d) $-8\pi xy^3 - 10\pi y^3$

**3.** Factorise each of the following expressions fully.
(a) $a^2 + 9a + 8$ (b) $b^2 + 8b + 15$
(c) $c^2 - 9c + 20$ (d) $d^2 - 16d + 28$
(e) $f^2 + 6f - 16$ (f) $h^2 + 2h - 120$
(g) $k^2 - 4k - 12$ (h) $m^2 - 20m - 21$

**4.** Factorise each of the following expressions completely.
(a) $3n^2 + 10n + 7$ (b) $4p^2 + 8p + 3$
(c) $6q^2 - 17q + 12$ (d) $4r^2 - 7r + 3$
(e) $8s^2 + 2s - 15$ (f) $6t^2 + 19t - 20$
(g) $4u^2 - 8u - 21$ (h) $18w^2 - w - 39$

**5.** Factorise each of the following expressions.
(a) $a^2 + 3ab - 4b^2$ (b) $c^2 - 4cd - 21d^2$
(c) $2h^2 + 7hk - 15k^2$ (d) $3m^2 - 16mn - 12n^2$

**6.** Factorise each of the following expressions fully.
(a) $a^3 + 5a^2 + 4a$ (b) $3b^3 - 8b^2 - 3b$
(c) $6c^3 - 11c^2 + 5c$ (d) $6d^3 - 13d^2 + 6d$

**7.** Factorise each of the following expressions completely.
(a) $-xy^2z^2 - x^2y^3$ (b) $12a^2b^3 + 6a^3b^2 - 2a^2b^2$
(c) $10\pi p^2r - 20\pi p^2q - 14\pi pqr^3$
(d) $3v^3w^2 - 18tv^2w^3 + \frac{1}{3}tv^2w$

**8.** Factorise each of the following expressions fully.
(a) $-a^2 + 2a + 35$ (b) $-3b^2 + 76b - 25$
(c) $4c^2 + 10c + 4$ (d) $5d^2 - 145d + 600$
(e) $8f^2 + 4f - 60$ (f) $24h^2 - 15h - 9$
(g) $30 + 14k - 4k^2$ (h) $35m^2 + 5m - 30$

**9.** The area of a rectangle with a length of $(2x - 3)$ cm is $(4x^2 + 8x + 15)$ cm$^2$. Without factorising the expression, explain if the breadth of the rectangle can be $(x + 5)$ cm.

**10.** Factorise each of the following expressions completely.
(a) $3p^2 + 15pq + 18q^2$ (b) $2r^2t - 9rst + 10s^2t$
(c) $x^2y^2 + 2xy - 15$ (d) $12x^2y^2 - 17xy - 40$
(e) $4x^2y^2z - 22xyz + 24z$

**11.** Factorise each of the following completely.
(a) $2x^3 + 6x^2 + 4x$ (b) $-3p^3 - 3p^2 + 18p$

**12.** Factorise each of the following expressions fully.
(a) $(x + y)(a + b) - (y + z)(a + b)$
(b) $(c + 2d)(c + 2d) - (c + 2d)(3c - 7d)$.

## Exercise 4C

13. Factorise each of the following expressions completely.
    (a) $\frac{4}{9}p^2 + p - 1$
    (b) $0.6r - 0.8qr - 12.8q^2r$

14. Each side of a cube is $(2x + 11)$ cm long. Without finding an expression for the total surface area of the cube, explain if the total surface area of the cube is $(4x^2 + 22x + 100)$ cm$^2$.

15. A car travels a distance of $\left(7x^2 - \frac{99}{2}x - 85\right)$ km, where $x$ is a positive integer.
    (i) Factorise the expression $7x^2 - \frac{99}{2}x - 85$ completely.
    OPEN (ii) Hence, by substituting an appropriate value of $x$ into the factorised expression, suggest a possible average speed of the car and the corresponding time taken.

# 4.5 Factorisation of algebraic expressions into the form $(a + b)(c + d)$

## A. Factorisation into the form $(a + b)(c + d)$

In Section 4.4, we have learnt how to factorise a quadratic expression into the form $(px + q)(rx + s)$.
Can we factorise other algebraic expressions into the form $(a + b)(c + d)$?
In Section 4.2B, we have learnt how to expand $(a + b)(c + d)$ to obtain $ac + ad + bc + bd$.
In this section, we will learn two methods to factorise algebraic expressions of the form $ac + ad + bc + bd$ into the form $(a + b)(c + d)$.

**Method 1: Factorisation using multiplication frame**

| | | extract common factor $c$ | extract common factor $d$ |
|---|---|---|---|
| | $\times$ | $c$ | $+d$ |
| extract common factor $a$ | $a$ | $ac$ | $+ad$ |
| extract common factor $b$ | $+b$ | $+bc$ | $+bd$ |

$\therefore ac + ad + bc + bd = (a + b)(c + d)$

## Class Discussion — Arrangement of terms for factorisation using multiplication frame

Does Method 1 work regardless of how we arrange the four terms in the rectangular array of the multiplication frame? The following shows some other arrangements of the four terms. Discuss with your classmates which arrangements work and which do not.

**(i)**

| $\times$ | | |
|---|---|---|
| | $ac$ | $+bc$ |
| | $+ad$ | $+bd$ |

**(ii)**

| $\times$ | | |
|---|---|---|
| | $ac$ | $+bd$ |
| | $+ad$ | $+bc$ |

**(iii)**

| $\times$ | | |
|---|---|---|
| | $ac$ | $+bd$ |
| | $+bc$ | $+ad$ |

From the above Class Discussion, we observe that for Method 1 to work, we have to be careful of how we arrange the four terms in the rectangular array of the multiplication frame.

**Method 2: Factorisation by grouping**

In the expansion of $(a + b)(c + d)$, the intermediate step is $a(c + d) + b(c + d)$:

$$\begin{aligned}(a + b)(c + d) &= a(c + d) + b(c + d)\\ &= ac + ad + bc + bd\end{aligned}$$

To factorise an algebraic expression of the form $ac + ad + bc + bd$, we should group the four terms into two appropriate groups, where the two terms in each group have a common factor. Then, we can extract the common factor of each group:

> **Attention**
> It does not help if we group $ac + ad + bc + bd$ as $(ac + bd) + (bc + ad)$. Why?

$$\begin{aligned}ac + ad + bc + bd &= (ac + ad) + (bc + bd) && \text{group the 4 terms into 2 groups}\\ &= a(c + d) + b(c + d) && \text{extract common factor from each group}\\ &= (a + b)(c + d) && \text{extract common factor } (c + d)\end{aligned}$$

This method is called **factorisation by grouping** because we *group* the four terms into two appropriate groups first.

## Class Discussion — Arrangement of terms for factorisation by grouping

Does Method 2 work regardless of how we arrange the four terms?
The following shows some other arrangements of the four terms. Discuss with your classmates which arrangements work and which do not.

**(i)** $ac + bc + ad + bd$
**(ii)** $ac + bd + ad + bc$
**(iii)** $ac + bd + bc + ad$

From the above Class Discussion, we observe that like Method 1, we also have to be careful of how we arrange the four terms for Method 2 to work.

Worked Example 19

## Factorising algebraic expressions into the form $(a + b)(c + d)$

Factorise each of the following expressions completely.

**(a)** $ab + ad + 2bc + 2cd$  **(b)** $12ax - 6by - 8bx + 9ay$

### Solution

**(a) Method 1:**

| $\times$ | $b$ | $+d$ |
|---|---|---|
| $a$ | $ab$ | $+ad$ |
| $+2c$ | $+2bc$ | $+2cd$ |

$\therefore ab + ad + 2bc + 2cd$
$= (a + 2c)(b + d)$

**Method 2:**

$ab + ad + 2bc + 2cd$
$= (ab + ad) + (2bc + 2cd)$  group the 4 terms into 2 groups
$= a(b + d) + 2c(b + d)$  extract common factors from each group
$= (a + 2c)(b + d)$  extract common factor $(b + d)$

**Reflection**

**(a)** Is there another way to arrange the four terms? E.g. in **Method 2**, can we arrange the terms as such: $ab + 2bc + ad + 2cd$?

**(b) Method 1(i):**

| $\times$ | $4x?$ | |
|---|---|---|
| $6$ | $12ax$ | $-6by$ |
| | $-8bx$ | $+9ay$ |

$6 \times 4x \neq 12ax$
This arrangement of the four terms does not help.

**Method 2(i):**

$12ax - 6by - 8bx + 9ay$
$= 6(2ax - by) - 8bx + 9ay$

This arrangement of the four terms does not help.

**Method 1(ii):**

| $\times$ | $3a$ | $-2b$ |
|---|---|---|
| $4x$ | $12ax$ | $-8bx$ |
| $+3y$ | $+9ay$ | $-6by$ |

If we only extract $2b$, $4x \times 2b \neq -8bx$.
Hence, it is a good practice to always check by expansion.

$\therefore 12ax - 6by - 8bx + 9ay$
$= (4x + 3y)(3a - 2b)$

**Method 2(ii):**

$12ax - 6by - 8bx + 9ay$
$= (12ax - 8bx) + (9ay - 6by)$  group the 4 terms into 2 groups
$= 4x(3a - 2b) + 3y(3a - 2b)$  extract common factors from each group
$= (3a - 2b)(4x + 3y)$  extract common factor $(3a - 2b)$

**Reflection**

**(b)** Which method do you prefer? Why?

**Practise Now 19**

Similar and Further Questions
Exercise 4D
Questions 1(a)–(f)

Factorise each of the following expressions completely.

**(a)** $ab + ac + 2bd + 2cd$  **(b)** $3pq + 7rs + 3pr + 7qs$
**(c)** $6ax - 20by - 8bx + 15ay$  **(d)** $3hp - 12kq + 18kp - 2hq$

## Worked Example 20

### Factorising algebraic expressions into the form $(a + b)(c + d)$

Find the two factors of $6xy - 9x + 15 - 10y$.

**Solution**

**Method 1(i):**

| $\times$ | $3?$ | |
|---|---|---|
| $3x$ | $6xy$ | $-9x$ |
| | $+15$ | $-10y$ |

$3x \times 3 \neq 6xy$

This arrangement of the four terms does not help.

**Method 2(i):**

$6xy - 9x + 15 - 10y$

$= (6xy - 9x) + (15 - 10y)$ group the 4 terms into 2 groups

$= 3x(2y - 3) + 5(3 - 2y)$ extract common factor(s) from each group

$= 3x(2y - 3) - 5(2y - 3)$ $k(y - x) = -k(x - y)$

$= (3x - 5)(2y - 3)$ extract common factor $(2y - 3)$

$\therefore$ the two factors are $(3x - 5)$ and $(2y - 3)$.

**Method 1(ii):**

| $\times$ | $3x$ | $-5$ |
|---|---|---|
| $2y$ | $6xy$ | $-10y$ |
| $-3$ | $-9x$ | $+15$ |

If we only extract 3 here, then $3 \times 3x \neq -9x$.

If we extract $-3x$ here, then $2y \times (-3x) \neq 6xy$.

Hence, it is a good practice to always check by expansion.

$\therefore$ $6xy - 9x + 15 - 10y$

$= (2y - 3)(3x - 5)$

$\therefore$ the two factors are $(2y - 3)$ and $(3x - 5)$.

**Method 2(ii):**

$6xy - 9x + 15 - 10y$

$= (6xy - 10y) + (-9x + 15)$ group the 4 terms into 2 groups

$= (6xy - 10y) - (9x - 15)$ $-x + y = -(x - y)$

$= 2y(3x - 5) - 3(3x - 5)$ extract common factor(s) from each group

$= (3x - 5)(2y - 3)$ extract common factor $(3x - 5)$

$\therefore$ the two factors are $(3x - 5)$ and $(2y - 3)$.

---

**Practise Now 20**

Similar and Further Questions
**Exercise 4D**
Questions 2(a)–(f), 3, 5(a), (b)

Find the two factors of each of the following expressions.

**(a)** $6xy - 15x + 20 - 8y$

**(b)** $6ab - 9ac + 21c - 14b$

## Worked Example 21

### Factorising quadratic expressions by grouping

Factorise $6x^2 - 9x + 15y - 10xy$ fully.

**Solution**

**Method 1:**

$6x^2 - 9x + 15y - 10xy = (6x^2 - 9x) + (15y - 10xy)$ group the 4 terms into 2 groups

$= 3x(2x - 3) + 5y(3 - 2x)$ extract common factors from each group

$= 3x(2x - 3) - 5y(2x - 3)$ $k(y - x) = -k(x - y)$

$= (2x - 3)(3x - 5y)$ extract common factor $(2x - 3)$

**Method 2:**

$$
\begin{aligned}
6x^2 - 9x + 15y - 10xy &= (6x^2 - 9x) + (-10xy + 15y) && \text{group the 4 terms into 2 groups} \\
&= (6x^2 - 9x) - (10xy - 15y) && -x + y = -(x - y) \\
&= 3x(2x - 3) - 5y(2x - 3) && \text{extract common factors from each group} \\
&= (2x - 3)(3x - 5y) && \text{extract common factor } (2x - 3)
\end{aligned}
$$

**Practise Now 21**

Similar and Further Questions
**Exercise 4D**
Questions 4(a)–(d), 6

Factorise each of the following expressions fully.

**(a)** $x^2 + xy - 3x - 3y$

**(b)** $15w^2 - 20w - 6wz + 8z$

## Thinking Time

In Section 4.4, we learnt how to factorise quadratic expressions using a multiplication frame.

In this section, we learnt a new method called factorisation by grouping to factorise algebraic expressions of the form $ab + ad + bc + bd$.

Can we also use factorisation by grouping to factorise quadratic expressions?

For example,
$$
\begin{aligned}
2x^2 + 7x - 15 &= 2x^2 + 10x - 3x - 15 \\
&= 2x(x + 5) - 3(x + 5) \\
&= (2x - 3)(x + 5)
\end{aligned}
$$

How do we know that we should rewrite $7x$ as $10x - 3x$, and not other possibilities, e.g. $2x + 5x$, which will not work?

One method is to use guess and check. Another method is to use the multiplication frame in Fig. 4.17, which shows why we use $10x - 3x$.

1. How is this method of grouping similar to the method of using a multiplication frame in Fig. 4.17?
2. Factorise the following expressions by grouping (without using a multiplication frame). Is it easy to do so?
   **(a)** $2x^2 + 7x + 6$ **(b)** $3x^2 + 10x - 8$

## Reflection

1. What have I learnt about using a multiplication frame to factorise quadratic expressions that could help me factorise algebraic expressions of the form $ac + ad + bc + bd$?
2. In this section, how many terms must there be in an algebraic expression before I can attempt to factorise the expression by grouping?
3. What have I learnt in this section or chapter that I am still unclear of?

Basic Intermediate Advanced

## Exercise 4D

1. Factorise each of the following expressions.
   (a) $xy + 4x + 3y + 12$
   (b) $ax - 5a + 4x - 20$
   (c) $12cy + 20c - 15 - 9y$
   (d) $3by + 4ax + 12ay + bx$
   (e) $6xy - 4x - 2z + 3yz$
   (f) $dy + fy - fz - dz$

2. Factorise each of the following expressions completely.
   (a) $2xy - 8x + 12 - 3y$
   (b) $6xy - 15y + 10 - 4x$
   (c) $10 - 14p + 7pq - 5q$
   (d) $kx + hy - hx - ky$
   (e) $2ab - 6ad - bc + 3cd$
   (f) $24mx + 8my - 6nx - 2ny$

3. Li Ting factorised the expression $2hk + 2hn + 3km - 3mn$ and her answer was $(2h - 3m)(k + n)$. Explain if her answer is correct.

4. Factorise each of the following expressions fully.
   (a) $x + xy + 2y + 2y^2$
   (b) $x^2 - 3x + 2xy - 6y$
   (c) $3x^2 + 6xy - 4xz - 8yz$
   (d) $x^2y^2 - 5x^2y - 5xy^2 + xy^3$

5. Factorise each of the following expressions fully.
   (a) $144p(y - 5x^2) - 12q(10x^2 - 2y)$
   (b) $2(5x + 10y)(2y - x)^2 - 4(6y + 3x)(x - 2y)$

6. The volume of a rectangular tank can be expressed as $(5xy - 25x^2 + 50x - 10y)$ m$^3$. Given that the height of the tank is 5 m,
   (i) express the base area of the tank in the form $(ay - bx)(ax - c)$, where $a$, $b$ and $c$ are integers to be determined.
   (ii) Hence, if $x = 6$ and $y = 40$, find the dimensions of the tank.

# 4.6 Expansion using special algebraic identities

In this section, we will learn how to use three special algebraic identities to expand certain algebraic expressions.

### Investigation — First special algebraic identity

1. We have learnt that $x^2$ means $x \times x$.
   What does $(a + b)^2$ mean?

2. Expand $(a + b)^2$ using the Distributive Law: $(a + b)^2 = (a + b)(a + b)$

   $=$

   $=$

3. Expand $(a + b)^2$ using a multiplication frame:

| $\times$ | $a$ | $+b$ |
| --- | --- | --- |
| $a$ | | |
| $+b$ | | |

$\therefore (a + b)^2 =$

4. (i) Do you get the same answer for Questions 2 and 3?
   (ii) Is $(a + b)^2 = a^2 + b^2$? Why or why not?
   Hint: Consider your answer to Questions 2 and 3.

5. Fig. 4.19 shows a square *PQRS* formed by two smaller squares and two rectangles, whose dimensions are given in the figure.

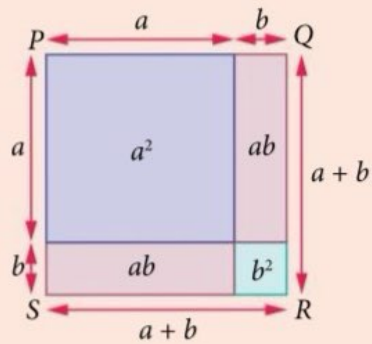



Fig. 4.19

   (i) What is the area of the square *PQRS* in terms of its length?
   (ii) What is the area of the square *PQRS* in terms of the total area of the two smaller squares and two rectangles?
   (iii) Are the two expressions in parts **(i)** and **(ii)** equal? Explain.
   (iv) From part **(iii)**, $(a + b)^2 =$
   (v) Is your answer in part **(iv)** the same as your answer in Questions 2 and 3?
   (vi) Using Fig. 4.19, explain why $(a + b)^2 \neq a^2 + b^2$.
   Hint: What is the area of the square *PQRS*?

6. In Book 1, we have learnt that numbers such as $7^2$ and $16^2$ are called perfect squares.
   In algebra, the expression $a^2$ is also called a **perfect square**.
   (i) By referring to Fig. 4.19, explain why $a^2$ is called a perfect square.
   (ii) Is $b^2$ a perfect square? Explain.
   (iii) Is $(a + b)^2$ a perfect square? Explain.

7. In Book 1, we have learnt that an **identity** is an equation that is true for all values of the variable, e.g. $3(x + 2) = 3x + 6$.
   Is $(a + b)^2 = a^2 + 2ab + b^2$ an identity? Explain.

From the Investigation on pages 134 and 135, we have discovered the first special algebraic identity:

**First special algebraic identity (or first perfect square identity)**

$$(a + b)^2 = a^2 + 2ab + b^2$$

**Big Idea**

**Equivalence**

In Book 1, we have learnt that two expressions are equivalent if the value of both expressions is the same for any value we substitute into the same variables in the expressions. In an *identity*, the expressions on both sides are *equivalent*. So, substituting the same values of each $a$ and $b$ in $(a + b)^2$ and in $a^2 + 2ab + b^2$ will always give the same result.

**Worked Example 22**

### Expanding algebraic expressions of the form $(a + b)^2$

Expand each of the following expressions.

**(a)** $(x + 4)^2$ **(b)** $\left(3y+\frac{1}{3}\right)^2$ **(c)** $(4a + 3b)^2$

**Solution**

**(a)** $(x + 4)^2 = x^2 + 2(x)(4) + 4^2$ apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = x$ and $b = 4$

$= x^2 + 8x + 16$

**(b)** $\left(3y+\frac{1}{3}\right)^2$

$= (3y)^2 + 2(3y)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2$ apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = 3y$ and $b = \frac{1}{3}$

$= 9y^2 + 2y + \frac{1}{9}$

**Attention**

**(b)** $(3y)^2 = 3y \times 3y$
$= 9y^2$
$(3y)^2 \neq 3y^2$

**(c)** $$(4a + 3b)^2 = \underbrace{(4a)^2}_{\text{square }1^{\text{st}}\text{ term}} + \underbrace{2(4a)(3b)}_{2 \times 1^{\text{st}}\text{ term} \times 2^{\text{nd}}\text{ term}} + \underbrace{(3b)^2}_{\text{square }2^{\text{nd}}\text{ term}}$$ $1^{\text{st}}$ term $= 4a$; $2^{\text{nd}}$ term $= 3b$

$= 16a^2 + 24ab + 9b^2$

**Practise Now 22**

Similar and Further Questions
**Exercise 4E**
Questions 1(a)–(f), 7(a), (b)

Expand each of the following expressions.

**(a)** $(x + 6)^2$ **(b)** $(4y + 3)^2$ **(c)** $(7 + 3a)^2$

**(d)** $\left(\frac{1}{2}x+8\right)^2$ **(e)** $(2x + 3y)^2$ **(f)** $(5a + 2b)^2$

## Investigation Second special algebraic identity

1. Expand $(a-b)^2$ using the Distributive Law: $(a-b)^2 = (a-b)(a-b)$

$=$

$=$

2. Expand $(a-b)^2$ using a multiplication frame:

| $\times$ | $a$ | $-b$ |
|---|---|---|
| $a$ | | |
| $-b$ | | |

$\therefore (a-b)^2 =$

3. Replace $b$ with $-b$ in the first special algebraic identity:

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 =$

$=$

**Attention**

$(-b)^2 = (-b) \times (-b)$
$= b^2$
$\neq -b^2$

4. **(i)** Do you get the same answer for Questions 1, 2 and 3?
   **(ii)** Is $(a-b)^2 = a^2 - b^2$? Why or why not?
   Hint: Consider your answer to Questions 1, 2 and 3.

5. Fig. 4.20 shows a square *PTUV* of length $a$, with two rectangles covering parts of the square.

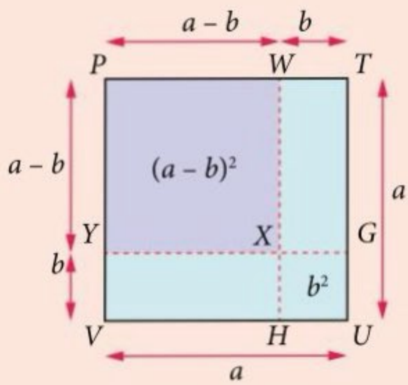



Fig. 4.20

**Attention**

Does Fig. 4.20 help you see why $(a-b)^2 = a^2 - 2ab + b^2$? Discuss with your classmates.

   **(i)** What is the length of the square *PWXY*?
   **(ii)** Do you think $(a-b)^2$ is a **perfect square**? Explain.
   **(iii)** Using Fig. 4.20, explain why $(a-b)^2 \neq a^2 - b^2$.
   Hint: What is the area of each of the squares *PTUV*, *XGUH* and *PWXY*?

6. Is $(a-b)^2 = a^2 - 2ab + b^2$ an **identity**? Explain.

From the Investigation on page 137, we have discovered the second special algebraic identity:

**Second special algebraic identity (or second perfect square identity)**

$$(a - b)^2 = a^2 - 2ab + b^2$$

**Worked Example 23**

### Expanding algebraic expressions of the form $(a - b)^2$

Expand each of the following expressions.

**(a)** $(x - 3)^2$ **(b)** $\left(\frac{7}{2} - 4y\right)^2$ **(c)** $(5a - 2b)^2$

**Solution**

**(a)** $(x - 3)^2 = x^2 - 2(x)(3) + 3^2$ apply $(a - b)^2 = a^2 - 2ab + b^2$, where $a = x$ and $b = 3$

$= x^2 - 6x + 9$

**(b)** $\left(\frac{7}{2} - 4y\right)^2 = \left(\frac{7}{2}\right)^2 - 2\left(\frac{7}{2}\right)(4y) + (4y)^2$ apply $(a - b)^2 = a^2 - 2ab + b^2$, where $a = \frac{7}{2}$ and $b = 4y$

$= \frac{49}{4} - 28y + 16y^2$

**Attention**

**(b)** $(4y)^2 = 4y \times 4y$
$= 16y^2$
$(4y)^2 \neq 4y^2$

**(c)** $(5a - 2b)^2 = \underbrace{(5a)^2}_{\text{square } 1^{\text{st}} \text{ term}} - \underbrace{2(5a)(2b)}_{2 \times 1^{\text{st}} \text{ term} \times 2^{\text{nd}} \text{ term}} + \underbrace{(2b)^2}_{\text{square } 2^{\text{nd}} \text{ term}}$ $1^{\text{st}}$ term $= 5a$; $2^{\text{nd}}$ term $= 2b$

$= 25a^2 - 20ab + 4b^2$

**Practise Now 23**

Similar and Further Questions
Exercise 4E
Questions 2(a)–(f), 8(a), (b)

Expand each of the following expressions.

**(a)** $(x - 4)^2$ **(b)** $(5y - 3)^2$ **(c)** $(8 - 2a)^2$

**(d)** $\left(\frac{2}{3}x - 6\right)^2$ **(e)** $(b - 3a)^2$ **(f)** $(3a - 4b)^2$

## Investigation

### Third special algebraic identity

1. Expand $(a + b)(a - b)$ using the Distributive Law: $(a + b)(a - b) =$ ______

$=$ ______

2. Expand $(a + b)(a - b)$ using a multiplication frame:

| $\times$ | $a$ | $-b$ |
|---|---|---|
| $a$ | | |
| $+b$ | | |

$\therefore (a + b)(a - b) =$ ______

3. Do you get the same answer for Questions 1 and 2?

4. Fig. 4.21(**a**) shows a polygon $PTGXHV$ formed by cutting the small square $XGUH$ (with length $b$) from the big square $PTUV$ (with length $a$).

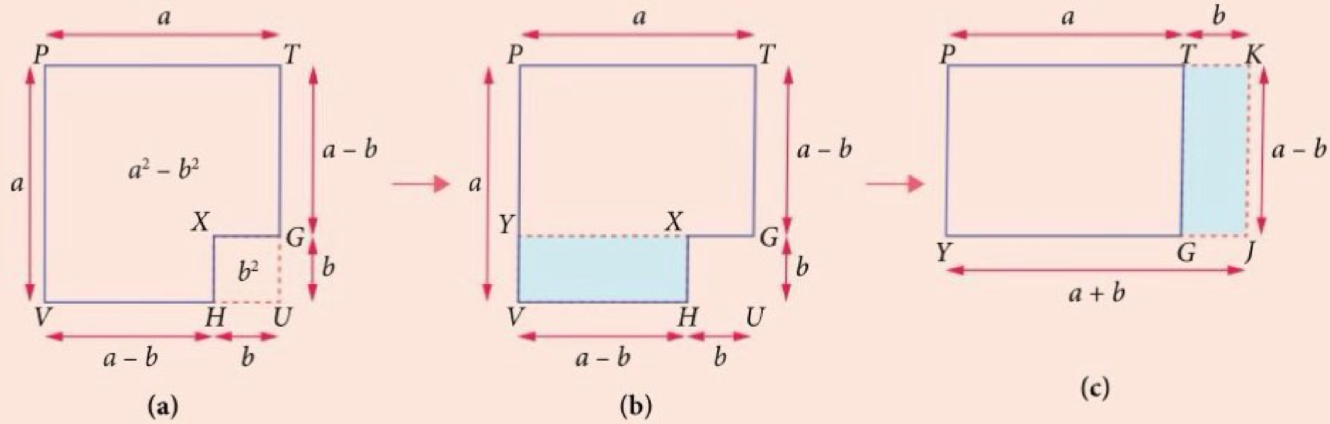



Fig. 4.21

(i) What is the area of the polygon $PTGXHV$ in Fig. 4.21(**a**)?
(ii) Do you think $a^2 - b^2$ is a perfect square or a **difference of two squares**? Explain.

5. Fig. 4.21(**b**) shows the same polygon $PTGXHV$ with the dotted line $YX$ parallel to $VH$.
What are the dimensions of the rectangle $YXHV$?

6. Remove the rectangle $YXHV$ from the polygon $PTGXHV$ in Fig. 4.21(**b**) and place it as shown in Fig. 4.21(**c**).
(i) Is the area of the rectangle $PKJY$ equal to the area of the polygon $PTGXHV$? Why?
(ii) What are the dimensions of the rectangle $PKJY$?
(iii) Find the area of the rectangle $PKJY$ in terms of its length and breadth.
(iv) By relating the area of rectangle $PKJY$ in parts **(i)** and **(iii)**, write down the third special algebraic **identity**:

$(a + b)(a - b) =$ ____________

From the above Investigation, we have discovered the third special algebraic identity:

**Third special algebraic identity (or difference of two squares identity)**

$$(a + b)(a - b) = a^2 - b^2$$

**Worked Example 24**

### Expanding algebraic expressions of the form $(a + b)(a - b)$

Expand each of the following expressions.

**(a)** $(x + 5)(x - 5)$ **(b)** $\left(4a - \frac{7}{2}b\right)\left(4a + \frac{7}{2}b\right)$

**Solution**

**(a)** $(x + 5)(x - 5) = x^2 - 5^2$ apply $(a + b)(a - b) = a^2 - b^2$, where $a = x$ and $b = 5$

$= x^2 - 25$

(b) $\left(4a-\frac{7}{2}b\right)\left(4a+\frac{7}{2}b\right) = \underbrace{(4a)^2}_{\text{square 1st term}} - \underbrace{\left(\frac{7}{2}b\right)^2}_{\text{square 2nd term}}$ 1st term = $4a$; 2nd term = $\frac{7}{2}b$

$= 16a^2 - \frac{49}{4}b^2$

**Practise Now 24**

Similar and Further Questions
**Exercise 4E**
Questions 3(a)–(d), 9(a)–(d), 10(a), (b), 13

Expand each of the following expressions.

**(a)** $(x + 3)(x - 3)$ **(b)** $(5y - 4)(5y + 4)$

**(c)** $(-3 + 2a)(-3 - 2a)$ **(d)** $\left(\frac{1}{4}x+8\right)\left(8-\frac{1}{4}x\right)$

**(e)** $(2x + 7y)(2x - 7y)$ **(f)** $(6b - a)(a + 6b)$

**Problem-solving Tip**

**(d)** Is it easier to re-order the terms in $\left(8-\frac{1}{4}x\right)$ or $\left(\frac{1}{4}x+8\right)$?

**Worked Example 25**

## Evaluating square and product of numbers using special algebraic identities

Without using a calculator, evaluate each of the following.

**(a)** $104^2$ **(b)** $78^2$ **(c)** $301 \times 299$

### Solution

**(a)** $104^2 = (100 + 4)^2$
$= 100^2 + 2(100)(4) + 4^2$ apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = 100$ and $b = 4$
$= 10\ 000 + 800 + 16$
$= 10\ 816$

**(b)** $78^2 = (80 - 2)^2$
$= 80^2 - 2(80)(2) + 2^2$ apply $(a - b)^2 = a^2 - 2ab + b^2$, where $a = 80$ and $b = 2$
$= 6400 - 320 + 4$
$= 6084$

**(c)** $301 \times 299 = (300 + 1)(300 - 1)$
$= 300^2 - 1^2$ apply $(a + b)(a - b) = a^2 - b^2$, where $a = 300$ and $b = 1$
$= 90\ 000 - 1$
$= 89\ 999$

**Practise Now 25**

Similar and Further Questions
**Exercise 4E**
Questions 4(a)–(d), 14, 15

Without using a calculator, evaluate each of the following.

**(a)** $103^2$ **(b)** $1001^2$ **(c)** $49^2$
**(d)** $197^2$ **(e)** $205 \times 195$ **(f)** $798 \times 802$

Worked Example 26

## Solving problem using special algebraic identity

If $(x + y)^2 = 147$ and $xy = -10$, find the value of $x^2 + y^2$.

**Solution**

$$(x + y)^2 = 147$$

$$x^2 + 2xy + y^2 = 147 \quad \text{apply } (a + b)^2 = a^2 + 2ab + b^2\text{, where } a = x \text{ and } b = y$$

Since $xy = -10$, then $x^2 + 2(-10) + y^2 = 147$

$$x^2 - 20 + y^2 = 147$$

$$x^2 + y^2 = 167$$

Practise Now 26

Similar and Further Questions
Exercise 4E
Questions 5, 6, 11, 12

1. If $(x + y)^2 = 38$ and $xy = -24$, find the value of $x^2 + y^2$.
2. If $(a - b)^2 = 296$ and $ab = -51$, find the value of $a^2 + b^2$.

Worked Example 27

## Solving problem using special algebraic identity

$n$ is a positive integer.

**(i)** Explain why $2n$ is an even number.

**(ii)** Write down an expression for the next even number which is greater than $2n$.

**(iii)** Find and simplify expressions for the squares of these two even numbers.

**(iv)** Hence explain why the difference between the squares of two consecutive even numbers is always a multiple of 4.

**Solution**

**(i)** Since $2n \div 2 = n$ (which is an integer), then $2n$ is divisible by 2.

$\therefore$ $2n$ is an even number.

**(ii)** $2n + 2$

**(iii)** $(2n)^2 = 4n^2$

$(2n + 2)^2 = (2n)^2 + 2(2n)(2) + 2^2$ apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = 2n$ and $b = 2$

$= 4n^2 + 8n + 4$

**(iv)** $(2n + 2)^2 - (2n)^2 = (4n^2 + 8n + 4) - 4n^2$

$= 8n + 4$

$= 4(2n + 1)$, which is a multiple of 4

$\therefore$ the difference between the squares of two consecutive even numbers is always a multiple of 4.

**Problem-solving Tip**

**(ii)** The next even number is greater than the previous even number by 2.

Practise Now 27

Similar and Further Questions
Exercise 4E
Questions 16, 17

$n$ is a positive integer.

**(i)** Explain why $(2n + 1)$ is an odd number.

**(ii)** Write down an expression for the next odd number which is greater than $2n + 1$.

**(iii)** Find and simplify expressions for the squares of these two odd numbers.

**(iv)** Hence explain why the difference between the squares of two consecutive odd numbers is always a multiple of 8.

## Reflection

1. What do I already know about the expansion of algebraic expressions using the Distributive Law or a multiplication frame that could help me learn the three special algebraic identities?
2. What have I learnt in this section that I am still unclear of?

Basic | Intermediate | Advanced

## Exercise 4E

**1.** Expand each of the following expressions.
(a) $(a+4)^2$ (b) $(3b+2)^2$
(c) $(c+4d)^2$ (d) $(9h+2k)^2$
(e) $(3a+4b)^2$ (f) $(2b+3a)^2$

**2.** Expand each of the following expressions.
(a) $(m-9)^2$ (b) $(5n-4)^2$
(c) $(9-5p)^2$ (d) $(3q-8r)^2$
(e) $(3a-4b)^2$ (f) $(5b-3a)^2$

**3.** Expand each of the following expressions.
(a) $(s+5)(s-5)$ (b) $(w-10x)(w+10x)$
(c) $(2t+11)(2t-11)$ (d) $(7-2u)(7+2u)$

**4.** Without using a calculator, evaluate each of the following.
(a) $1203^2$ (b) $892^2$
(c) $403 \times 397$ (d) $1998 \times 2002$

**5.** If $x^2+y^2=80$ and $xy=12$, find the value of $(x-y)^2$.

**6.** If $x+y=10$ and $x-y=4$, find the value of $x^2-y^2$.

**7.** Expand each of the following expressions.
(a) $\left(\frac{1}{5}a+3b\right)^2$ (b) $\left(\frac{1}{2}c+\frac{2}{3}d\right)^2$

**8.** Expand each of the following expressions.
(a) $\left(\frac{3}{2}h-5k\right)^2$ (b) $\left(-\frac{6}{5}m-3n\right)^2$

**9.** Expand each of the following expressions.
(a) $(6p+5)(5-6p)$ (b) $\left(9r-\frac{4}{5}q\right)\left(9r+\frac{4}{5}q\right)$
(c) $\left(\frac{s}{2}+\frac{t}{3}\right)\left(\frac{t}{3}-\frac{s}{2}\right)$ (d) $(u+2)(u-2)(u^2+4)$

**10.** Simplify each of the following expressions.
(a) $4(x+3)^2-3(x+4)(x-4)$
(b) $(5x-7y)(5x+7y)-2(x-2y)^2$

**11.** If $x^2+y^2=14$ and $xy=5$, find the value of $\left(\frac{1}{2}x+\frac{1}{2}y\right)^2$.

**12.** If $2x^2-2y^2=125$ and $x-y=2.5$, find the value of $x+y$.

**13.** Expand the expression $\left(\frac{1}{16}x^2+\frac{1}{25}y^2\right)\left(\frac{1}{4}x+\frac{1}{5}y\right)\left(\frac{1}{4}x-\frac{1}{5}y\right)$.

**14.** (i) Simplify the expression $(p-2q)^2-p(p-4q)$.
(ii) Hence, by substituting a suitable value of $p$ and of $q$, find the value of $5310^2-5330 \times 5290$.

**15.** (i) Simplify the expression $n^2-(n-a)(n+a)$.
(ii) Hence, by substituting a suitable value of $n$ and of $a$, find the value of $16\,947^2-16\,944 \times 16\,950$.

## Exercise 4E

16. $m$ is a positive integer.
    **(i)** Explain why $2m$ is an even number.
    **(ii)** Write down an expression for the next even number which is greater than $2m$.
    **(iii)** Find and simplify expressions for the squares of these two even numbers.
    **(iv)** Hence explain why the sum of the squares of two consecutive even numbers is always a multiple of 4.

17. $m$ is a positive integer.
    **(i)** Explain why $(2m + 1)$ is an odd number.
    **(ii)** Write down an expression for the next odd number which is greater than $(2m + 1)$.
    **(iii)** Find and simplify expressions for the squares of these two odd numbers.
    **(iv)** Hence, or otherwise, explain why the sum of the squares of two consecutive odd numbers is always an even number.

# 4.7 Factorisation using special algebraic identities

In Section 4.6, we have learnt how to *expand* certain algebraic expressions using the three special algebraic identities. Since factorisation is the reverse of expansion, we can use these same identities to *factorise* certain algebraic expressions.

$$(a+b)^2 \underset{\text{factorise}}{\overset{\text{expand}}{\rightleftharpoons}} a^2 + 2ab + b^2 \qquad (a-b)^2 \underset{\text{factorise}}{\overset{\text{expand}}{\rightleftharpoons}} a^2 - 2ab + b^2 \qquad (a+b)(a-b) \underset{\text{factorise}}{\overset{\text{expand}}{\rightleftharpoons}} a^2 - b^2$$

**Attention**

$(a + b)^2 \neq a^2 + b^2$
$(a - b)^2 \neq a^2 - b^2$

**Worked Example 28**

**Factorising algebraic expressions of the form $a^2 + 2ab + b^2$**

Factorise each of the following expressions completely.

**(a)** $x^2 + 6x + 9$ **(b)** $16y^2 + 20y + \dfrac{25}{4}$ **(c)** $4a^2 + 12ab + 9b^2$

**Solution**

**(a)**
$$\begin{aligned} x^2 + 6x + 9 &= x^2 + 2(x)(3) + 3^2 && \text{express as } a^2 + 2ab + b^2, \text{ where } a = x \text{ and } b = 3 \\ &= (x + 3)^2 && \text{apply } a^2 + 2ab + b^2 = (a + b)^2 \end{aligned}$$

**Attention**

**(a)** $x^2$ and 9 are *perfect squares*. Can we can write $6x$ as $2 \times x \times \sqrt{9}$? If yes, we can make use of $a^2 + 2ab + b^2 = (a + b)^2$ to factorise the expression.

**(b)**
$$\begin{aligned} 16y^2 + 20y + \frac{25}{4} &= (4y)^2 + 2(4y)\left(\frac{5}{2}\right) + \left(\frac{5}{2}\right)^2 && \text{express as } a^2 + 2ab + b^2, \text{ where } a = 4y \text{ and } b = \frac{5}{2} \\ &= \left(4y + \frac{5}{2}\right)^2 && \text{apply } a^2 + 2ab + b^2 = (a + b)^2 \end{aligned}$$

$$\begin{array}{l} \text{(c)}\ 4a^2 + 12ab + 9b^2 = \underbrace{(2a)^2}_{\text{square 1st term}} + \underbrace{2(2a)(3b)}_{2\times\text{1st term}\times\text{2nd term}} + \underbrace{(3b)^2}_{\text{square 2nd term}} \quad \text{1st term} = 2a;\ \text{2nd term} = 3b \\ \qquad = (\underbrace{2a}_{\text{1st term}} + \underbrace{3b}_{\text{2nd term}})^2 \end{array}$$

**Practise Now 28**

Similar and Further Questions
**Exercise 4F**
Questions 1(a)–(e), 5(a)–(f), 6

If possible, factorise each of the following expressions completely using an algebraic identity. If it is not possible to do so, state N.A. (for not applicable).

**(a)** $x^2 + 10x + 25$ **(b)** $x^2 + 18x + 36$ **(c)** $9y^2 + 24y + 16$

**(d)** $36a^2 + 8a + \frac{4}{9}$ **(e)** $25a^2 + 40ab + 16b^2$ **(f)** $16x^2 + 28xy + 49y^2$

**Worked Example 29**

### Factorising algebraic expressions of the form $a^2 - 2ab + b^2$

Factorise each of the following expressions completely.

**(a)** $2x^2 - 16x + 32$ **(b)** $36 - 8y + \frac{4y^2}{9}$ **(c)** $81a^2 - 36ab + 4b^2$

**Solution**

**(a)** $2x^2 - 16x + 32 = 2(x^2 - 8x + 16)$ HCF of 2, 16 and 32 = 2

$= 2[x^2 - 2(x)(4) + 4^2]$ express as $a^2 - 2ab + b^2$, where $a = x$ and $b = 4$

$= 2(x - 4)^2$ apply $a^2 - 2ab + b^2 = (a - b)^2$

> **Attention**
> **(a)** $x^2$ and 16 are *perfect squares*. Can we write $8x$ as $2 \times x \times \sqrt{16}$? If yes, we can make use of $a^2 - 2ab + b^2 = (a - b)^2$ to factorise the expression.

**(b)** $36 - 8y + \frac{4y^2}{9} = 6^2 - 2(6)\left(\frac{2y}{3}\right) + \left(\frac{2y}{3}\right)^2$ express as $a^2 - 2ab + b^2$, where $a = 6$ and $b = \frac{2y}{3}$

$= \left(6 - \frac{2y}{3}\right)^2$ apply $a^2 - 2ab + b^2 = (a - b)^2$

$$\begin{array}{l} \text{(c)}\ 81a^2 - 36ab + 4b^2 = \underbrace{(9a)^2}_{\text{square 1st term}} - \underbrace{2(9a)(2b)}_{2\times\text{1st term}\times\text{2nd term}} + \underbrace{(2b)^2}_{\text{square 2nd term}} \quad \text{1st term} = 9a;\ \text{2nd term} = 2b \\ \qquad = (\underbrace{9a}_{\text{1st term}} - \underbrace{2b}_{\text{2nd term}})^2 \end{array}$$

**Practise Now 29**

Similar and Further Questions
**Exercise 4F**
Questions 2(a)–(d), 7(a)–(f)

If possible, factorise each of the following expressions completely using an algebraic identity. If it is not possible to use an algebraic identity to factorise, state N.A. (for not applicable).

**(a)** $8x^2 - 56x + 98$ **(b)** $\frac{4}{3}t^2 - 4t + 3$ **(c)** $1 - \frac{2}{3}q + \frac{1}{9}q^2$

**(d)** $\frac{16}{25} - \frac{24}{5}n + 9n^2$ **(e)** $25x^2 - 10xy + y^2$ **(f)** $49h^2 - 42hk + 36k^2$

**Worked Example 30**

### Factorising algebraic expressions of the form $a^2 - b^2$

Factorise each of the following expressions completely.

**(a)** $9x^2 - 49$ **(b)** $48b^2 - \frac{25}{3}a^2$

**Solution**

**(a)** $9x^2 - 49 = (3x)^2 - (7)^2$ express as $a^2 - b^2$, where $a = 3x$ and $b = 7$

$= (3x + 7)(3x - 7)$ apply $a^2 - b^2 = (a + b)(a - b)$

**(b)** $48b^2 - \frac{25}{3}a^2 = \frac{1}{3}(144b^2 - 25a^2)$ extract common factor $\frac{1}{3}$

$= \frac{1}{3}[(12b)^2 - (5a)^2]$ 1st term = $12b$; 2nd term = $5a$

$= \frac{1}{3}(12b + 5a)(12b - 5a)$

**Problem-solving Tip**

**(b)** How can we obtain perfect squares from 48 and $\frac{25}{3}$? We can extract the common factor $\frac{1}{3}$ to get 144 and 25, which are perfect squares.

**Practise Now 30**

Similar and Further Questions
**Exercise 4F**
Questions 3(a)–(d), 8(a)–(f), 9(a)–(f), 12(a)–(d)

If possible, factorise each of the following expressions completely using an algebraic identity. If it is not possible to use an algebraic identity to factorise, state N.A. (for not applicable).

**(a)** $81x^2 - 16$ **(b)** $-25y^2 + 9$ **(c)** $64x^2 - 27$

**(d)** $4a^2 - 64b^2$ **(e)** $\frac{8}{25}b^2 - 18a^2$ **(f)** $(4x + 1)^2 - 49$

**Worked Example 31**

### Evaluating difference of two square numbers using special algebraic identity

Without using a calculator, evaluate $104^2 - 16$.

**Solution**

$104^2 - 16 = 104^2 - 4^2$

$= (104 + 4)(104 - 4)$ apply $a^2 - b^2 = (a + b)(a - b)$, where $a = 104$ and $b = 4$

$= 108 \times 100$

$= 10\,800$

**Practise Now 31**

Similar and Further Questions
**Exercise 4F**
Questions 4(a)–(d), 10

Without using a calculator, evaluate

**(a)** $103^2 - 9$, **(b)** $211^2 - 121$,

**(c)** $49 - 107^2$, **(d)** $247^2 - 147^2$.

Worked Example 32

## Solving problem using special algebraic identity

**(i)** Factorise $x^2 - 9y^2$.

**(ii)** Given that $x$ and $y$ are positive integers, solve the equation $x^2 - 9y^2 = 13$.

### Solution

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

**Stage 1: Understand the problem**

*For part **(i)**, what does the term 'factorise' imply?*
*For part **(ii)**, what can we understand from the term 'positive integers'? Based on the word 'solve', what do we have to find?*

**Stage 2: Think of a plan**

*For part **(i)**, what are some methods of factorisation that we have learnt? Is there any special algebraic identity that we can use?*

*For part **(ii)**, have we encountered such an equation before?*
We have learnt how to solve a linear equation in one variable as well as two simultaneous linear equations. However, the given equation is a quadratic equation in two variables, which we have not learnt how to solve.
*How can we use the answer from part **(i)**?*

**Stage 3: Carry out the plan**

**(i)** $x^2 - 9y^2 = x^2 - (3y)^2$
$= (x + 3y)(x - 3y)$ — apply $a^2 - b^2 = (a + b)(a - b)$, where $a = x$ and $b = 3y$

**(ii)** $x^2 - 9y^2 = 13$
$(x + 3y)(x - 3y) = 13$

**Go back to Stage 2: Think of a plan**

*Is there a way to convert this equation into two simultaneous linear equations? What have we not used? Is there anything special about 13?*
Since $x$ and $y$ are positive integers, then $(x + 3y)$ and $(x - 3y)$ are also integers.
13 is a prime number, so it has exactly two integer factors: 1 and 13. As $(x + 3y)(x - 3y) = 13$, then the larger factor $(x + 3y)$ must be 13 and the smaller factor $(x - 3y)$ must be 1.

**Stage 3: Carry out the plan**

**(ii)** $x^2 - 9y^2 = 13$
$(x + 3y)(x - 3y) = 13$
Since 13 is a prime number, it has exactly two factors: 1 and 13.
Since $x$ and $y$ are positive integers, $x - 3y$ is smaller than $x + 3y$.
$\therefore x - 3y = 1$ —— (1)
$x + 3y = 13$ —— (2)
(1) + (2): $2x = 14$
$x = 7$
Substitute into (2): $7 + 3y = 13$
$3y = 6$
$y = 2$
$\therefore x = 7$ and $y = 2$

> **Attention**
> If $(x + 3y)(x - 3y) = 6$, which is not a prime number, can we conclude that $x + 3y = 6$ and $x - 3y = 1$? Or is it possible for $x + 3y = 3$ and $x - 3y = 2$?

**Stage 4: Look back**

**(ii)** *How can we check that the answer is correct?*

Substitute $x = 7$ and $y = 2$ into the LHS of the given equation:

$$\begin{aligned}\text{LHS} &= x^2 - 9y^2 \\ &= 7^2 - 9(2)^2 \\ &= 49 - 36 \\ &= 13 \\ &= \text{RHS}\end{aligned}$$

**Reflection**

What have you learnt from solving this question? If you see a similar question again, are you confident enough to solve it?

**Practise Now 32**

Similar and Further Questions
**Exercise 4F**
Questions 11, 13(a), (b)

1. **(i)** Factorise $x^2 - 4y^2$.
   **(ii)** Given that $x$ and $y$ are positive integers, solve the equation $x^2 - 4y^2 = 5$.
2. **(a)** Factorise $x^2 - 9$.
   **(b)** Use your answer to part **(a)** to find two factors of 2491, other than 1 and 2491.

## Class Discussion

### Equivalent expressions

We have learnt that two expressions are equivalent if the value of both expressions is the same for any value we substitute for the same variables in the expressions. Find all the pairs of **equivalent expressions** in Table 4.2 and justify why each pair is equivalent. If your classmate does not obtain the correct answer, explain to him or her what he or she has done wrong.

| | | | | |
|---|---|---|---|---|
| **A** $(x - y)^2$ | **B** $(x + y)(x + y)$ | **C** $x^2 + y^2$ | **D** $(2w - x)(z - 3y)$ | **E** $-5x^2 + 28x - 24$ |
| **F** $2wz - 6wy + 3xy - xz$ | **G** $(x + y)^2$ | **H** $(2w + x)(z - 3y)$ | **I** $(x - y)(x - y)$ | **J** $x^2 - y^2$ |
| **K** $(x + y)(x - y)$ | **L** $2x - (x - 4)(5x - 6)$ | **M** $x^2 - 2xy + y^2$ | **N** $-5x^2 - 24x + 24$ | **O** $x^2 + 2xy + y^2$ |

Table 4.2

## Reflection

1. How do I explain to my classmates that $(a + b)^2 \neq a^2 + b^2$ and $(a - b)^2 \neq a^2 - b^2$?
2. What have I learnt in this section or chapter that I am still unclear of?

## Exercise 4F

**1.** Factorise each of the following expressions completely.

(a) $a^2 + 14a + 49$ (b) $4b^2 + 4b + 1$
(c) $c^2 + 2cd + d^2$ (d) $4h^2 + 20hk + 25k^2$
(e) $9a^2 + 30ab + 25b^2$

**2.** Factorise each of the following expressions completely.

(a) $m^2 - 10m + 25$ (b) $169n^2 - 52n + 4$
(c) $81 - 180p + 100p^2$ (d) $49q^2 - 42qr + 9r^2$

**3.** Factorise each of the following expressions completely.

(a) $s^2 - 144$ (b) $36t^2 - 25$
(c) $225 - 49u^2$ (d) $49w^2 - 81x^2$

**4.** Without using a calculator, evaluate each of the following.

(a) $59^2 - 41^2$ (b) $29^2 - 39^2$
(c) $7.7^2 - 2.3^2$ (d) $81 - 91^2$

**5.** If possible, factorise each of the following expressions completely using an algebraic identity. If it is not possible do so, state N.A.

(a) $3a^2 + 12a + 12$ (b) $25b^2 + 5bc + \frac{1}{4}c^2$
(c) $8x^2 + 20xy + 50y^2$ (d) $\frac{16}{49}w^2 + \frac{8}{35}wv + \frac{1}{25}v^2$
(e) $h^4 + 2h^2k + k^2$ (f) $p^2q^2 + 10pq + 100$

**6.** The surface area of each face of a cube is $(x^2 + 4x + 4)$ cm². Find
(i) the length,
(ii) the volume, in expanded form,
of the cube.

**7.** If possible, factorise each of the following expressions completely using an algebraic identity. If it is not possible to do so, state N.A.

(a) $36m^2 - 48mn + 16n^2$ (b) $\frac{1}{4}h^2 - 4hk + 4k^2$
(c) $\frac{1}{3}p^2 - \frac{2}{3}pq + \frac{1}{3}q^2$ (d) $16r^2 - rs + \frac{1}{64}s^2$
(e) $25 - 10tu + t^2u^2$ (f) $75w^2 - \frac{15}{2}wz + \frac{3}{16}z^2$

**8.** If possible, factorise each of the following expressions completely using an algebraic identity. If it is not possible to do so, state N.A.

(a) $32a^2 - 98b^2$ (b) $c^2 - \frac{1}{4}d^2$
(c) $m^2 - 64n^4$ (d) $100y^2 - 125z^2$
(e) $\frac{9h^2}{100} - 16k^2$ (f) $60p^2 - \frac{15}{64}q^2$

**9.** Factorise each of the following expressions completely.

(a) $(a + 3)^2 - 9$ (b) $16 - 25(b + 3)^2$
(c) $c^2 - (d + 2)^2$ (d) $(2h - 1)^2 - 4k^2$
(e) $(3x - 5)^2 - 169$ (f) $(p + 1)^2 - (p - 1)^2$

**10.** Without using a calculator, evaluate $5 \times 88^2 - 720$.

**11.** (i) Factorise $x^2 - 4y^2$.
(ii) Given that $x$ and $y$ are positive integers, solve the equation $x^2 - 4y^2 = 13$.

**12.** Factorise each of the following expressions completely.

(a) $4(x - 1)^2 - 81(x + 1)^2$
(b) $16x^2 + 8x + 1 - 9y^2$
(c) $4x^2 - y^2 + 4y - 4$
(d) $13x^2 + 26xy + 13y^2 - 13$

**13.** (a) Factorise $x^2 - 121$.
(b) Use your answer to part **(a)** to find two factors of 7979, other than 1 and 7979.

## Looking Back

In this chapter, we continue our journey of learning algebra by extending our knowledge of algebraic expressions through working with products of linear expressions and the factorisation of quadratic expressions. This knowledge provides us with additional tools to **model** more complicated real-world situations and enables us to use these algebraic techniques to obtain mathematical solutions that can be interpreted in relation to the problems. As we learn more techniques in algebra, let us always keep in mind that these 'new' techniques are still built on the ideas of **equivalence**, whereby the product of a given pair of linear algebraic expressions can be written in different ways.

## Summary

**1. Addition and subtraction of like terms in $x^2$**

To add or subtract like terms in $x^2$, we add or subtract their coefficients in the same way we would add or subtract real numbers.

| Real numbers | Terms in $x^2$ |
|---|---|
| $(-2) + (-3) = -5$ | $(-2x^2) + (-3x^2) = -5x^2$ |
| $5 + (-2) = 5 - 2 = 3$<br>$-2 + 5 = 5 - 2 = 3$ | $5x^2 + (-2x^2) = 5x^2 - 2x^2 = 3x^2$<br>$-2x^2 + 5x^2 = 5x^2 - 2x^2 = 3x^2$ |
| $2 - 5 = -3$ | $2x^2 - 5x^2 = -3x^2$ |
| $-5 - 2 = -7$ | $-5x^2 - 2x^2 = -7x^2$ |
| $5 - (-2) = 5 + 2 = 7$ | $5x^2 - (-2x^2) = 5x^2 + 2x^2 = 7x^2$ |

**2. The Distributive Law for $(a + b)(c + d)$**

$$(a + b)(c + d) = ac + ad + bc + bd$$

**3. Expansion of quadratic expressions using multiplication frame**

For example, to expand $(3x + 4)(7 - 8x)$:

| $\times$ | $7$ | $-8x$ |
|---|---|---|
| $3x$ | | |
| $+4$ | | |

$\longrightarrow$

| $\times$ | $7$ | $-8x$ |
|---|---|---|
| $3x$ | $21x$ | $-24x^2$ |
| $+4$ | $+28$ | $-32x$ |

$$\begin{aligned} \therefore (3x + 4)(7 - 8x) &= 21x - 24x^2 + 28 - 32x \\ &= -24x^2 - 11x + 28 \end{aligned}$$

- Expand $(2x + 5)(3 - x)$ using a multiplication frame.

## Summary

**4. Factorisation of quadratic expressions using multiplication frame**

For example, to factorise $2x^2 + 7x - 15$:

**Method 1: Guess and Check**

$-15 = 1 \times (-15)$ or $(-1) \times 15$
$\quad = 3 \times (-5)$ or $(-3) \times 5$

| × | $x$ | ? |
|---|---|---|
| $2x$ | $2x^2$ | |
| ? | | $-15$ |

guess →

| × | $x$ | $+5$ |
|---|---|---|
| $2x$ | $2x^2$ | |
| $-3$ | | $-15$ |

check →

| × | $x$ | $+5$ |
|---|---|---|
| $2x$ | $2x^2$ | $+10x$ |
| $-3$ | $-3x$ | $-15$ |

$\therefore 2x^2 + 7x - 15 = (2x - 3)(x + 5)$

**Method 2: Use some reasoning**

$-30 = 1 \times (-30)$ or $(-1) \times 30$
$\quad = 2 \times (-15)$ or $(-2) \times 15$
$\quad = 3 \times (-10)$ or $(-3) \times 10$ $\longrightarrow$ $(-3) + 10 = 7$ (coefficient of $x$ term)
$\quad = 5 \times (-6)$ or $(-5) \times 6$

| × | | |
|---|---|---|
| | $2x^2$ | $?x$ |
| | $?x$ | $-15$ |

guess →

| × | | |
|---|---|---|
| | $2x^2$ | $-3x$ |
| | $+10x$ | $-15$ |

check →

| × | $2x$ | $-3$ |
|---|---|---|
| $x$ | $2x^2$ | $-3x$ |
| $+5$ | $+10x$ | $-15$ |

$\therefore 2x^2 + 7x - 15 = (2x - 3)(x + 5)$

- Factorise $15 + x - 2x^2$ using a multiplication frame.

**5. Factorisation of algebraic expression into the form $(a + b)(c + d)$**

**Method 1: Multiplication frame**

| × | $c$ | $+d$ |
|---|---|---|
| $a$ | $ac$ | $+ad$ |
| $+b$ | $+bc$ | $+bd$ |

$\therefore ac + ad + bc + bd = (a + b)(c + d)$

**Method 2: By grouping**

$$\begin{aligned} ac + ad + bc + bd &= (ac + ad) + (bc + bd) \\ &= a(c + d) + b(c + d) \\ &= (a + b)(c + d) \end{aligned}$$

**6. Special algebraic identities**

$(a + b)^2 = a^2 + 2ab + b^2$ (expand → ; ← factorise)

$(a - b)^2 = a^2 - 2ab + b^2$ (expand → ; ← factorise)

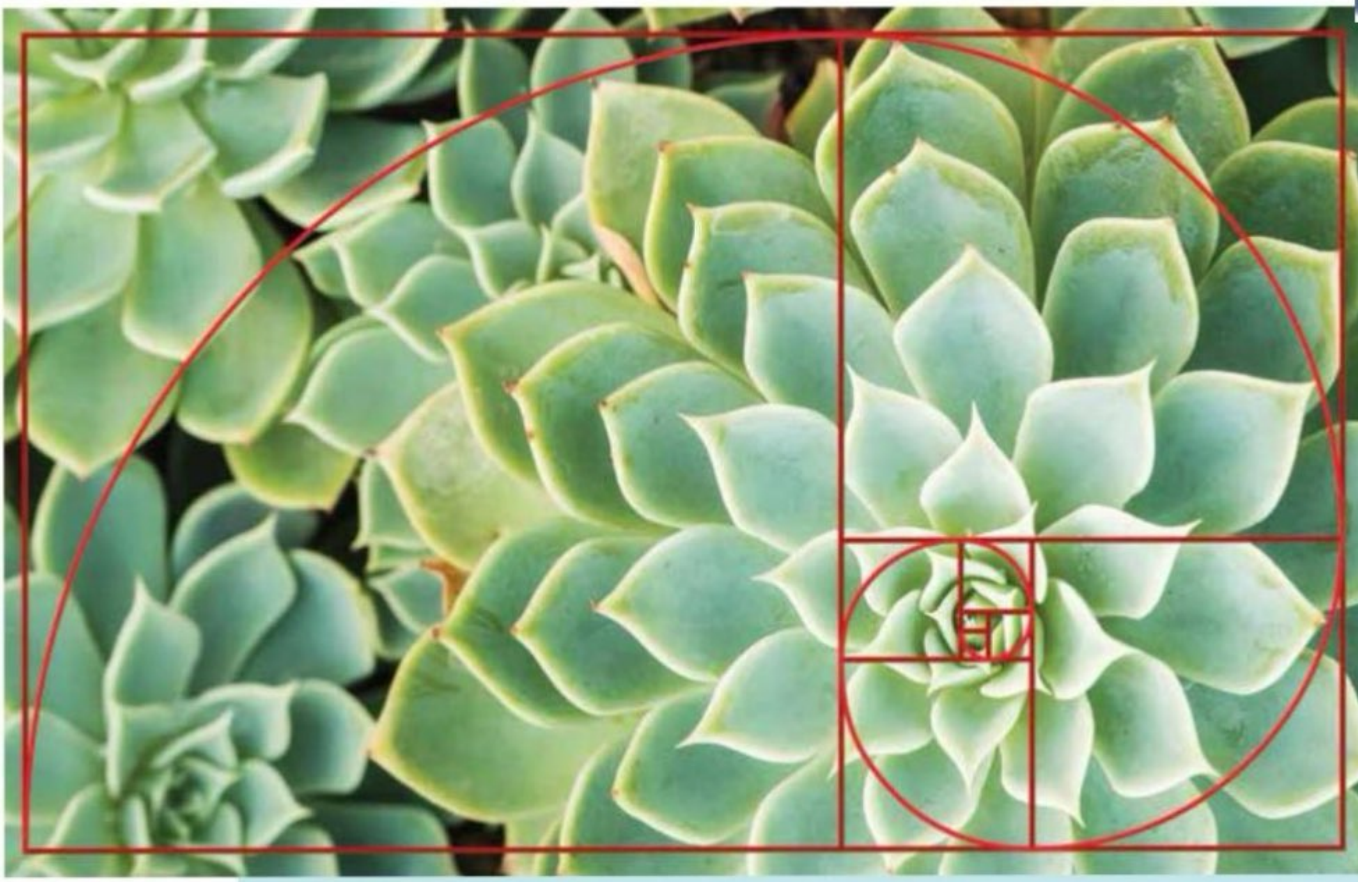
$(a + b)(a - b) = a^2 - b^2$ (expand → ; ← factorise)

**Attention**

$(a + b)^2 \neq a^2 + b^2$
$(a - b)^2 \neq a^2 - b^2$

# CHAPTER 5

## Number Patterns



Mathematics equips us with the skill to observe, recognise and create patterns, which is useful in helping us make good predictions in the real world. Patterns allow us to see relationships and form generalisations. Patterns are found everywhere! For example, they can be found in art, music and nature. In this chapter, we will look at number patterns and their applications in daily life.

### Learning Outcomes

What will we learn in this chapter?

- What number sequences are
- How to recognise patterns in number sequences
- How to find a formula for the general term of a number sequence
- Why number patterns have useful applications in real life

## Introductory Problem

There is an interesting number sequence found in the family tree of bees.

In every bee hive, one female queen bee lays eggs. A male bee is hatched from an unfertilised egg. In other words, a male bee has only one parent (the mother). A female bee is hatched from an egg which is fertilised by a male bee. So a female bee has two parents (a mother and a father).

**1.** Let us look into the family tree of a male bee called Buzz.
   - **(a)** How many parents and grandparents does Buzz have?
   - **(b)** How many great-grandparents, great-great-grandparents and great-great-great-grandparents does Buzz have?
   - **(c)** What pattern(s) do you observe about these numbers?

**2.** Let us also look into the family tree of a female bee called Betty.
   - **(a)** How many parents and grandparents does Betty have?
   - **(b)** How many great-grandparents, great-great-grandparents and great-great-great-grandparents does Betty have?
   - **(c)** What pattern(s) do you observe about these numbers?

In this chapter, we will learn about number sequences and how to find a formula for the general term of a number sequence.

# 5.1 Number sequences

## A. Patterns in number sequences

Consider the following whole numbers:

| 1st term | | 2nd term | | 3rd term | | 4th term | | 5th term |
|---|---|---|---|---|---|---|---|---|
| 2, | | 5, | | 8, | | 11, | | 14, ... |
| | + 3 | | + 3 | | + 3 | | + 3 | |

The set of numbers 2, 5, 8, 11, 14, … forms a **number sequence**.

The numbers in the number sequence are known as the **terms** of the sequence. The 1st, 2nd, 3rd, 4th and 5th terms are 2, 5, 8, 11 and 14 respectively.

The numbers are governed by a *specific rule*: start with 2, then add 3 to each term to get the next term.

## Class Discussion

**Number sequences**

1. Table 5.1 shows a few examples of number sequences. Copy and complete Table 5.1.

| | Sequence | Rule |
|---|---|---|
| Positive even numbers | 2, 4, 6, 8, 10, ___, ___, …<br>+ 2, + 2, + 2, + 2, ___, ___ | Start with ___, then add ___ to each term to get the next term. |
| Positive odd numbers | 1, 3, 5, 7, 9, ___, ___, …<br>+ 2, + 2, + 2, + 2, ___, ___ | Start with ___, then add ___ to each term to get the next term. |
| Multiples of 3 | 3, 6, 9, 12, 15, ___, ___, …<br>+ 3, + 3, + 3, + 3, ___, ___ | Start with ___, then add ___ to each term to get the next term. |
| Powers of 2 | 1, 2, 4, 8, 16, ___, ___, …<br>× 2, × 2, × 2, × 2, ___, ___ | Start with ___, then multiply each term by ___ to get the next term. |
| Powers of 3 | 1, 3, 9, 27, 81, ___, ___, …<br>× 3, × 3, × 3, × 3, ___, ___ | Start with ___, then multiply each term by ___ to get the next term. |

Table 5.1

2. The sequence of positive even numbers can also be obtained by multiplying each term of the sequence of positive integers (i.e. 1, 2, 3, 4, 5, …) by 2. Can you think of a different rule to obtain the sequence of positive odd numbers?

**Worked Example 1**

### Finding rule that generates sequence

For each of the following sequences, state the rule of the sequence and write down the next two terms.

**(a)** 42, 39, 36, 33, 30, … **(b)** –22, –18, –14, –10, –6, …
**(c)** 256, 128, 64, 32, 16, … **(d)** –1, 1, –1, 1, –1, …

**Solution**

**(a)** 42, 39, 36, 33, 30, …
– 3, – 3, – 3, – 3

Rule: Start with 42, then subtract 3 from each term to get the next term. The next two terms are 27 and 24.

**(b)** –22, –18, –14, –10, –6, …
+ 4, + 4, + 4, + 4

Rule: Start with –22, then add 4 to each term to get the next term. The next two terms are –2 and 2.

**Problem-solving Tip**

Check
- if the sequence is a common one that you recognise,
- how two consecutive terms in the sequence might be related through addition, subtraction, multiplication or division.

**(c)** 256, 128, 64, 32, 16, …

$\div 2$ $\div 2$ $\div 2$ $\div 2$

Rule: Start with 256, then divide each term by 2 to get the next term. The next two terms are 8 and 4.

**(d)** –1, 1, –1, 1, –1, …

$\times(-1)$ $\times(-1)$ $\times(-1)$ $\times(-1)$

Rule: Start with –1, then multiply each term by –1 to get the next term. The next two terms are 1 and –1.

**Attention**

An alternative rule for **(c)** is: Start with 256. Multiply each term by $\frac{1}{2}$ to get the next term.

Can you think of an alternative rule for **(d)**?

**Practise Now 1**

Similar and Further Questions
**Exercise 5A**
Questions 1(a)–(h), 10(a)–(d), 18(a)–(d)

1. For each of the following sequences, state the rule of the sequence and write down the next two terms.
   **(a)** 3, 8, 13, 18, 23, …
   **(b)** –20, –26, –32, –38, –44, …
   **(c)** 5, 15, 45, 135, 405, …
   **(d)** 4374, –1458, 486, –162, 54, …
2. Write down the next two terms of each of the following sequences.
   **(a)** 1, 2, 4, 7, 11, 16, …
   **(b)** 26, 25, 21, 20, 16, …

**Attention**

When listing the terms of a number sequence, we need to *list at least 4 terms* so that others can observe the *intended pattern*.

For example, what is the rule of the sequence 1, 2, 3, …?

It can be: Start with 1, then add 1 to each term to get the next term, so the term after 3 will be 4.

But it can also be: Start with 1 and 2, then add the preceding two terms to get the next term, 3. The term after 3 will be 2 + 3 = 5.

What other rules can you think of, so that the next term will be another number?

## B. General term of simple number sequences

Consider the sequence of positive even numbers: 2, 4, 6, 8, 10, …

The terms of the sequence can be denoted by $T_1, T_2, T_3, T_4, T_5, \ldots, T_n$, where

$T_1 = 1^{\text{st}}$ term = 2,
$T_2 = 2^{\text{nd}}$ term = 4,
$T_3 = 3^{\text{rd}}$ term = 6,
$T_4 = 4^{\text{th}}$ term = 8,
$T_5 = 5^{\text{th}}$ term = 10,
⋮
$T_n = n^{\text{th}}$ term (**general term**).

From Table 5.2, observe that each term in the sequence can be obtained by multiplying its position $n$ by 2.

| **Position $n$** | 1 | 2 | 3 | 4 | 5 | … | $n$ |
|---|---|---|---|---|---|---|---|
| **Term $T_n$** | $2 \times 1 = 2$ | $2 \times 2 = 4$ | $2 \times 3 = 6$ | $2 \times 4 = 8$ | $2 \times 5 = 10$ | … | $2n$ |

Table 5.2

Therefore, the general term of the sequence is $T_n = 2n$.

The general term, $T_n$, of a sequence can be represented by an algebraic expression in terms of its term position, $n$, where $n$ is a variable.

**Big Idea**

**Functions**

We have learnt in Chapter 1 that a function is a relationship that expresses how the input (in this case, the term position $n$) follows a general rule ($T_n = 2n$) to *uniquely* determine the output ($T_n$).

In every sequence, there is only one value of $T_n$ for every value of $n$. We say that $T_n$ is a function of $n$.

A function may be represented in the form of a table, e.g. Table 5.2 shows the relationship between the input $n$ and its corresponding unique output $T_n$.

By substituting different values of $n$, we are able to generate the terms of the sequence.
For example, the $68^{th}$ term of the above sequence is given by $T_{68} = 2(68) = 136$.
As such, knowing the formula for the general term enables us to accurately predict any term in the sequence withou having to work out all the previous terms.

## Investigation — Finding general term of simple sequences

For each of the following sequences, use the table provided to find a formula for the general term and then state the $100^{th}$ term, $T_{100}$.

**(a)** **Multiples of 3**: 3, 6, 9, 12, 15, …

From Table 5.3, observe that each term in the sequence can be obtained by multiplying its position $n$ by 3.

| **Position $n$** | 1 | 2 | 3 | 4 | 5 | … | $n$ |
|---|---|---|---|---|---|---|---|
| **Term $T_n$** | $3 \times 1 = 3$ | $3 \times 2 = 6$ | $3 \times 3 = 9$ | $3 \times 4 = 12$ | $3 \times 5 = 15$ | … | |

Table 5.3

Therefore, $T_n =$ ☐.

$100^{th}$ term, $T_{100} =$ ☐

**(b)** **Perfect squares**: 1, 4, 9, 16, 25, …

From Table 5.4, observe that each term in the sequence can be obtained by squaring its position $n$.

| **Position $n$** | 1 | 2 | 3 | 4 | 5 | … | $n$ |
|---|---|---|---|---|---|---|---|
| **Term $T_n$** | $1^2 = 1$ | $2^2 = 4$ | $3^2 = 9$ | $4^2 = 16$ | $5^2 = 25$ | … | |

Table 5.4

Therefore, $T_n =$ ☐.

$100^{th}$ term, $T_{100} =$ ☐

**(c)** **Perfect cubes**: 1, 8, 27, 64, 125, …

From Table 5.5, observe that each term in the sequence can be obtained by cubing its position $n$.

| **Position $n$** | 1 | 2 | 3 | 4 | 5 | … | $n$ |
|---|---|---|---|---|---|---|---|
| **Term $T_n$** | $1^3 = 1$ | $2^3 = 8$ | $3^3 = 27$ | $4^3 = 64$ | $5^3 = 125$ | … | |

Table 5.5

Therefore, $T_n =$ ☐.

$100^{th}$ term, $T_{100} =$ ☐

**Worked Example 2**

### Finding specific term

Given that the $n^{\text{th}}$ term, $T_n$, of a sequence is $T_n = 5n - 3$, find

**(i)** the $3^{\text{rd}}$ term,

**(ii)** the difference between the $3^{\text{rd}}$ term and the $5^{\text{th}}$ term,

of the sequence.

**Solution**

**(i)** $T_3 = 5(3) - 3$
$= 15 - 3$
$= 12$

**(ii)** $T_5 = 5(5) - 3$
$= 25 - 3$
$= 22$

Difference between the $3^{\text{rd}}$ term and the $5^{\text{th}}$ term of the sequence $= T_5 - T_3$
$= 22 - 12$
$= 10$

**Practise Now 2**

Similar and Further Questions
**Exercise 5A**
Questions 2, 11

Given that the $n^{\text{th}}$ term, $T_n$, of a sequence is $T_n = 4n + 7$, find

**(i)** the $4^{\text{th}}$ term,

**(ii)** the sum of the $4^{\text{th}}$ term and the $7^{\text{th}}$ term,

of the sequence.

## C. General term of linear sequences

Consider the sequence 2, 5, 8, 11, 14, …

Notice that the difference between *consecutive* (or successive) terms are all equal to a constant.

The **common difference** of this sequence is 3.

| | | | | | |
|---|---|---|---|---|---|
| Position $n$ | 1 | 2 | 3 | 4 | 5 |
| Term $T_n$ | 2, | 5, | 8, | 11, | 14, ... |

$$2 \xrightarrow{+3} 5 \xrightarrow{+3} 8 \xrightarrow{+3} 11 \xrightarrow{+3} 14$$

Such a sequence is called a **linear sequence** because the formula for the general term is a linear expression. But how do we find a formula for the general term?

**Method 1: Looking for pattern**

We can express each term as follows:

$$\begin{aligned}
T_1 &= 2 &&= 2 &&= 2 + 0 \times 3\\
T_2 &= 5 &&= 2 + 3 &&= 2 + 1 \times 3\\
T_3 &= 8 &&= 2 + 3 + 3 &&= 2 + 2 \times 3\\
T_4 &= 11 &&= 2 + 3 + 3 + 3 &&= 2 + 3 \times 3\\
T_5 &= 14 &&= 2 + 3 + 3 + 3 + 3 &&= 2 + 4 \times 3\\
T_n &= &&2 + \underbrace{3 + 3 + 3 + \ldots + 3}_{(n-1) \text{ terms}} &&= 2 + (n - 1) \times 3
\end{aligned}$$

**Problem-solving Tip**

To find a formula for the general term, observe

**(i)** which numbers or operations remain the same (in this case, "2 +" and "× 3"), and

**(ii)** which numbers change and how they change with the position $n$ of the term (in this case, the number before "× 3", which is 1 less than $n$).

By looking at the pattern on the previous page, we can infer that $T_n = 2 + (n - 1) \times 3$
$= 2 + 3n - 3$
$= 3n - 1.$

**Method 2: Comparing with another more familiar sequence**

Let us try another method to find a formula for the general term of the same sequence 2, 5, 8, 11, 14, …

Since the *common difference is 3*, we can compare the original sequence with a sequence which consists of terms that are *multiples of 3*:

| Position $n$ | 1 | 2 | 3 | 4 | 5 | $n$ |
|---|---|---|---|---|---|---|
| Term $T_n$ | 2, | 5, | 8, | 11, | 14, ... | $3n - 1$ |
| | ↑ − 1 | ↑ − 1 | ↑ − 1 | ↑ − 1 | ↑ − 1 | ↑ − 1 |
| | 3, | 6, | 9, | 12, | 15, ... | $3n$ |

Therefore, the general term of the original sequence is $T_n = 3n - 1$.

**Method 3: Connecting the sequence to its graph (or by observation)**

This is the *fastest* method. After understanding how it works, we will be able to write down the formula for the general term of a sequence straightaway *without drawing the graph*.

Consider the same sequence 2, 5, 8, 11, 14, …

If we plot $T_n$ against $n$, we will get points that lie on a straight line as shown in Fig. 5.1**(a)**. This is why this sequence is called a **linear sequence**.

Let the equation of the straight line be $T_n = mn + c$ (in the form of $y = mx + c$).

From Fig. 5.1**(b)**, gradient $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{3}{1} = 3 = d$, where $d$ is the common difference,

and $T_n$-intercept, $c = 2 - 3 = -1$.

Is there a way to find $c$ without looking at the graph?

When $n = 0$, $T_0 = c$, where $T_0$ is the term just before the first term $T_1$.

Thus we can find $c = T_0 = 2 - 3 = -1$ *mentally* as follows:

| $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | |
|---|---|---|---|---|---|---|
| ?, | 2, | 5, | 8, | 11, | 14, | ... |
| − 3 | + 3 | + 3 | + 3 | + 3 | + 3 | |

Hence, the general term is $T_n = 3n - 1$.

Worked Example 3 shows how to find a formula for the general term of a sequence without drawing the graph.

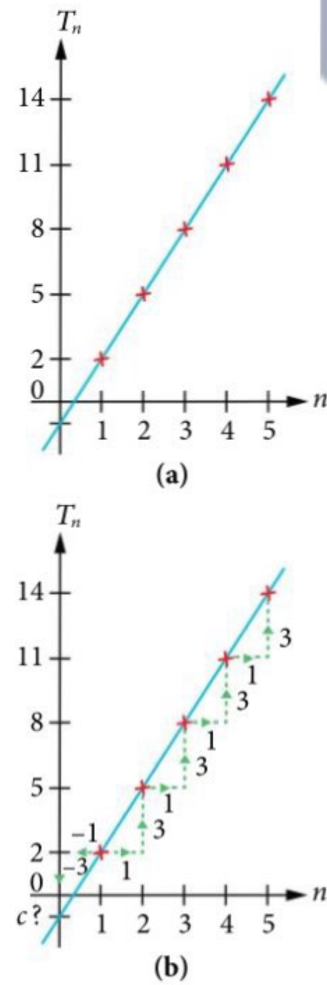



Fig. 5.1

**Big Idea**

**Functions**

A function may be represented in the form of a straight line graph.

The graphs in Fig. 5.1 show that each input $n$ has a unique output $T_n$.

Worked Example 3

### Finding general term of linear sequence by observation

Find a formula for the general term of the sequence 7, 11, 15, 19, 23, …

**Solution**

| $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | |
|---|---|---|---|---|---|---|
| ?, | 7, | 11, | 15, | 19, | 23, | … |
| − 4 | + 4 | + 4 | + 4 | + 4 | + 4 | |

Since the common difference is 4, $T_n = 4n + ?$.

The term before $T_1$ is $c = T_0$

$= 7 - 4$

$= 3.$

$\therefore$ general term of the sequence, $T_n = 4n + 3$

**Problem-solving Tip**

You can work out the solution *mentally* and write down the formula straightaway.

**Reflection**

Do you prefer to use this Method 3 (by observation), or the other methods from pages 156-157? Why?

**Practise Now 3**

Similar and Further Questions
Exercise 5A
Questions 3(a)–(d), 4, 5, 6(a), (b), 12, 13(a), (b)

1. Find a formula for the general term of each of the following sequences.
   (a) 5, 9, 13, 17, 21, …
   (b) 7, 12, 17, 22, 27, …
   (c) 2, 8, 14, 20, 26, …
   (d) 1, 4, 7, 10, 13, …
2. Consider the sequence −10, −2, 6, 14, 22, …
   (i) Write down the next two terms of the sequence.
   (ii) Find, in terms of $n$, a formula for the $n^{th}$ term of the sequence.
   (iii) Hence, find the $50^{th}$ term.

## D. General term of quadratic sequences

Consider the sequence 0, 3, 8, 15, 24, …

The difference between consecutive terms is not constant. Thus, this is not a linear sequence.

However, if we consider the sequence of first differences, we will notice that the *second difference* is 2:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Position $n$ | 1 | | 2 | | 3 | | 4 | | 5 |
| Term $T_n$ | 0, | | 3, | | 8, | | 15, | | 24, … |
| First Difference | | + 3 | | + 5 | | + 7 | | + 9 | |
| Second Difference | | | + 2 | | + 2 | | + 2 | | |

Number sequences with a **common (non-zero) second difference** are **quadratic sequences**.

Quadratic sequences are so called because the general term of the sequence is a quadratic expression in the form $an^2 + bn + c$, where $a$, $b$ and $c$ are constants and $a \neq 0$.

We can determine the general term of a quadratic sequence using the following methods.

**Method 1: Looking for pattern**

We can express each term as follows:

$$\begin{aligned}
T_1 &= 0 &&= 0 \times 2 = (1 - 1) \times (1 + 1)\\
T_2 &= 3 &&= 1 \times 3 = (2 - 1) \times (2 + 1)\\
T_3 &= 8 &&= 2 \times 4 = (3 - 1) \times (3 + 1)\\
T_4 &= 15 &&= 3 \times 5 = (4 - 1) \times (4 + 1)\\
T_5 &= 24 &&= 4 \times 6 = (5 - 1) \times (5 + 1)\\
T_n &= &&= (n - 1) \times (n + 1)
\end{aligned}$$

We can thus infer that $T_n = (n - 1)(n + 1) = n^2 - 1$.

**Method 2: Comparing with another more familiar sequence**

If we compare the sequence with a sequence of perfect squares, we can see that each term in the original sequence can be obtained by subtracting 1 from the corresponding perfect square:

$$\begin{array}{lcccccc}
\text{Position } n & 1 & 2 & 3 & 4 & 5 & n\\
\text{Term } T_n & 0, & 3, & 8, & 15, & 24, \ldots & n^2 - 1\\
 & \uparrow -1 & \uparrow -1 & \uparrow -1 & \uparrow -1 & \uparrow -1 & \uparrow -1\\
 & 1, & 4, & 9, & 16, & 25, \ldots & n^2
\end{array}$$

Therefore, the general term of the original sequence is $T_n = n^2 - 1$.

**Method 3: Connecting the sequence to the general expression $an^2 + bn + c$**

It was mentioned that $T_n = an^2 + bn + c$ for a quadratic sequence. We can obtain the following algebraic expressions for the first five terms of the sequence by substituting $n = 1$ to 5 into $T_n = an^2 + bn + c$. The first and second differences in terms of $a$, $b$ and $c$ can also be found.

$$\begin{array}{lrcc}
 & & \underline{\text{First Difference}} & \underline{\text{Second Difference}}\\
T_1 = (1)^2a + (1)b + c = & a + b + c & & \\
 & & +(3a + b) & \\
T_2 = (2)^2a + (2)b + c = & 4a + 2b + c & & +2a\\
 & & +(5a + b) & \\
T_3 = (3)^2a + (3)b + c = & 9a + 3b + c & & +2a\\
 & & +(7a + b) & \\
T_4 = (4)^2a + (4)b + c = & 16a + 4b + c & & +2a\\
 & & +(9a + b) & \\
T_5 = (5)^2a + (5)b + c = & 25a + 5b + c & &
\end{array}$$

Now let us consider the same sequence 0, 3, 8, 15, 24, …, which has a second difference of 2 as we have seen on page 158. We can form a set of equations by equating

- the second difference: $2a = 2$
- the first difference between $T_1$ and $T_2$: $3a + b = 3$
- $T_1$: $a + b + c = 0$

The solutions to the three equations are $a = 1$, $b = 0$ and $c = -1$, and the general term of the sequence is

$$\begin{aligned}
T_n &= an^2 + bn + c\\
&= (1)n^2 + (0)n + (-1)\\
&= n^2 - 1.
\end{aligned}$$

This method is especially useful for quadratic sequences where the pattern is not apparent, or where it is challenging to find a familiar sequence to which the terms can be compared. The sequence in Worked Example 4 is one such example.

Worked Example 4

### Finding general term of quadratic sequence

Find a formula for the general term of the sequence 6, 13, 24, 39, 58, …

**Solution**

The first and second differences of the sequence are shown below:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Position $n$ | 1 | | 2 | | 3 | | 4 | | 5 |
| Term $T_n$ | 6, | | 13, | | 24, | | 39, | | 58, ... |
| First Difference | | + 7 | | + 11 | | + 15 | | + 19 | |
| Second Difference | | | + 4 | | + 4 | | + 4 | | |

$\therefore 2a = 4$    equate second difference
$a = 2$

$3a + b = 7$    equate first difference between $T_1$ and $T_2$
$3(2) + b = 7$    substitute value of $a$
$6 + b - 6 = 7 - 6$    subtract 6 from both sides
$b = 1$

$a + b + c = 6$    equate $T_1$
$2 + 1 + c = 6$    substitute values of $a$ and $b$
$3 + c - 3 = 6 - 3$    subtract 3 from both sides
$c = 3$

$\therefore$ general term of the sequence, $T_n = 2n^2 + n + 3$.

**Problem-solving Tip**

It is a good practice to substitute a value of $n$ into the formula to check if it is correct.

By substituting $n = 4$, we get $T_4 = 2(4)^2 + 4 + 3 = 39$, which is the fourth term of the given sequence.

**Practise Now 4**

Similar and Further Questions
Exercise 5A
Questions 7, 14(a), (b), 15

1. Find a formula for the general term of each of the following sequences.
   (a) 3, 7, 13, 21, 31, …
   (b) $1\frac{1}{2}$, 5, $9\frac{1}{2}$, 15, $21\frac{1}{2}$, …
   (c) −6, −16, −30, −48, −70, …
   (d) −0.25, −1, −2.25, −4, −6.25, …
2. The third term of a quadratic sequence is 25.
   If the difference between the first and second terms of the sequence is 2, and the sequence has a second difference of 4, determine
   (i) the first term, and
   (ii) the general term
   of the sequence.

## E. General term of cubic sequences

In Section 5.1D, we have learnt that a quadratic sequence has a common second difference. How do we identify a cubic sequence?

## Investigation — Identifying cubic sequence

The general term of a cubic sequence has the form $an^3 + bn^2 + cn + d$, where $a$, $b$, $c$ and $d$ are constants and $a \neq 0$.

**1.** Fill in the blanks.

| | | First Difference | Second Difference |
|---|---|---|---|
| $T_1 = (1)^3a + (1)^2b + (1)c + d$ | $= a + b + c + d$ | | |
| | | $+ (7a + 3b + c)$ | |
| $T_2 = (2)^3a + (2)^2b + (2)c + d$ | $= \underline{\qquad}$ | | $+ (12a + 2b)$ |
| | | $+ (19a + 5b + c)$ | |
| $T_3 = (\underline{\quad})^3a + (\underline{\quad})^2b + (\underline{\quad})c + d$ | $= \underline{\qquad}$ | | $+ (\underline{\quad}a + \underline{\quad}b)$ |
| | | $+ (\underline{\quad}a + \underline{\quad}b + c)$ | |
| $T_4 = \underline{\qquad\qquad}$ | $= \underline{\qquad}$ | | $+ (\underline{\qquad\qquad})$ |
| | | $+ (\underline{\qquad\qquad})$ | |
| $T_5 = \underline{\qquad\qquad}$ | $= \underline{\qquad}$ | | |

**2.** Write down the algebraic expression for the third difference of a cubic sequence. What can you conclude about the value of the third difference of a cubic sequence?

From the above Investigation, we see that cubic sequences have a **common (non-zero) third** difference. For a cubic sequence of the form $an^3 + bn^2 + cn + d$, the algebraic expression for the third difference is $6a$.

Like quadratic sequences, we can form a set of equations to determine the values of the coefficients $a$, $b$, $c$ and $d$, and hence the general term of a cubic sequence. This is illustrated in Worked Example 5.

### Worked Example 5 — Finding general term of cubic sequence

Find a formula for the general term of the sequence 9, 36, 107, 246, 477, …

**Solution**

The first and second differences of the sequence are shown below:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Position $n$ | 1 | | 2 | | 3 | | 4 | | 5 | |
| Term $T_n$ | 9, | | 36, | | 107, | | 246, | | 477, | … |
| First Difference | | + 27 | | + 71 | | + 139 | | + 231 | | |
| Second Difference | | | + 44 | | + 68 | | + 92 | | | |
| Third Difference | | | | + 24 | | + 24 | | | | |

$$6a = 24$$ equate third difference
$$a = 4$$

$$12a + 2b = 44$$ equate second difference between first differences of $T_1$ and $T_2$ (= 27), and of $T_2$ and $T_3$ (= 71)
$$12(4) + 2b = 44$$ substitute value of $a$
$$12(4) + 2b - 48 = 44 - 48$$ subtract 48 from both sides
$$2b = -4$$
$$b = -2$$

$$\begin{aligned} 7a + 3b + c &= 27 && \text{equate first difference between } T_1 \text{ and } T_2 \\ 7(4) + 3(-2) + c &= 27 && \text{substitute values of } a \text{ and } b \\ c &= 5 \end{aligned}$$

$$\begin{aligned} a + b + c + d &= 9 && \text{equate } T_1 \\ 4 + (-2) + 5 + d &= 9 && \text{substitute values of } a, b \text{ and } c \\ d &= 2 \end{aligned}$$

$\therefore$ general term of the sequence, $T_n = 4n^3 - 2n^2 + 5n + 2$.

**Practise Now 5**

Similar and Further Questions
**Exercise 5A**
Questions 8, 16(a), (b)

**1.** Find a formula for the general term of each of the following sequences.
**(a)** 10, 26, 58, 112, 194, … **(b)** 2, 0, −18, −64, −150, …
**(c)** 7, 6, −7, −38, −93, … **(d)** 4.5, 20, 48.5, 96, 168.5, …

**2.** Given the sequence 0, −7, −31, −81, −166, …
**(i)** Show that the sequence is cubic.
**(ii)** Find, in terms of $n$, a formula for the general term of the sequence.
**(iii)** Hence, find the 15th term.

## F. General term of sequences with exponential patterns

In the Class Discussion on page 153, we came across the sequence 1, 3, 9, 27, 81, …

Each successive number in this sequence increases by the *same factor*, 3.

$$1 \xrightarrow{\times 3} 3 \xrightarrow{\times 3} 9 \xrightarrow{\times 3} 27 \xrightarrow{\times 3} 81, \quad \ldots$$

In another words, the sequence has a **common ratio** of 3:

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots = \frac{T_n}{T_{n-1}} = 3.$$

As a result, the terms in this sequence increases *exponentially*.
How do we find the general term of a number sequence with an exponential pattern?

**Attention**

A number sequence also exhibits an exponential pattern if each term *decreases* by the same factor.

## Investigation

**1.** We can express each term as follows:

$$\begin{aligned} T_1 &= 1 &&= 1 &&= 1 \times 3^0 \\ T_2 &= 3 &&= 1 \times 3 &&= 1 \times 3^1 \\ T_3 &= 9 &&= 1 \times 3 \times 3 &&= 1 \times 3^2 \\ T_4 &= 27 &&= 1 \times 3 \times 3 \times 3 &&= 1 \times 3^3 \\ T_5 &= 81 &&= 1 \times 3 \times 3 \times 3 \times 3 &&= 1 \times 3^4 \\ T_n &= &&= 1 \times 3 \times 3 \times \ldots \times 3 &&= 1 \times 3^{(n-1)}. \end{aligned}$$

$\therefore$ the general term of the sequence is $T_n = 1 \times 3^{(n-1)} = \boxed{\phantom{3^{n-1}}}$.

2. Let us now consider the sequence 2, 6, 18, 54, 162, ….
   (a) The common ratio in this sequence is also 3 (same as the sequence in Question 1), but the terms in this sequence are different from those in Question 1. How are they different?
   (b) Copy and complete the following:

$$
\begin{aligned}
T_1 &= 2 &&= 2 &&= 2 \times 3^0\\
T_2 &= 6 &&= 2 \times 3 &&= 2 \times 3^1\\
T_3 &= 18 &&= 2 \times \square \times \square &&= 2 \times \square\\
T_4 &= 54 &&= 2 \times \square \times \square \times \square &&= \square \times \square\\
T_5 &= 162 &&= \square &&= \square \times \square\\
T_n &= &&= \square &&= \square \times \square
\end{aligned}
$$

   $\therefore$ the general term of the sequence is $T_n = \square \times \square = \square$.

3. A third sequence consists of the terms 4, 20, 100, 500, 2500, ...
   (a) What is the common ratio of this sequence?
   (b) From your observation in Questions 1 and 2, write down the general term of this sequence.
   (c) Use your answer in Question 3(**b**) to determine the value of $T_5$. Does this value agree with the fifth term of the given sequence?

From the above Investigation, we see that for a number sequence with an exponential pattern, the formula for its general term, $T_n$, is given by $T_n = ar^{(n-1)}$, where $a$ is the first term of the sequence and $a \neq 0$, and $r$ is the common ratio.

**Worked Example 6**

### Finding general term of exponential sequence

Find a formula for the general term of each of the following sequences.

**(a)** 8, 40, 200, 1000, 5000, …
**(b)** 22, 2.2, 0.22, 0.022, 0.0022, …

**Solution**

**(a)**

| Position $n$ | 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|
| Term $T_n$ | 8, | 40, | 200, | 1000, | 5000, | ... |

$8 \xrightarrow{\times 5} 40 \xrightarrow{\times 5} 200 \xrightarrow{\times 5} 1000 \xrightarrow{\times 5} 5000$

The common ratio is 5.

$$
\begin{aligned}
\therefore T_n &= ar^{(n-1)}\\
&= 8[5^{(n-1)}]
\end{aligned}
$$

**Attention**

The general term cannot be simplified to $T_n = 40^{(n-1)}$ because terms with powers, i.e. $5^{(n-1)}$, are evaluated before multiplication.

**(b)**

| Position $n$ | 1 | | 2 | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| Term $T_n$ | 22, | | 2.2, | | 0.22, | | 0.022, | | 0.0022, | ... |
| | | $\times \frac{1}{10}$ | | $\times \frac{1}{10}$ | | $\times \frac{1}{10}$ | | $\times \frac{1}{10}$ | | |

The common ratio is $\frac{1}{10}$.

$$\therefore T_n = ar^{(n-1)}$$
$$= 22\left[\left(\frac{1}{10}\right)^{(n-1)}\right]$$

**Practise Now 6**

Similar and Further Questions
**Exercise 5A**
Questions 9, 17(a), (b)

1. Find a formula for the general term of each of the following sequences.
   **(a)** 1, 2, 4, 8, 16, ...
   **(b)** $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
   **(c)** −3, −4.5, −6.75, −10.125, −15.1875, …
   **(d)** −1, −0.2, −0.04, −0.008, −0.0016, …
2. Given that the third to fifth terms of an exponential sequence are 3, 6 and 12 respectively, determine
   **(i)** the first term,
   **(ii)** a formula for the general term,
   **(iii)** the tenth term,
   of the sequence.

## Reflection

1. What do I already know about number sequences that could guide my learning in this section?
2. How do I tell if a number sequence is linear, quadratic, cubic or exponential?
3. What are some methods to find the general term of a number sequence?

Advanced Intermediate Basic

## Exercise 5A

**1.** For each of the following sequences, state the rule and write down the next two terms.
(a) 14, 19, 24, 29, 34, …
(b) 80, 72, 64, 56, 48, …
(c) 6, 12, 24, 48, 96, 192, …
(d) 1600, 800, 400, 200, 100, …
(e) −16 384, 4096, −1024, 256, −64, …
(f) 9, −18, 36, −72, 144, …
(g) −52, −59, −66, −73, −80, …
(h) −100, −90, −80, −70, −60, …

**2.** Given that the $n^{th}$ term, $T_n$, of a sequence is $T_n = 2n + 5$, find
(i) the $5^{th}$ term,
(ii) the $8^{th}$ term,
(iii) the lowest common multiple of the $5^{th}$ term and the $8^{th}$ term,
of the sequence.

**3.** Find a formula for the general term of each of the following sequences.
(a) 7, 13, 19, 25, 31, …
(b) −4, −1, 2, 5, 8, …
(c) 60, 67, 74, 81, 88, …
(d) 14, 11, 8, 5, 2, …

**4.** Consider the sequence 7, 14, 21, 28, 35, …
(i) Write down the next two terms of the sequence.
(ii) Find, in terms of $n$, a formula for the $n^{th}$ term of the sequence.
(iii) Hence, find the $105^{th}$ term.

**5.** Consider the sequence 10, 14, 18, 22, 26, …
(i) Write down the next two terms of the sequence.
(ii) Find, in terms of $n$, a formula for the $n^{th}$ term of the sequence.
(iii) Hence, find the $200^{th}$ term.

**6.** (a) The $n^{th}$ term of a sequence is given by $12 - 5n$. Write down the first three terms of the sequence.
(b) The first four terms of a different sequence are 5, 12, 19 and 26. Find
(i) an expression for the $n^{th}$ term of this sequence,
(ii) the $15^{th}$ term,
(iii) the value of $k$ if the $k^{th}$ term in the sequence is 222.

**7.** Consider the sequence 0, 3, 10, 21, 36, …
(i) Show that this sequence is quadratic.
(ii) Find, in terms of $n$, a formula for the general term of the sequence.
(iii) Hence, find the difference between the $80^{th}$ and $81^{st}$ terms.

**8.** The first five terms of a sequence are 4, 6, 16, 40 and 84.
(i) Explain if the terms belong to a quadratic or a cubic sequence.
(ii) Find, in terms of $n$, a formula for the general term of the sequence.
(iii) Determine the $21^{st}$ term.

**9.** The $n^{th}$ term of a sequence is given by $ar^{(n-1)}$. Find
(i) the values of $a$ and $r$ if the first two terms of the sequence are 3 and 48; and
(ii) the third, fourth and fifth terms of the sequence.

**10.** Write down the next two terms of each of the following sequences.
(a) −6, −5, −3, 0, 4, …
(b) 47, 38, 30, 23, 17, …
(c) −50, −45, −44, −39, −38, …
(d) 100, 98, 95, 93, 90, …

## Exercise 5A

**11.** Given that the $n^{\text{th}}$ term, $T_n$, of a sequence is $T_n = 8n + 3$,
**(i)** find the $57^{\text{th}}$ term of the sequence.
**(ii)** Joyce says that all the terms in the sequence are odd numbers. Do you agree? Explain your answer.

**12.** Consider the sequence 1, 8, 27, 64, 125, …
**(i)** Write down the next two terms of the sequence.
**(ii)** Find, in terms of $n$, an expression for the $n^{\text{th}}$ term of the sequence.
**(iii)** The $p^{\text{th}}$ term of the sequence is 3375. Find the value of $p$.

**13.** **(a)** Consider the sequence −3, −10, −17, −24, −31, …
**(i)** Write down the next two terms of the sequence.
**(ii)** Find, in terms of $n$, a formula for the $n^{\text{th}}$ term of the sequence.
**(b)** The first four terms of another sequence are −93, −100, −107, −114.
**(i)** By comparing this sequence with the sequence in part **(a)**, write down, in terms of $n$, a formula for the $n^{\text{th}}$ term of the sequence.
**(ii)** Is −268 a term in the sequence? Explain your answer.

**14.** **(a)** The $n^{\text{th}}$ term of a sequence is given by $2n^2 + 1$. Write down the first four terms of the sequence.
**(b)** The first four terms of another sequence are 1, 7, 17, 31.
**(i)** By comparing this sequence with the sequence in part **(a)**, write down, in terms of $n$, a formula for the $n^{\text{th}}$ term of the sequence.
**(ii)** Hence, find the $38^{\text{th}}$ term.

**15.** The $n^{\text{th}}$ term of a sequence is $an^2 + bn + c$ where $a$, $b$ and $c$ are constants. It is given that the first two terms of the sequence are both 1, and the sequence has a common second difference of 8.
**(i)** Find the values of $a$, $b$ and $c$.
**(ii)** Hence write down the next three terms of the sequence.

**16.** The $n^{\text{th}}$ term of a sequence is $an^3 - 2n^2 + n + d$, where $a$ and $d$ are constants. The sequence has a common third difference of 12.
**(a)** If the first term of the sequence is 4,
**(i)** find the values of $a$ and $d$,
**(ii)** write down the next four terms of the sequence.
**(b)** The first five terms of another sequence are 0, 7, 38, 99 and 204.
**(i)** By comparing this sequence with the one in part **(a)**, write down, in terms of $n$, a formula for the $n^{\text{th}}$ term of the sequence.
**(ii)** Hence, find the $75^{\text{th}}$ term of this sequence.

**17.** **(a)** The first five terms of a sequence are 3, 12, 48, 192 and 768. Find, in term of $n$, a formula for the general term of this sequence.
**(b)** The first five terms of another sequence are 4, 16, 57, 208 and 793.
**(i)** By comparing this sequence with the one in part **(a)**, write down, in terms of $n$, a formula for the $n^{\text{th}}$ term of this sequence.
**(ii)** Hence, find the $10^{\text{th}}$ term of this sequence.

**18.** Write down the next two terms of each of the following sequences.
**(a)** −5, −7, −11, −19, −35, …
**(b)** 1, 1, 2, 3, 5, …
**(c)** 4, 16, 36, 64, 100, …
**(d)** 1, −8, 27, −64, 125, …

# 5.2 Number sequences and patterns

In this section, we shall apply what we have learnt about number sequences to sequences involving geometrical figures and other types of number patterns, including patterns in real-world contexts.

## A. Sequences involving geometrical figures and other types of number patterns

**Worked Example 7**

**Sequence involving geometrical figures**

The first four figures of a sequence are as shown.

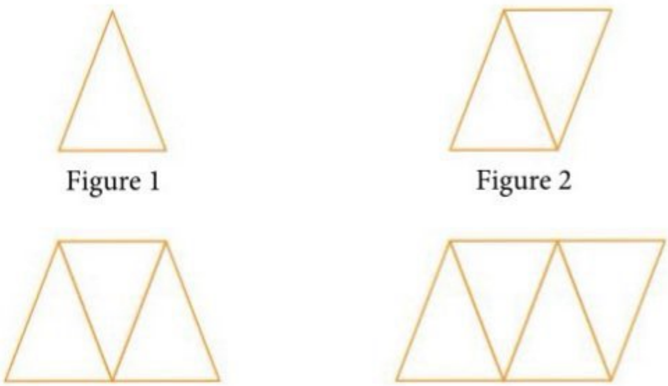


Figure 1 Figure 2 Figure 3 Figure 4

**(i)** Draw the next two figures of the sequence.

**(ii)** Copy and complete the table.

| Figure number | Number of triangles | Number of lines |
|---|---|---|
| 1 | 1 | $1 + 1 \times 2 = 3$ |
| 2 | 2 | $1 + 2 \times 2 = 5$ |
| 3 | 3 | $1 + 3 \times 2 = 7$ |
| 4 | 4 | $1 + 4 \times 2 = 9$ |
| 5 | | |
| 6 | | |
| ⋮ | ⋮ | ⋮ |
| $n$ | | |

**(iii)** Calculate the number of triangles and lines in the 121st figure.

**(iv)** Write down a formula connecting the number of triangles, $T$, and the number of lines, $L$, in the sequence shown above.

**(v)** Will there be a figure in this sequence that has 80 024 lines? Explain your answer.

**Solution**

(i)

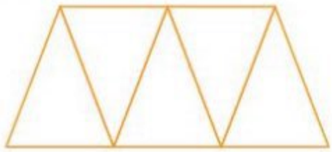


Figure 5

Figure 6

(ii)

| Figure number | Number of triangles | Number of lines |
|---|---|---|
| 1 | 1 | $1 + 1 \times 2 = 3$ |
| 2 | 2 | $1 + 2 \times 2 = 5$ |
| 3 | 3 | $1 + 3 \times 2 = 7$ |
| 4 | 4 | $1 + 4 \times 2 = 9$ |
| 5 | 5 | $1 + 5 \times 2 = 11$ |
| 6 | 6 | $1 + 6 \times 2 = 13$ |
| ⋮ | ⋮ | ⋮ |
| $n$ | $n$ | $1 + n \times 2 = 2n + 1$ |

**Problem-solving Tip**

To find a formula for the general term, observe

**(i)** which numbers or operations remain the same (in this case "1 +" and "× 2"), and

**(ii)** which numbers change and how they change with the position $n$ of the term (in this case, the number before "× 2" is the same as $n$).

**(iii)** In the 121st figure, $n = 121$.

$$\begin{aligned}\text{Number of triangles} &= n \\ &= 121\end{aligned}$$

$$\begin{aligned}\text{Number of lines} &= 2n + 1 \\ &= 2(121) + 1 \\ &= 243\end{aligned}$$

**(iv)** From part **(ii)**, number of triangles, $T = n$
and number of lines, $L = 2n + 1$.

$\therefore L = 2T + 1$

**(v)** **Method 1:**

For any value of $n$, $L = 2n + 1$ will always be an odd number.
This means that every figure in the sequence has an odd number of lines.
Since 80 024 is an even number, then no figure in this sequence has 80 024 lines.

**Method 2:**

$$\begin{aligned}2n + 1 &= 80\,024 \\ 2n &= 80\,024 - 1 \\ n &= 80\,023 \div 2 \\ &= 40\,011\frac{1}{2}\end{aligned}$$

Since $n$ is not a positive integer, no figure in this sequence has 80 024 lines.

**Reflection**

**(v)** Which method do you prefer? Why?

**Practise Now 7**

Similar and Further Questions
**Exercise 5B**
Questions 1–7

1. The first four figures of a sequence are as shown.

Figure 1 Figure 2 Figure 3 Figure 4

**(i)** Draw the next two figures of the sequence.
**(ii)** Copy and complete the table below.

| Figure number | Number of dots |
|---|---|
| 1 | $2 + 1 \times 4 = 6$ |
| 2 | $2 + 2 \times 4 = 10$ |
| 3 | $2 + 3 \times 4 = 14$ |
| 4 | $2 + 4 \times 4 = 18$ |
| 5 | |
| 6 | |
| $\vdots$ | $\vdots$ |
| $n$ | |

**(iii)** Find the number of dots in the 2020$^{\text{th}}$ figure.
**(iv)** Will there be a figure in this sequence that has 80 000 dots? Explain your answer.

2. Consider the following number sequence:

$$\begin{aligned} 2 &= 1 \times 2 \\ 6 &= 2 \times 3 \\ 12 &= 3 \times 4 \\ 20 &= 4 \times 5 \\ &\vdots \\ 110 &= k(k+1) \end{aligned}$$

**(i)** Write down the 8$^{\text{th}}$ line in the pattern.
**(ii)** Deduce the value of $k$.
**(iii)** Will there be a line with 342 on the left-hand side? Explain your answer.

3. The first four figures of a sequence are as shown.
This sequence is called the triangular number sequence because each term consists of equally-spaced dots in a triangle.

Figure 1 Figure 2 Figure 3 Figure 4

**(i)** Draw the next two figures of the sequence.

**(ii)** Copy and complete the table below.

Hint: Use the pattern in Question 2 to help you.

| Figure number, $n$ | Number of dots at the base of the triangle, $n$ | Total number of dots, $T_n$ |
|---|---|---|
| 1 | 1 | $1 = 1$ |
| 2 | 2 | $1 + 2 = 3$ |
| 3 | 3 | $1 + 2 + 3 = 6$ |
| 4 | 4 | $1 + 2 + 3 + 4 = 10$ |
| 5 | | |
| 6 | | |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | | |

**(iii)** Find the total number of dots needed to form a triangle with a base that has 100 dots.

## B. Number patterns in real-world contexts

Let us look at number patterns that can be found in the real world.

### Investigation — Fibonacci sequence

The number of petals of a flower follows a special type of sequence called the **Fibonacci sequence**, which is named after an Italian mathematician, also known as Leonardo of Pisa. He introduced this sequence in his book *'Liber Abaci'* in 1202.

Let us explore this phenomenon by looking at the flowers below.

**1.** Write down the number of petals each of the following flowers has. Some of them have been done for you.

Picture A: White Calla Lily ☐

Picture B: Euphorbia 2
(Note: There are 8 flowers in the picture.)

Picture C: Mariposa Lily 3

Picture D: Madagascar Periwinkle ☐

Picture E: Moonbeam Coreopsis 8

Picture F: Black-eyed Susan ☐

Picture G: Shasta Daisy ☐

Picture H: Sunflower 34

Fig. 5.2

2. The number of petals the flowers have forms a sequence.
   Fill in the next 6 terms of the sequence.
   1, 1, 2, ___, ___, ___, ___, ___, ___.
   This is known as the Fibonacci sequence. Explain the rule that generates this sequence.

3. A sunflower has 34 petals. Using your answer in Question 2, predict the number of petals for the flower next in the sequence. This flower is the Michaelmas Daisy (see Fig. 5.3 below).

Fig. 5.3

4. Note that there are four common exceptions to the Fibonacci sequence.
   Write down the number of petals each of the following flowers has.

Picture I: Ixora ___

Picture J: Daylily ___

Picture K: Anemone Nemorosa ___

Picture L: Passion Flower ___

Fig. 5.4

**Information**

There are flowers with the following number of petals: 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13 and 14 (except for 9). So where is the pattern?

Actually, the Fibonacci pattern comes mainly from daisies: 13, 21, 34, 55 and 89 petals.

What about roses? The roses we commonly encounter are hybrids with many number of petals, but naturally-occurring wild roses have only 5 petals.

## Introductory Problem Revisited

Buzz has 1 parent – the female queen bee. The female queen bee was hatched from a fertilised egg and so she has two parents – a female and a male. Therefore, Buzz has 2 grandparents – a female and a male.

His grandfather has only 1 parent and his grandmother has 2 parents. So Buzz will have 3 great-grandparents – 2 female and 1 male.

Since every male bee has only 1 female parent and every female bee has 2 parents, Buzz will have 5 great-great-grandparents – 3 female and 2 male. It also follows that Buzz will have 8 great-great-great-grandparents – 5 female and 3 male.

We can extend a similar reasoning to Betty's family tree.

Organising our observations in a table will help us to recognise the pattern better.

| | Number of parents | Number of grandparents | Number of great-grandparents | Number of great-great-grandparents | Number of great-great-great-grandparents |
|---|---|---|---|---|---|
| **Buzz** | 1 | 2 | 3 | 5 | 8 |
| **Betty** | 2 | 3 | 5 | 8 | 13 |

Table 5.8

Can you see that the family tree of bees follows the Fibonacci sequence exactly? The Fibonacci sequence is interesting as it is not merely a theoretical sequence but is also prevalent in nature such as the number of petals of a flower and the growth of bees. This is indeed a fascinating application of number patterns in real life!

**Worked Example 8**

### Number patterns in chemistry

The members in a family of chemical compounds are made up of carbon atoms and hydrogen atoms.

**(i)** The number of carbon atom(s) and hydrogen atoms of the first four members in the family are given in the table. Copy and complete the table.

| Member number | Number of carbon atom(s) | Number of hydrogen atoms |
|---|---|---|
| 1 | 1 | 4 |
| 2 | 2 | 6 |
| 3 | 3 | 8 |
| 4 | 4 | 10 |
| 5 | | |
| 6 | | |
| ⋮ | ⋮ | ⋮ |
| $n$ | | |

**(ii)** If a member of the family has 30 carbon atoms, how many hydrogen atoms does it have?

OPEN **(iii)** A member of the family has 52 hydrogen atoms and $(a + b)$ carbon atoms. Given that $a$ is an even number and $b$ is an odd number, find a possible value of $a$ and of $b$.

**Solution**

**(i)**

| Member number | Number of carbon atom(s) | Number of hydrogen atoms |
|---|---|---|
| 1 | 1 | 4 |
| 2 | 2 | 6 |
| 3 | 3 | 8 |
| 4 | 4 | 10 |
| 5 | 5 | 12 |
| 6 | 6 | 14 |
| ⋮ | ⋮ | ⋮ |
| $n$ | $n$ | $2n + 2$ |

**(ii)** When $n = 30$, $T_{30} = 2(30) + 2$
$= 62$

The member of the family has 62 hydrogen atoms.

**(iii)** Let $2n + 2 = 52$.

Then $2n = 52 - 2$
$= 50$
$n = 25$

$\therefore$ the member of the family has $a + b = 25$ carbon atoms.

So a possible value of $a$ and of $b$ is 2 and 23 respectively.

**Problem-solving Tip**

Consider the sequence 4, 6, 8, 10, …

| $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | |
|---|---|---|---|---|---|
| ?, | 4, | 6, | 8, | 10, | … |

$-2$ $\quad +2 \quad +2 \quad +2$

Since the common difference is 2, $T_n = 2n + ?$.

The term before $T_1$ is $c = T_0$
$= 4 - 2$
$= 2$.

$\therefore$ general term of the sequence, $T_n = 2n + 2$

**Practise Now 8**

Similar and Further Questions
**Exercise 5B**
Question 8

The members in a family of chemical compounds are made up of carbon atoms and hydrogen atoms.

**(i)** The number of carbon atoms and hydrogen atoms of the first four members in the family are given in the table. Copy and complete the table.

| Member number | Number of carbon atoms | Number of hydrogen atoms |
|---|---|---|
| 1 | 2 | 4 |
| 2 | 3 | 6 |
| 3 | 4 | 8 |
| 4 | 5 | 10 |
| 5 | | |
| 6 | | |
| ⋮ | ⋮ | ⋮ |
| $n$ | | |

**(ii)** If the $h^{\text{th}}$ member of the family has 55 carbon atoms, find the value of $h$. Hence, find the number of hydrogen atoms the member has.

**(iii)** If the $k^{\text{th}}$ member of the family has 120 hydrogen atoms, find the value of $k$. Hence, find the number of carbon atoms the member has.

OPEN Q **(iv)** A member of the family has 72 hydrogen atoms and $3(a + b)$ carbon atoms. Given that $a \geqslant 0$ and $a < b$, find a possible value of $a$ and of $b$.

## Reflection

1. How are the methods to find the general term of a sequence of geometrical figures the same as, or different from, those used to find the general term of a number sequence?
2. What have I learnt in this section or chapter that I am still unclear of?

Advanced | Intermediate | Basic

## Exercise 5B

**1.** The first four figures of a sequence are as shown.

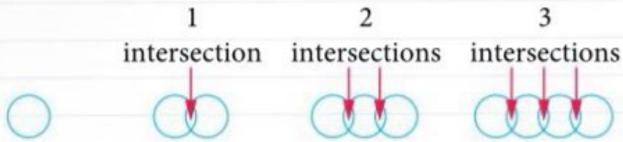



Figure 1 Figure 2 Figure 3 Figure 4

**(i)** Draw the next two figures of the sequence.

**(ii)** Copy and complete the table.

| Figure number | Number of intersection(s) between the circles |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| ⋮ | ⋮ |
| $n$ | |

**(iii)** Find the value of $n$ for which the circles in the figure have 28 intersections.

**2.** The first four figures of a sequence are as shown.

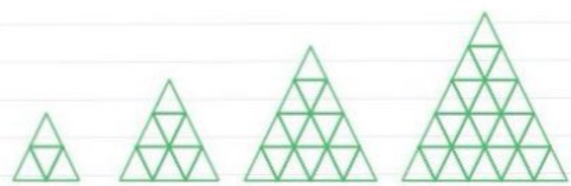


Figure 1 Figure 2 Figure 3 Figure 4

**(i)** Copy and complete the table.

| Figure number | Number of small triangles |
|---|---|
| 1 | 4 |
| 2 | 9 |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| ⋮ | ⋮ |
| $n$ | |

**(ii)** Find the number of small triangles in the $20^{\text{th}}$ figure.

**(iii)** Find the value of $n$ for which the figure has 121 small triangles.

**(iv)** Will there be a figure in this sequence that has 2400 small triangles? Explain your answer.

**3.** The diagram shows some patterns made from triangular tiles.

Pattern 1 Pattern 2 Pattern 3 Pattern 4

**(i)** Copy and complete the table.

| Pattern number | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Number of purple tiles | 1 | 2 | 3 | 4 | |
| Number of white tiles | 2 | 3 | 4 | 5 | |
| Total number of tiles | 3 | 5 | 7 | 9 | |

Basic | Intermediate | Advanced

## Exercise 5B

(ii) Find an expression, in terms of $n$, for
(a) the number of white tiles in Pattern $n$,
(b) the total number of tiles in Pattern $n$.
(iii) Find the total number of tiles in Pattern 32.
(iv) Will there be a pattern with a total of 501 tiles? Explain your answer.

**4.** Consider the following number pattern:

$$\begin{aligned} 4 &= 1 \times 4 \\ 10 &= 2 \times 5 \\ 18 &= 3 \times 6 \\ 28 &= 4 \times 7 \\ &\vdots \\ 208 &= k(k+3) \\ &\vdots \end{aligned}$$

(i) Write down the 6th line in the pattern.
(ii) Deduce the value of $k$.
(iii) Will there be a line with 1777 on the left-hand side? Explain your answer.

**5.** Consider the following number pattern:

$$\begin{aligned} 1+3 = {} & 4 = 2^2 = (1+1)^2 \\ 1+3+5 = {} & 9 = 3^2 = (2+1)^2 \\ 1+3+5+7 = {} & 16 = 4^2 = (3+1)^2 \\ 1+3+5+7+9 = {} & 25 = 5^2 = (4+1)^2 \\ & \vdots \\ 1+3+5+7+\ldots+a = {} & b = c^2 = (d+1)^2 \\ & \vdots \end{aligned}$$

(i) Write down the 5th line in the pattern.
(ii) Given that $b = 169$, find the values of $a$, $c$ and $d$.
(iii) Will there be a line in the pattern where $a = 86\ 868$? Explain your answer.

**6.** Albert opens an account with an investment fund that pays a fixed amount of bonus at the end of each year. The table below shows the amount of money in his account at the end of each year.

| Year | Amount of money ($) |
|---|---|
| 1 | |
| 2 | |
| 3 | 93 280 |
| 4 | 95 040 |
| 5 | 96 800 |
| 6 | 98 560 |
| ⋮ | ⋮ |
| $n$ | |

(i) Copy and complete the table.
(ii) Albert wants to close the investment account once the amount in it exceeds $120 000. After how many years would he close the account?

**7.** The first five rows of Pascal's Triangle are as shown.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \end{array}$$

(i) Write down the next line in the pattern.
(ii) Copy and complete the table.

| Row | Sum |
|---|---|
| 1 | $1 = 1 = 2^0$ |
| 2 | $1 + 1 = 2 = 2^1$ |
| 3 | $1 + 2 + 1 = 4 = 2^2$ |
| 4 | $1 + 3 + 3 + 1 = 8 = 2^3$ |
| 5 | $1 + 4 + 6 + 4 + 1 = 16 = 2^4$ |
| 6 | |
| ⋮ | ⋮ |
| $n$ | |

(iii) Find, in terms of $n$, for the sum of numbers in row $n$.
(iv) Will there be a line in the pattern with a sum of 3072? Explain your answer.

## Exercise 5B

8. The members in a family of chemical compounds are made up of carbon atoms and hydrogen atoms. The number of carbon atoms and hydrogen atoms of the first four members in the family are given in the table below.

| Member number | Number of carbon atoms | Number of hydrogen atoms |
|---|---|---|
| 1 | 3 | 4 |
| 2 | 4 | 6 |
| 3 | 5 | 8 |
| 4 | 6 | 10 |
| 5 | | |
| 6 | | |
| ⋮ | ⋮ | ⋮ |
| $n$ | | |

(i) Copy and complete the table.

(ii) If the $h^{\text{th}}$ member of the family has 25 carbon atoms, find the value of $h$. Hence, find the number of hydrogen atoms the member has.

OPEN (iii) If the $(j + k)^{\text{th}}$ member of the family has 64 hydrogen atoms, where $j$ is a prime number and $k$ is a composite number, find a possible set of values of $j$ and $k$. Hence, find the number of carbon atoms the member has.

## Looking Back

A famous mathematician, G. H. Hardy once said, "A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."

Although what happens around us may seem random, there are many regularities — things that repeat themselves or follow a rule — that we can observe. Figuring out the patterns behind a problem or phenomenon allows us to make predictions about the world. In a way, patterns **model** the regularity in the world we live in. Mathematics allows us to make sense of these patterns by enabling us to see relationships and form generalisations about patterns. For example, in Section 5.1, we have learnt how to find a formula for the general term of a number sequence. Knowing the general term enables us to generate the terms in the sequence and helps us to accurately predict any term in the sequence without having to find all the previous terms. In Section 5.2, we see how the Fibonacci sequence is embedded in nature. Understanding the Fibonacci sequence helps us to foresee the way things grow in nature. The ability to observe and recognise patterns is an important skill in helping us make good predictions.

In this chapter, we see that sequences can be viewed as **functions** because each term in the sequence (output) is uniquely determined by its term position (input) based on a specific rule. As shown in Section 5.1, functions may be presented in the form of a table and a straight line graph. These different representations show us different perspectives of the same patterns we observe in the real world or in mathematics.

## Summary

1. **Number sequence**
   A number sequence is formed by a set of **terms**, which are governed by a *specific rule*.
   An example of a number sequence is –3, 1, 5, 9, 13, 17, …
   The rule that generates this sequence is: start with –3, then add 4 to each term to get the next term.
   - Give another example of a number sequence and write down the rule that generates it.

2. **General term**
   The general term, $T_n$, of a number sequence can be represented by an algebraic expression in terms of its term position, $n$. We say that $T_n$ is a *function* of $n$.
   - Give an example of a number sequence and write down a formula for the general term.

3. **Methods of finding the general formula of a sequence:**
   **(a)** Looking for a pattern by examining the common difference or common ratio between consecutive terms in the sequence.
   **(b)** Comparing the original sequence with another more familiar sequence.
   **(c)** By connecting the sequence to its graph or a general expression.
   - Find, by observation, a formula for the general term of the sequence 2, 9, 16, 23, 30, …

4. Number patterns have useful applications in real life. For example, the Fibonacci sequence appears in patterns of growth in nature.

# Financial Transactions


In Book 1, we learnt the concepts of percentage, ratio, and rate. In the topic "Percentages", we were introduced to the symbol, %, that is used to represent 'percent'. The symbol is believed to evolve from a sign that was introduced in an anonymous Italian manuscript around 1425. Prior to this, scribes used abbreviations such as 'p 100' or 'p cento' to represent the Italian term *per cento*, which means 'for a hundred'. In the 1425 manuscript, the abbreviation was presented as "pc" with a tiny circle at the end. This eventually became two circles separated with a horizontal line by the mid-17th century, a symbol that is similar to the current %.

Percentages, together with ratios and rates, are frequently used in financial transactions such as in calculating profit, loss, inheritance and bank interest. What are some other financial transactions that involve ratios, rates and percentages that you have encountered?

## Learning Outcomes

What will we learn in this chapter?

- What percentage, ratio and rate are
- Why percentage has useful applications in real life, e.g. profit and loss, discount, General Sales Tax, commission, Zakat and Ushr
- Why ratio and rate have useful applications in real life, e.g. inheritance, partnership, insurance premium rate, interest rate and income tax rate

## Introductory Problem

Imran is interested in purchasing a new laptop and is considering a hire purchase option instead of making a one-time payment.

In the hire purchase option, Imran will pay a portion of the laptop price first, known as down payment. The remaining price of the laptop is then divided into monthly instalments, including interest.

The details of the hire purchase plan offered by the shop are as follows:

- Price of laptop = PKR 460 000
- Down payment = 20% of laptop price
- Interest rate = 8% per annum of remaining laptop price
- Repayment period = 12 months

**1.** If Imran decides to proceed with the hire purchase plan, calculate
   **(i)** the down payment for the laptop, and hence the remaining price of the laptop that will be repaid over 12 months,
   **(ii)** the total interest Imran will pay altogether,
   **(iii)** the monthly instalment Imran will need to pay, including interest.

**2.** Would you recommend that Imran take up the hire purchase plan? Explain the considerations in your recommendation.

# 6.1 Percentage, ratio and rate

## A. Percentage (recap)

In Book 1, we learnt that *percentage* can be expressed as a fraction and vice versa.

- To express a percentage to a fraction: $x\% = \frac{x}{100}$
- To express a fraction $\frac{a}{b}$ to a percentage: $\frac{a}{b} \times 100\%$

Using the equivalence of percentage and fraction, we can determine a quantity when given its percentage. For example, if 80% of a bouquet of 5 flowers are roses, we can determine that the number of roses = $\frac{80}{100} \times 5 = 4$.

If there is 1 lily in the bouquet, we can also determine the percentage of flowers that are lilies:

$$\text{percentage of lilies} = \frac{\text{number of lilies}}{\text{number of flowers}} \times 100\% = \frac{1}{5} \times 100\% = 20\%.$$

**Practise Now 1A**

1. Every month, Albert sets aside 32% of his salary as savings, 15% on transportation, PKR 180 000 on food and the rest on shopping. If his monthly salary is PKR 720 000, fir
   **(i)** the amount of money Albert sets aside as savings and transportation respectively,
   **(ii)** the percentage of his salary he spends on food.

2. At a concert, 40% of the audience are men and the rest are women. 24% of the men wear spectacles. If 60 men in the audience wear spectacles, determine
   **(i)** the number of men,
   **(ii)** the total number of people,
   in the audience.

## B. Ratio and rate (recap)

We have learnt in Book 1 that *ratios* are used to compare numbers or quantities of the same kind. In Section 6.1A, the bouquet of flowers consists of 4 roses and 1 lily. Therefore, the ratio of roses to lilies = 4 : 1.

We can also say there are 4 roses per lily. If there are 12 lilies, how many roses will there be?

The statement "4 roses per lily" is a *rate*. It measures how the number of roses changes with the number of lilies.

- A ratio $c : d$ compares two quantities $c$ and $d$ that either have no units or have the same units.
- A rate compares how one quantity changes with another quantity. The two quantities may or may not have the same unit.

**Practise Now 1B**

1. In a stationery store, the respective number of pens, pencils and notebooks are in the ratio 5 : 3 : 6. If there are a total of 126 pens, pencils and notebooks in the store, how many pens, pencils and notebooks are there respectively?

2. A machine in a toy factory produces 35 toys per hour.
   **(i)** Determine the number of toys the machine produces in 6 hours.
   **(ii)** A second machine in the same factory produces toys at a faster rate. If the second machine produces 92 more toys than the first machine after 4.6 hours, determine the rate at which the second machine produces toys.

In the following sections, we will be applying the concepts of percentage, ratio and rate in some financial transactions that we frequently encounter in our daily lives.

# 6.2 Profit, loss, discount, General Sales Tax and commission

## A. Profit and loss

A manufacturer produces goods at a certain cost.
If the goods are sold at a price *higher* than the cost price, the manufacturer makes a **profit** (or **gain**).
However, if the manufacturer sells the goods at a price *lower* than the cost price, he suffers a **loss**.

**Profit** = selling price – cost price
**Loss** = cost price – selling price

**Attention**

If the profit is negative, what does it mean?

We can also express the profit or loss as a percentage of the cost price:

$$\frac{\text{profit}}{\text{cost price}}\times 100\% \quad \text{or} \quad \frac{\text{loss}}{\text{cost price}}\times 100\%$$

In the real world, profit or loss can also be expressed as a percentage of the selling price:

$$\frac{\text{profit}}{\text{selling price}}\times 100\% \quad \text{or} \quad \frac{\text{loss}}{\text{selling price}}\times 100\%$$

**Information**

In the real world, $\frac{\text{profit}}{\text{selling price}}\times 100\%$ is called the profit margin. If the profit margin of an item is 30%, a shopkeeper will know that he cannot give a customer a discount of more than 30%. Why?

**Worked Example 1**

### Profit / loss as a percentage of cost price / selling price

A bag costs \$28.

**(a)** If the bag is sold for \$35, express the profit as a percentage of the cost price.

**(b)** If the bag is sold for \$24.50, express the loss as a percentage of the selling price.

**Solution**

**(a)** Profit = selling price – cost price
= \$35 – \$28
= \$7

Profit as percentage of cost price $= \frac{\$7}{\$28}\times 100\%$
$= 25\%$

**(b)** Loss = cost price – selling price
= \$28 – \$24.50
= \$3.50

Loss as percentage of selling price $= \frac{\$3.50}{\$24.50}\times 100\%$
$= 14.3\%$ (to 3 s.f.)

**Practise Now 1C**

Similar and Further Questions
**Exercise 6A**
Questions 1(a)-(d), 2, 11

1. **(a)** A bicycle which costs PKR 18 000 is sold for PKR 24 000. Express the as percentage of the cost price.
   **(b)** An antique chest which costs \$6000 is sold for \$5000. Express the loss as a percentage of the selling price.
2. **(a)** A gold chain which costs PKR 112 000 is sold at a profit of 27% on the cost price. Find its selling price.
   **(b)** A car which costs \$78 400 is sold at a loss of 6% on the cost price. Find its selling price.

**Worked Example 2**

### Finding cost price

A bookseller gains 30% on the cost price by selling a book for \$65. Calculate the cost price of the book.

**Solution**

$$130\% \text{ of the cost price} = \$65$$

$$1\% \text{ of the cost price} = \frac{\$65}{130}$$

$$100\% \text{ of the cost price} = \frac{\$65}{130} \times 100$$

$$= \$50$$

$\therefore$ cost price of book = \$50

**Practise Now 2**

Similar and Further Questions
**Exercise 6A**
Questions 1(e), (f), 3, 4, 12-15

1. A shopkeeper gains 35% on the cost price by selling a smartphone for PKR 287 550. Find the cost price of the smartphone.
2. A shopkeeper buys 1800 eggs at \$1.20 per dozen. 5% of the eggs are rotten and cannot be sold. Find the selling price of each egg if he wants to earn a 33% profit on the cost price.

**Thinking Time**

Waseem sells his camera at \$250.
What is his profit/loss expressed as a percentage of the cost price?
*To find the answer, what other information is necessary?*

## B. Discount

Very often, retailers are not able to sell defective merchandise, overstocked items or discontinued models at retail selling prices.

To clear the merchandise in stock, the items are usually sold at a lower price, called the **sale price**.

The original selling price is also known as the **marked price**.

The **discount** is the difference between the marked price and the sale price:

**Discount** = marked price – sale price

The discount is often given as a percentage of the marked price:

$$\textbf{Percentage discount} = \frac{\text{discount}}{\text{marked price}} \times 100\%$$

**Worked Example 3**

**Percentage discount**

A watch priced at \$160 is sold for \$100. Find the percentage discount.

**Solution**

Discount = marked price – sale price
= \$160 – \$100
= \$60

$$\text{Percentage discount} = \frac{\$60}{\$160} \times 100\%$$
$$= 37.5\%$$

**Practise Now 3**

Similar and Further Questions
Exercise 6A
Questions 5, 6, 20

1. A scarf priced at PKR 1000 is sold for PKR 880. Find the percentage discount.
2. The marked price of a washing machine is \$600. A discount of 6% is given during a sale. Find the sale price of the washing machine.

**Worked Example 4**

### Finding marked price

A sculpture is sold for \$533 after a discount of 18%.

**(i)** Calculate the marked price of the sculpture.

**(ii)** If a 10% discount is given on the marked price of the sculpture before it is sold at a further discount of 8%, would the sale price still be \$533? Show your working clearly.

**Solution**

**(i)**

$$\begin{aligned}
82\% \text{ of the marked price} &= \$533 \\
1\% \text{ of the marked price} &= \frac{\$533}{82} \\
100\% \text{ of the marked price} &= \frac{\$533}{82} \times 100 \\
&= \$650
\end{aligned}$$

$\therefore$ marked price of sculpture = \$650

**(ii)** Sale price of sculpture after a 10% discount $= \frac{90}{100} \times \$650$

$= \$585$

Sale price of sculpture after a further discount of 8%

$= \frac{92}{100} \times \$585$

$= \$538.20$

No, the sale price would not be \$533.

**Attention**

The price used for the first 10% discount is different from the price used for the further 8% discount. Note that the further 8% discount is only given after the 10% discount has been given. Hence, the discount is not the same as taking 10% + 8% = 18%.

**Practise Now 4**

Similar and Further Questions
**Exercise 6A**
Questions 7, 16, 17

1. A laptop is sold for PKR 111 020 after a discount of 9%.
   **(i)** Find the marked price of the laptop.
   **(ii)** If a 5% discount is given on the marked price of the laptop before it is sold at a further discount of 4%, would the sale price still be PKR 111 020? Show your working clearly.
2. During a sale, a department store offers a 10% discount on all items. Members of the store have an additional 15% discount.
   **(i)** A non-member bought a handbag for \$180 during the sale. Find the marked price of the handbag.
   **(ii)** A member buys the same handbag during the sale. How much does she pay?

## C. General Sales Tax (GST)

The various types of taxes collected by the Pakistani government go towards funding government expenditure such as national defence and education.

Examples of taxes include the **General Sales Tax (GST)** and income tax.

The GST is levied on the consumption of goods and services. It is paid in addition to the price of goods and services and is usually expressed as a percentage of the selling price. The current GST rate in Pakistan is 18%.

GST is also known as value-added tax (VAT) in some countries.

Worked Example 5

## Finding selling price inclusive of GST

A piece of decoration costs PKR 640 before GST. Assuming that GST is at 18%, calculate the total amount of money David has to pay for the piece of decoration.

**Solution**

**Method 1:**

$$\begin{aligned} \text{GST} &= 18\% \times \text{PKR } 640 \\ &= \frac{18}{100} \times \text{PKR } 640 \\ &= \text{PKR } 115.20 \end{aligned}$$

Total amount of money David has to pay for the piece of decoration
= PKR 640 + PKR 115.20
= PKR 755 (to the nearest PKR 1)

**Method 2:**

100% + 18% = 118%

Total amount of money David has to pay for the piece of decoration

$$\begin{aligned} &= 118\% \times \text{PKR } 640 \\ &= \frac{118}{100} \times \text{PKR } 640 \\ &= \text{PKR } 755.20 \\ &= \text{PKR } 755 \text{ (to the nearest PKR 1)} \end{aligned}$$

**Reflection**

Which method do you prefer? Why?

**Practise Now 5**

Similar and Further Questions
Exercise 6A
Questions 8, 9

1. An article costs PKR 19 000 before GST. Assuming that GST is at 18%, find the total amount of money a man has to pay for the article.
2. A printer is sold for $642 inclusive of 18% GST. Find the marked price of the printer.

## Investigation — Discount, service charge and GST

In this Investigation, we shall examine a bill to find out how discount, *service charge* and GST are calculated. A service charge is a fee collected to pay for services related to the main product being purchased. Fig. 6.1 shows the bill that Cheryl, Vasi and Waseem received after patronising a restaurant that is offering a 20% discount.

1. Show clearly how the restaurant calculates the GST.
2. Cheryl claims that the GST of 18% should be charged on the subtotal of PKR 1040, and thus the GST would be PKR 187 and not PKR 206. Do you agree? Explain your answer.
3. Waseem is unhappy that the discount is given for the food only and not for the service charge. He believes that the total bill will be less if the discount is given after adding the service charge (and before GST).
   Vasi says that it makes no difference either way. Show that Vasi is correct.
4. Waseem does not understand why applying the discount after adding the service charge does not give them a bigger discount.
   How would you explain to him why his argument was incorrect?

**Yummy Restaurant**

| | |
|---|---|
| Fish and Chips: | PKR 600 |
| Chicken Chop: | PKR 700 |
| Subtotal: | PKR 1300 |
| Discount 20%: | – PKR 260 |
| Subtotal: | PKR 1040 |
| Service Charge 10%: | PKR 104 |
| GST 18%: | PKR 206 |
| **Total:** | **PKR 1350** |

Fig. 6.1

Worked Example 6

## Problem involving discount, service charge and GST

Imran has dinner with his family at a restaurant which offers a 10% discount.
The marked price of the food that they order is PKR 6400.
Given that there is a service charge of 10% and GST is at 18%, calculate the total amount of money he has to pay to the nearest PKR 1.

**Solution**

**Method 1:**

$$\text{Discount} = \frac{10}{100} \times \text{PKR } 6400$$
$$= \text{PKR } 640$$

$$\text{Service charge} = 10\% \times (\text{marked price} - \text{discount})$$
$$= \frac{10}{100} \times (\text{PKR } 6400 - \text{PKR } 640)$$
$$= \frac{10}{100} \times \text{PKR } 5760$$
$$= \text{PKR } 576$$

$$\text{GST payable} = 18\% \times (\text{marked price} - \text{discount} + \text{service charge})$$
$$= \frac{18}{100} \times (\text{PKR } 6400 - \text{PKR } 640 + \text{PKR } 576)$$
$$= \frac{18}{100} \times \text{PKR } 6336$$
$$= \text{PKR } 1140.48$$

$$\text{Total amount payable} = \text{marked price} - \text{discount} + \text{service charge} + \text{GST payable}$$
$$= \text{PKR } 6400 - \text{PKR } 640 + \text{PKR } 576 + \text{PKR } 1140.48$$
$$= \text{PKR } 7476.48$$
$$= \text{PKR } 7476 \text{ (to the nearest PKR 1)}$$

**Method 2:**

$$\text{Total amount payable} = \text{PKR } 6400 \times \underbrace{0.9}_{\text{10\% discount}} \times \underbrace{1.1}_{\text{10\% service charge}} \times \underbrace{1.18}_{\text{18\% GST}}$$
$$= \text{PKR } 7476.48$$
$$= \text{PKR } 7476 \text{ (to the nearest PKR 1)}$$

**Reflection**

Which method do you prefer? Why?

**Practise Now 6**

Similar and Further Questions
Exercise 6A
Questions 18, 21

1. Li Ting orders a bowl of chicken soup at a restaurant which offers a 15% discount. The marked price of the chicken soup is PKR 640. Given that there is a service charge of 10% and GST is at 18%, find the total amount of money she has to pay.
2. Ken orders one set meal at a restaurant which offers a 20% discount. There is a service charge of 10% and GST is at 18%. Given that he pays a total of PKR 2600, find the marked price of the set meal.

## D. Commission

A commission is the payment an agent receives for selling or buying something on behalf of another party. It is usually given as a percentage of the cost price or the selling price.

**Worked Example 7**

**Finding commission**

A property agent sells a house for \$320 000, of which he receives a commission of 1.5%. Calculate the amount of commission the agent receives.

**Solution**

$$\text{Amount of commission the agent receives} = \frac{1.5}{100} \times \$320\ 000$$
$$= \$4800$$

**Practise Now 7**

Similar and Further Questions
Exercise 6A
Questions 10(a), (b), 19

1. A property agent sells a house for PKR 14 000 000, of which he receives a commission of 2%. Find the amount of commission the agent receives.
2. A property agent charges a commission of 3.5% on the selling price of a piece of property. Given that he receives a commission of \$25 375, find the selling price of the property.

### Reflection

1. How does what I have learnt from the chapter “Percentage” in Book 1 help me in calculating profit, loss, discount, GST and commission?
2. What have I learnt in this section that I am still unclear of?

## Exercise 6A

1. Copy and complete the table.

| | Cost price | Selling price | Profit/loss | Profit/loss as a percentage of cost price | Profit/loss as a percentage of selling price |
|---|---|---|---|---|---|
| (a) | \$40 | \$45 | | | |
| (b) | \$600 | \$480 | | | |
| (c) | PKR 88 000 | | | Profit of 4% | |
| (d) | PKR 5680 | | | Loss of 22.5% | |
| (e) | | \$28.14 | | Profit of $17\frac{1}{4}$% | |
| (f) | | \$506.85 | | Loss of 7% | |

2. A florist buys roses at \$18 per dozen. If she sells them at \$1.20 each, express her loss as a percentage of her selling price.

3. The profit on a refrigerator is 35% of the cost price. If the profit is PKR 217 000, find
**(i)** the cost price, **(ii)** the selling price,
of the refrigerator.

4. By selling a book for \$16.50, Albert loses 12% on the cost price. Find the cost price of the book.

5. A necklace priced at PKR 129 800 is sold for PKR 103 840. Find the percentage discount.

6. The marked price of a folding table at a hypermarket is PKR 3200. The hypermarket gives a 12% discount during a sale. Find the sale price of the folding table.

7. During a sale, there is a discount of 7% on a television set. If the discount is \$49, find
**(i)** the marked price, **(ii)** the sale price,
of the television set.

8. A microwave oven costs PKR 20 000 before GST. Assuming that GST is at 18%, find the total amount of money Ali has to pay for the microwave oven.

9. An electronic gadget is sold for PKR 343 380 inclusive of 18% GST. Find the marked price of the gadget.

10. A property agent charges a commission of 2.5% on the selling price of a house.
**(a)** Given that the agent sells a house for \$650 000, find the amount of commission he receives.
**(b)** On another occasion, he receives a commission of \$12 000. Find the selling price of the house.

11. A trader mixes 2 kg of butter which costs \$8 per kg with 3 kg of butter which costs \$6 per kg. He sells the mixture at \$2.55 per 250 g. Express his profit as a percentage of his selling price.

12. Sara buys a printer and sells it to Li Ting at a gain of 25% on the cost price. Li Ting then sells the printer to Nadia at a loss of 25% on the price at which she buys it from Sara. If Nadia pays PKR 36 000 for the printer, how much did Sara pay for it?

Basic Intermediate Advanced

## Exercise 6A

13. Raju buys 200 boxes of apples at \$28 per box. There are 60 apples in each box. 15% of the apples are rotten. Find the selling price per apple if he wants to earn an 80% profit on the cost price.

14. A shopkeeper buys 300 identical articles at a total cost of \$1500. He sells 260 articles at a price that is 20% above the cost price. Each of the remaining articles is sold at a price that is 50% of the selling price of each of the 260 articles. Express the shopkeeper's profit as a percentage of his cost price.

15. A publisher typically prices his books between \$10 and \$20. Give a possible cost price and a selling price of a book such that the publisher's profit as a percentage of the cost price is 30%.

OPEN Q

16. An air conditioner is sold for PKR 423 500 after a discount of 12.5%.
    (i) Find the marked price of the air conditioner.
    (ii) If a 10% discount is given on the marked price of the air conditioner before it is sold at a further discount of 2.5%, would the sale price still be PKR 423 500? Show your working clearly.

17. On its 16th anniversary, Brand A offers a 16% discount on all items. Members of Brand A are given an additional 14% discount.
    (i) A non-member buys an item for \$420 during the sale. Find the marked price of that item.
    (ii) Find the sale price of the same item if a member buys it.

18. Yasir orders one plate of chicken fried rice at a restaurant which offers a 25% discount. The marked price of the chicken fried rice is PKR 900. Given that there is a service charge of 10% and GST is at 18%, find the total amount of money he has to pay.

19. Joyce's monthly income consists of a basic salary of \$500 and a commission of 4% on her sales for the month. If her income is \$1220 for a particular month, find her sales for that month.

20. Ken wants to buy a sofa set. He is offered 3 successive discounts of 10%, 20% and 25%, and he can arrange them in any order he wants. Which order will benefit him the most? Explain your answer.

21. David orders one bowl of ramen at a japanese restaurant which offers an 18% discount. There is a service charge of 10% and GST is at 18%. Given that he pays a total of PKR 2395, find the marked price of the ramen.

# 6.3 Insurance, hire purchase and interest

## A. Health and vehicle insurance

Health insurance policies are designed to cover the cost of private medical treatment for illnesses and injuries. Depending on the policy, the insured pays a monthly or yearly premium to keep the insurance in force.

Vehicle insurance is another example of insurance. The policyholder pays regular premiums to the insurance company, which then repays the amount in the event of vehicular damage or loss. The insured amount is determined based the value of the vehicle, which *depreciates*, or decreases, over time.

**Worked Example 8**

### Finding premium of health insurance

Albert has a life insurance policy of PKR 100 000. If the insurance premium rate is 3% per annum, what is the premium amount paid by Albert every year?

> **Attention**
> '*Per annum*' means per year.

**Solution**

$$\begin{aligned} \text{Policy amount} &= \text{PKR } 100\,000 \\ \text{Amount of yearly premium} &= 3\% \times \text{insurance amount} \\ &= \frac{3}{100} \times \text{PKR } 100\,000 \\ &= \text{PKR } 3000 \end{aligned}$$

**Practise Now 8**

Similar and Further Questions
Exercise 6B
Questions 1, 7

1. Raju buys a life insurance valued at PKR 400 000. If the premium rate is 2.5% per annum, what is the premium Raju pays every year?
2. Joyce has a life insurance of PKR 600 000 on which she pays PKR 21 000 every year. Determine the premium rate.

**Worked Example 9**

### Finding premium of vehicle insurance

Cheryl insures her car that is valued at PKR 500 000. She pays 4% of the value of the car every year as premium. For three years, the car is valued at PKR 500 000. At the beginning of the fourth year, her car is worth PKR 475 000. What is the total amount Cheryl paid in premiums by the end of the fourth year?

**Solution**

$$\begin{aligned} \text{Value of car (first 3 years)} &= \text{PKR } 500\,000 \\ \text{Premium paid (first 3 years)} &= 3 \times \text{amount of yearly premium for first 3 years} \\ &= 3 \times 4\% \times \text{value of car} \\ &= 3 \times \frac{4}{100} \times \text{PKR } 500\,000 \\ &= \text{PKR } 60\,000 \end{aligned}$$

$$\begin{aligned} \text{Value of car (fourth year)} &= \text{PKR } 475\,000 \\ \text{Amount of yearly premium (fourth year)} &= 4\% \times \text{value of car} \\ &= \frac{4}{100} \times \text{PKR } 475\,000 \\ &= \text{PKR } 19\,000 \end{aligned}$$

$$\begin{aligned} \text{Total premiums paid} &= \text{PKR } 60\,000 + \text{PKR } 19\,000 \\ &= \text{PKR } 79\,000 \end{aligned}$$

**Practise Now 9**

Similar and Further Questions
**Exercise 6B**
Questions 2, 8

1. A man insured his car with an annual premium of 2.6%. Calculate the annual premium he pays if the car is valued at PKR 860 000.
2. A car was bought for PKR 650 000. Given that the value of a car depreciates by 4% the following year, determine the total insurance premium paid in the first 2 years if the car was insured at a premium rate of 2% per annum.

## B. Hire purchase

Customers who wish to purchase an expensive item such as a car may not have the money to pay for the item in full, or may be unwilling to fork out such a large sum of money. Very often, stores offer the option of **hire purchase**, where part of the purchase price is paid first (called the **down payment** or **deposit**) and the remaining amount is paid in monthly instalments. We need to pay *interest* on the monthly instalments. Why?

**Information**

As customers still owe the seller money, they are not the owners but the *hirers* of the item or the *purchase* (thus the term 'hire purchase').

**Worked Example 10**

### Hire purchase

A washing machine is priced at \$450. A man buys the washing machine on hire purchase with these terms: a down payment of 15% and the remaining to be paid in monthly instalments over 2 years at a simple interest rate of $10\frac{2}{3}\%$ per annum. Calculate

**(i)** his monthly instalment,
**(ii)** the total amount the man pays for the washing machine.

**Solution**

**(i)** Down payment $= \frac{15}{100} \times \$450$
$= \$67.50$

Remaining amount $= \$450 - \$67.50$
$= \$382.50$

Amount of interest the man has to pay at the end of 2 years

$= \$382.50 \times 10\frac{2}{3}\% \times 2$

$= \$382.50 \times \frac{10\frac{2}{3}}{100} \times 2$

$= \$81.60$

Total amount to be paid in monthly instalments
$= \$382.50 + \$81.60$
$= \$464.10$

Monthly instalment $= \frac{\$464.10}{24}$ 2 years = 24 months
$= \$19.34$ (to the nearest cent)

**Attention**

The total amount that the man has to pay in instalments includes the interest charged.

**(ii)** Total amount the man pays for the washing machine $= \$81.60 + \$450$
$= \$531.60$

**Practise Now 10**

Similar and Further Questions
**Exercise 6B**
Questions 3, 9

An air conditioner is priced at PKR 484 500. Nadia buys the air conditioner on hire purchase with these terms: a down payment of 20% and the remaining to be paid in monthly instalments over 4 years at a simple interest rate of 10% per annum.

**(i)** Find her monthly instalment.

**(ii)** Find the total amount Nadia pays for the air conditioner.

**(iii)** How much more does she have to pay for buying the air conditioner on hire purchase?

## C. Simple interest

If we deposit our money (called the principal amount, or simply, the **principal**) in a bank or a financial institution, we will earn interest on the principal. If we borrow money from a bank or other financial institutions, we will also be charged **interest**, which is a fee for borrowing money.

For example, if a bank charges a simple interest rate of 5% per annum and Yasir borrows $1000 from the bank for two years, we can calculate the amount of interest Yasir has to pay in this manner:

$$\text{Amount of interest} = \underbrace{\$1000}_{\text{Principal}} \times \underbrace{5\%}_{\text{Interest rate per year}} \times \underbrace{2}_{\text{Number of years}}$$

For simple interest, the interest remains the same each year.

> **Information**
>
> The following formula for simple interest is based on the reasoning in the main text:
> $I = \frac{PRT}{100}$, where $I$ = interest, $P$ = principal, $R\%$ = interest rate per year, $T$ = number of years.

**Worked Example 11**

### Simple interest

**(a)** Cheryl borrows $1000 from a bank that charges simple interest at a rate of 4% per annum. Calculate the amount of interest she has to pay at the end of 3 years.

**(b)** Waseem invests $4000 in a savings scheme that pays simple interest at a rate of 2% per annum. Calculate the time taken for his investment to grow to $4400.

**Solution**

**(a)** Amount of interest Cheryl has to pay at the end of 3 years $= \$1000 \times 4\% \times 3$

$$= \$1000 \times \frac{4}{100} \times 3$$

$$= \$120$$

**(b)** Total amount of interest Waseem earns $= \$4400 - \$4000$

$$= \$400$$

Amount of interest Waseem earns per year $= \$4000 \times \frac{2}{100}$

$$= \$80$$

Time taken for his investment to grow to $\$4400 = \frac{\$400}{\$80}$

$$= 5 \text{ years}$$

**Practise Now 11**

Similar and Further Questions
**Exercise 6B**
Questions 4, 5, 10, 11

1. Bernard takes a loan of \$150 000 from a bank to start a business venture. The bank charges him simple interest at a rate of 5.5% per annum. If he plans to repay his loan at the end of 3 years, find the amount of interest he has to pay and hence, the total amount he owes the bank.
2. Joyce invests PKR 60 000 in a savings plan that pays simple interest at a rate of 3% per annum. Find the time taken for her investment to grow to PKR 67 200.

## D. Compound interest

We have learnt that the amount of simple interest $I$ a person earns from a bank depends on the principal $P$, the interest rate $R$% per annum and the duration in years, $T$.

Hence, for *simple interest*, the interest earned every year is the same because it is calculated based on the original principal.

Sometimes the interest earned each year is added to the principal, so that interest is also earned on the interest accumulated. This is called **compound interest**.

### Investigation — Exploring simple interest and compound interest

Shaha wants to place \$1000 in a bank as a fixed deposit for 3 years.
Table 6.1 shows what two banks offer.

| Bank A | Bank B |
|---|---|
| *Simple* interest rate of 2% per annum | Interest rate of 2% per annum *compounded* yearly |

Table 6.1

1. Calculate the interest earned from Bank A and the total amount of money he would have after 3 years.
2. Copy and complete the following to find the interest earned from Bank B and the total amount of money he would have at the end of each year.

**1st year:**

Principal $P_1 = \$1000$,
Interest $I_1 = \$1000 \times 2\%$
$= \$\square$

Total amount at the end of the 1st year, $A_1 = P_1 + I_1$
$= \$1000 + \$\square$
$= \$1020$

**2nd year:**

Principal $P_2 = A_1 = \$1020$
Interest $I_2 = \$\square \times 2\%$
$= \$\square$

Total amount at the end of the 2nd year, $A_2 = P_2 + I_2$
$= \$1020 + \$\square$
$= \$\square$

**3rd year:**

Principal $P_3 = A_2 = \$\boxed{\phantom{000}}$,

Interest $I_3 = \$\boxed{\phantom{000}} \times 2\%$

$= \$\boxed{\phantom{000}}$

Total amount at the end of the 3rd year, $A_3 = P_3 + I_3$

$= \$\boxed{\phantom{000}} + \$\boxed{\phantom{000}}$

$= \$\boxed{\phantom{000}}$ (to the nearest cent)

3. Which bank offers a higher interest and by how much?

The above Investigation shows that a compound interest gives more interest than a simple interest of the same rate.

We can find a formula to calculate compound interest by observing some patterns from the above Investigation.

Total amount at the end of the 1st year, $A_1 = P_1 + I_1$

$$= \$1000 + \$1000 \times \frac{2}{100}$$

$$= \$1000\left(1+\frac{2}{100}\right)$$

Total amount at the end of the 2nd year, $A_2 = P_2 + I_2$

$$= \$1000\left(1+\frac{2}{100}\right) + \$1000\left(1+\frac{2}{100}\right) \times \frac{2}{100}$$

$$= \$1000\left(1+\frac{2}{100}\right)\left(1+\frac{2}{100}\right) \qquad \text{extract common factor } 1000\left(1+\frac{2}{100}\right)$$

$$= \$1000\left(1+\frac{2}{100}\right)^2$$

Total amount at the end of the 3rd year, $A_3 = P_3 + I_3$

$$= \$1000\left(1+\frac{2}{100}\right)^2 + \$1000\left(1+\frac{2}{100}\right)^2 \times \frac{2}{100}$$

$$= \$1000\left(1+\frac{2}{100}\right)^2\left(1+\frac{2}{100}\right) \qquad \text{extract common factor } 1000\left(1+\frac{2}{100}\right)^2$$

$$= \$1000\left(1+\frac{2}{100}\right)^3$$

Based on the pattern above,

> if the interest paid on a sum of money is **compounded** annually (or yearly), then
> $$A = P\left(1+\frac{r}{100}\right)^n,$$

where $A$ is the total amount, $P$ is the principal, $r\%$ is the interest rate per annum and $n$ is the number of years. If the interest is compounded *monthly*, then $r\%$ is the interest rate per *month* and $n$ is the number of *months*.

**Worked Example 12**

### Finding compound interest

Find the compound interest on \$5000 for 7 years at 3% per annum, compounded annually.

**Solution**

$$A = P\left(1+\frac{r}{100}\right)^n$$

$$= \$5000\left(1+\frac{3}{100}\right)^7 \qquad P = \$5000,\ r = 3,\ n = 7$$

$$= \$6149.37 \text{ (to the nearest cent)}$$

Compound interest $I = A - P$

$$= \$6149.37 - \$5000$$

$$= \$1149.37 \text{ (to the nearest cent)}$$

**Reflection**

The simple interest formula $I = \frac{PRT}{100}$ allows us to calculate the interest $I$ directly. Can you calculate the compound interest directly from its formula? Explain.

**Attention**

Unless otherwise stated, we usually leave money to the nearest cent if it is not exact.

**Practise Now 12**

Similar and Further Questions
**Exercise 6B**
Questions 6, 12–15

1. Find the compound interest on PKR 30 000 for 4 years at 5% per annum, compounded annually.

2. Find the compound interest on \$1500 for 2 years at 2% per annum, compounded
   **(a)** annually,
   **(b)** monthly.

3. Vasi places PKR 40 000 in a bank as a fixed deposit for 2 years. The bank offers an interest compounded yearly. At the end of 2 years, he receives a total of PKR 42 436. Find the interest rate.

**Problem-solving Tip**

For Question 2(**b**), find the interest rate per month first. Why?

## Reflection

1. Why are insurance premiums, interest charged on hire purchase and bank interest expressed as percentages?
2. Although insurance premiums, interest charged on hire purchase and bank interest in this section are expressed as percentages, they differ from the applications of percentages in Section 6.1. How do they differ? Why is there a difference?

## Exercise 6B

1. Li Ting buys a life insurance policy of PKR 620 000 at the rate of 2.8% per annum. Find the amount of annual premium she has to pay.

2. Sara insured her car at the rate of 6.5% per annum. The estimated value of the car is PKR 750 000. Find the annual premium payable.

3. A computer system is priced at $3200. A man buys the computer system on hire purchase according to the following terms: a down payment of $480 and the remaining to be paid in monthly instalments over 2 years at a simple interest rate of 9.5% per annum.
   **(i)** Find his monthly instalment.
   **(ii)** Find the total amount the man pays for the computer system.
   **(iii)** How much more does he have to pay for buying the computer system on hire purchase?

4. Albert borrows PKR 48 000 from a bank that charges simple interest at a rate of 6% per annum. Find the total amount of money he has to pay the bank at the end of 2 years.

5. A man invests $16 800 in a savings plan that pays simple interest at a rate of 5% per annum. Find the time taken for his investment to grow to $18 900.

6. Ken places PKR 50 000 in his bank account. The bank offers an interest of 8% per annum compounded yearly. Find the total interest in his account at the end of 3 years.

7. Nadia pays a premium rate of 3% per annum on a life insurance. If her total premium payable over 5 years is PKR 67 500, calculate the value of her life insurance.

8. A car bought for PKR 1 000 000 depreciates each year by 5%. If the premium rate is 3.5% per annum, what is the total insurance premium paid in the first 3 years of purchase?

9. The price of a sofa is $\$x$. A man buys the sofa on hire purchase with these terms: a down payment of 25% and the remaining to be paid in monthly instalments over 30 months at a simple interest rate of 12% per annum. Given that his monthly instalment is $52, find the value of $x$.

10. Li Ting deposits $20 000 in a bank that pays a simple interest rate of 2.75% per annum. If the interest rate decreases to $x$% per annum, she will receive $50 less every year. Find the value of $x$.

11. The following information is advertised by a bank.

    10 Year High-Yield Account
    **2.35%**
    Simple Interest

    Cheryl found the interest rate attractive and decided to deposit PKR 400 000 in the bank. One year later, she only received an interest of PKR 940. Upon reading the above advertisement carefully, she realised that she had been misled by it.
    **(i)** How much interest had Cheryl expected to receive after one year?
    **(ii)** Explain how Cheryl had been misled by the advertisement.

12. Raju deposited PKR 15 000 in an account that pays 5.68% compound interest per year. Find the total amount in the account after 6 years if the interest is compounded
    **(i)** monthly, **(ii)** half-yearly.

13. Cheryl invested PKR 50 000 in an endowment fund for 5 years. The fund pays an interest compounded yearly. At the end of 5 years, she received a total of PKR 58 000. Find the interest rate.

Basic | Intermediate | Advanced

## Exercise 6B

14. Bernard borrowed a sum of money from the bank which charges a compound interest of 4.2% per annum, compounded quarterly. Given that Bernard had to pay \$96.60 in interest payments at the end of the first year, find the original sum of money borrowed, giving your answer correct to the nearest dollar.

15. Joyce deposited $\$x$ in an account that pays 2% compound interest per year. Assuming that no withdrawal was made, find the value of $x$ if she has \$36 757.94 in the account at the end of 7 years.

# 6.4 Zakat, ushr and income tax

## A. Zakat and ushr

In the Islamic 'socio-economic system', every Muslim contributes 'zakat' to the poor and needy Muslims of the society from his or her yearly savings. The savings may be in the form of money, gold, and silver, and the animals kept for the purpose of income. The amount of zakat is calculated as 2.5% of the total yearly savings.

Ushr is a tax that is imposed on a Muslim's agricultural assets. If the land receives water from natural water sources such as streams, rivers, or underground springs, the ushr payable is **10%** on the value of agricultural output. If the land is irrigated by artificial means, i.e. water is obtained from a tube well, ushr forms **5%** of the agricultural output.

**Worked Example 13**

### Finding zakat and ushr

**(a)** Determine the amount of zakat Sara paid in 2022 if she had an annual savings of PKR 75 000 that year.

**(b)** Raju sells his rice harvest for PKR 150 000. Determine the amount of ushr payable by Raju if his land is naturally irrigated.

**Solution**

**(a)** Yearly savings in 2022 = PKR 75 000

$$\begin{aligned} \text{Zakat paid in 2022} &= 2.5\% \times \text{yearly savings} \\ &= \frac{2.5}{100} \times \text{PKR } 75\,000 \\ &= \text{PKR } 1875 \end{aligned}$$

**(b)**

$$\begin{aligned} \text{Value of agricultural output} &= \text{PKR } 150\,000 \\ \text{Percentage of output payable as ushr} &= 10\% \\ \text{Payable ushr} &= \frac{10}{100} \times \text{PKR } 150\,000 \\ &= \text{PKR } 15\,000 \end{aligned}$$

**Practise Now 13**

Similar and Further Questions
**Exercise 6C**
Questions 1, 2, 6

1. A farmer owns a land that is artificially irrigated. If he sells his harvest from the land at PKR 74 200, determine the amount of ushr he should pay.
2. Ken contributed PKR 1255 in zakat in 2021. Determine the amount of yearly savings he had that year.

## B. Income tax

In the previous sections, we have already discussed two types of taxes, the General Sales Tax (GST) and ushr. Another type of tax is the income tax.

Income tax is charged on all incomes derived from Pakistan or received in Pakistan from sources outside Pakistan during the year starting from 1 July and ending on 30 June. The income tax payable is calculated based on the **chargeable income** (or **taxable income**), where chargeable income = total income – tax reliefs.

Tax reliefs include personal relief, foreign source salary, agriculture income, contributions to the General Provident Fund and gifts to charitable organisations in the form of cash, etc.

### Class Discussion

**What is a reasonable way to tax income?**

Although income tax is charged on the yearly income, for ease of discussion here, we will use monthly rate. Assume the taxable incomes of Cheryl and Joyce are PKR 200 000 and PKR 1 600 000 respectively. What is a reasonable way to tax their incomes?

**Option A: Charge a flat rate of 5% per month**

1. How much income tax do Cheryl and Joyce have to pay? Who pays more income tax?
2. How much money do Cheryl and Joyce have left to spend? Who has more money left to spend?

**Option B: Charge a progressive tax**

First PKR 400 000 a month: 0%
Above PKR 400 000 a month: 10%

3. How much income tax does Cheryl and Joyce each have to pay? Who pays more income tax?
4. How much money does Cheryl and Joyce each have left to spend? Who still has more money left to spend?
5. Which tax option is more favourable for people with lower incomes? Explain your reasoning.
6. Does Pakistan follow a flat tax rate or a progressive tax rate? Why?

Worked Example 14

## Income tax

In 2023, Vasi earned a gross annual income of PKR 2 784 000. Of this income, the amount that will not be subjected to income tax is agricultural produce worth PKR 300 000. Find his payable income tax given the following table containing the rates for different income tax *slabs*.

| Slab | Chargeable income | Income tax rate |
|---|---|---|
| 1 | Up to PKR 600 000. | 0% |
| 2 | More than PKR 600 000 and up to PKR 1 200 000. | 2.5% of the amount exceeding PKR 600 000 |
| 3 | More than PKR 1 200 000 and up to PKR 2 400 000. | PKR 15 000 + 12.5% of the amount exceeding PKR 1 200 000 |
| 4 | More than PKR 2 400 000 and up to PKR 3 600 000. | PKR 165 000 + 22.5% of the amount exceeding PKR 2 400 000 |
| 5 | More than PKR 3 600 000 and up to PKR 6 000 000. | PKR 435 000 + 27.5% of the amount exceeding PKR 3 600 000 |
| 6 | More than PKR 6 000 000 and up to PKR 12 000 000. | PKR 1 095 000 + 35% of the amount exceeding PKR 6 000 000 |
| 7 | More than PKR 12 000 000. | PKR 2 955 000 + 35% of the amount exceeding PKR 12 000 000 |

### Solution

Taxable income = PKR 2 784 000 – PKR 300 000
= PKR 2 484 000

Amount exceeding PKR 2 400 000
= PKR 2 484 000 – PKR 2 400 000
= PKR 84 000

Income tax payable = PKR 165 000 + (22.5% × PKR 84 000)
= PKR 165 000 + PKR 18 900
= PKR 183 900

**Problem-solving Tip**

The taxable income of PKR 2 484 000 falls in income tax slab 4.
To calculate the income tax payable, we need to first find the amount exceeding the *lower limit* of the tax slab.

**Practise Now 14**

Similar and Further Questions
Exercise 6C
Questions 3(a)-(d), 4, 7, 11

In a particular country, the income tax payable for the first \$20 000 is \$0 and the tax rate for the rest is 18%. In 2022, Bernard earned a gross annual income of \$75 600. Of this \$75 600, the amount not subjected to tax is shown in the following table:

| | |
|---|---|
| Personal relief | \$1000 |
| Child relief | \$4000 per child |
| Course Fees Relief | \$4500 |
| Donations | \$1500 |

If Bernard has 2 children, calculate his income tax payable for the year 2022.

# 6.5 Inheritance and partnership

## A. Inheritance

According to the Islamic Laws of Inheritance, the asset that a deceased man leaves behind is distributed among his widow and children according to the following:

- His widow inherits $\frac{1}{8}$ of the asset.
- The remaining $\frac{7}{8}$ of the asset is divided among the children such that a son inherits twice the share of a daughter.

If the Islamic Laws of Inheritance does not apply to a person, he or she may choose to name beneficiaries or to divide assets based on ratios of their own choosing in their will.

**Worked Example 15**

**Calculating amounts distributed in inheritance**

A man left savings of PKR 50 000 in the bank. Divide the amount between his widow, 2 sons and a daughter according to the Islamic Laws of Inheritance.

**Solution**

$$\begin{aligned}\text{Widow's share} &= \frac{1}{8} \times \text{PKR } 50\,000 \\ &= \text{PKR } 6250 \\ \text{Remaining amount} &= \text{PKR } 50\,000 - \text{PKR } 6250 \\ &= \text{PKR } 43\,750\end{aligned}$$

$$\begin{aligned}\text{Sons' share} &: \text{Daughter's share} \\ 2 \times 2 &: 1 \\ 4 &: 1\end{aligned}$$

$\therefore$ sons' share forms $\frac{4}{5}$ of the remaining amount.

$$\begin{aligned}\text{Sons' share} &= \frac{4}{5} \times \text{PKR } 43\,750 \\ &= \text{PKR } 35\,000 \\ \text{1 son's share} &= \text{PKR } 35\,000 \div 2 \\ &= \text{PKR } 17\,500 \\ \text{Daughter's share} &= \frac{1}{5} \times \text{PKR } 43\,750 \\ &= \text{PKR } 8750\end{aligned}$$

**Practise Now 15**

Similar and Further Questions
**Exercise 6C**
Questions 5, 8, 9

1. A man left savings of PKR 288 000 to his wife, 2 sons and 2 daughters. Find the amount each person gets under the Islamic Laws of Inheritance.
2. David left a property which was sold to give money to his 3 sons and 1 daughter. If each son received PKR 220 000 under the Islamic Laws of Inheritance, how much was the property worth?

## B. Partnership

A partnership occurs between two or more people who are running a business. Unless otherwise agreed upon, the profit earned from the business is divided based on the ratio of the investments made by each partner.

**Worked Example 16**

### Calculating amounts distributed in partnership

Waseem, Nadia and Albert each invested some money to start a company. In a particular month, the company earned a profit of PKR 42 210. Determine the amount of profit each of them will receive in that month if

**(a)** the amount invested by each of them is equal,

**(b)** the amount invested by Waseem, Nadia and Albert is in the ratio 2 : 1 : 4.

**Solution**

**(a)** Amount of profit each of them receives $= \dfrac{\text{PKR } 42\ 210}{3}$

$= \text{PKR } 14\ 070$

**(b)** Amount of profit Waseem receives $= \dfrac{\text{PKR } 42\ 210}{7} \times 2$

$= \text{PKR } 12\ 060$

Amount of profit Nadia receives $= \dfrac{\text{PKR } 42\ 210}{7} \times 1$

$= \text{PKR } 6030$

Amount of profit Albert receives $= \dfrac{\text{PKR } 42\ 210}{7} \times 4$

$= \text{PKR } 24\ 120$

**Practise Now 16**

Similar and Further Questions
**Exercise 6C**
Questions 10, 12, 13

1. Imran, Cheryl and Joyce invested PKR 44 100, PKR 14 700 and PKR 88 200 respectively to start a business. At the end of the year, the business reported a profit of PKR 80 100.
   **(i)** Determine the ratio of the investments made by Imran, Cheryl and Joyce.
   **(ii)** Calculate the amount of money each of them received from the profit.

2. Joyce, Li Ting and Nadia entered into a business partnership by investing in the ratio of 4 : 3 : 5. At the end of the first month, Nadia received a profit of $850. Calculate the total amount of profit the business generated in the first month.

## Reflection

1. Income tax is an example of a rate that uses categories or brackets, where the percentage of income tax payable differs in each income category. What advantage does this system have over charging a flat income tax rate?
2. How is the concept of equivalent ratios applied in asset distribution in inheritance and partnerships?

## Exercise 6C

1. How much zakat does Albert have to pay on his yearly savings of PKR 400 000?

2. How much ushr should Waseem pay if he sold crops harvested from his artificially-irrigated land for PKR 71 000?

3. Using the table in Worked Example 14 on page 200, calculate the income tax payable if the chargeable income is:
   (a) PKR 685 000  (b) PKR 3 050 000
   (c) PKR 6 000 000  (d) PKR 12 150 000

4. **Property tax** is a tax on land, houses, flats or buildings, which is computed as follows:

   property tax payable yearly
   = annual value × tax rate,

   where the annual value of a property is the estimated amount of money the owner will get if he or she rents out the property for a year. Given that the annual value of a flat is PKR 938 952 and the tax rate is 25% per annum, find the property tax payable for 6 months.

5. A man left savings of PKR 249 600 in the bank. Determine the amount of inheritance his widow, a son and 2 daughters will receive under the Islamic Laws of Inheritance.

6. A farmer paid PKR 6700 as ushr on his harvested wheat last year. If his land is naturally irrigated how much did his crop sell for?

7. In a particular country, the gross tax payable for the first $40 000 is $550 and the tax rate for the rest is 7%. In a particular year, Nadia earned a gross annual income of $80 000. Of this $80 000, the amount that will not be subjected to income tax is shown in the following table:

| | |
|---|---|
| Personal Relief | $3000 |
| Parent Relief | $5000 per parent |
| Course Fee Relief | $16 000 |
| Donations | $750 |

Given that she lives with 2 parents, find her income tax payable.

8. Raju left some savings to his wife and 3 sons as inheritance. If each son gets PKR 84 630 under the Islamic Laws of Inheritance, how, how much savings did he leave?

9. Sara willed PKR 12 850 000 to her 3 heirs in the ratio of 4 : 3 : 3. How much does each heir receive?

10. Albert, Imran and Sara invested $427 000, $671 000 and $305 000 in a property respectively and they agreed to share the profit in the ratio of their investments.
    After a few years, they sold the property for $1 897 500.
    Find the amount of profit each of them received.

11. Bernard paid an income tax of $1474 for the year 2023 in a particular country. Of his gross annual income, the amount that will not be subjected to income tax is shown in the following table:

| | |
|---|---|
| Personal Relief | $3000 |
| Wife Relief | $2000 |
| Child Relief | $4000 per child |
| Parent Relief | $5000 per parent |
| Provident Fund contributions | 20% of gross income |
| Donations | $200 |

Given that he has a wife, 4 children and 2 parents, and that for the remaining income that will be taxed, the gross tax payable for the first $40 000 is $550 and the tax rate for the rest is 7%, find his gross annual income.

Basic Intermediate Advanced

## Exercise 6C

12. Nadia, Joyce and Waseem invested a total of PKR 978 000 to start a business. If Nadia, Joyce and Waseem received profits of PKR 29 680, PKR 44 520 and PKR 37 100 at the end of the month respectively, determine the amount each partner invested.

13. Ken, Shaha and David invested some money in a start-up company in the ratio 3 : 5 : 4. At the end of the year, Shaha decided to give PKR 85 000 of his share of the profit to Ken. In doing so, the ratio of the profits received by Ken, Shaha and David became 4 : 6 : 5.
Determine
**(i)** the amount of profit they received altogether,
**(ii)** the amount of profit each received originally.

## Summary

1. **Profit and loss**

Profit = selling price – cost price

Loss = cost price – selling price

**(a)** Profit/loss as a percentage of cost price = $\frac{\text{profit / loss}}{\text{cost price}} \times 100\%$

**(b)** Profit/loss as a percentage of selling price = $\frac{\text{profit / loss}}{\text{selling price}} \times 100\%$

- Think of one real-life problem involving profit or loss, and solve it.

2. **Discount**

Discount = marked price – sale price

Percentage discount = $\frac{\text{discount}}{\text{marked price}} \times 100\%$

- Think of one real-life problem involving discount, and solve it.

3. **General Sales Tax (GST)**

In Pakistan, GST is paid in addition to the price of goods and services.

- Give an example of how GST is used in a real-world context.

4. **Commission**

A commission is the payment an agent receives for selling or buying something on behalf of another party.

- Give an example of how commission is used in a real-world context.

5. **Insurance**

- Health insurance, which covers the medical costs incurred in the event of illness or injuries, requires the insurer to pay a month or yearly premium. The amount of premium payable per month or year is expressed as a percentage of the policy amount.
- Vehicle insurance covers the cost of repair or replacement in the event of vehicular damage or loss. The monthly or yearly premium payable is expressed as a percentage of the market value of the vehicle, which depreciates over time.

6. **Hire purchase**

In hire purchase, customers pay a down payment for a product and the remaining amount together with interest, as monthly instalments.

Amount of interest accrued = remaining amount × interest rate × number of years

Monthly instalment = $\frac{\text{remaining amount + interest amount}}{\text{total number of months}}$

- Give an example of an item bought using hire purchase. Determine how much more a customer has to pay for an item using hire purchase.

## Summary

**7. Interest**

When a sum of money (or principal $P$) is deposited or loaned from a bank, an interest will be earned (or charged in a loan). The two types of interests are:

- Simple interest, which is calculated based on the principal $P$, the interest rate $R\%$ per year, and the number of years $T$, i.e.

  Interest earned $I = \dfrac{PRT}{100}$ .

- Compound interest, which is calculated based on principal sum $P$ and the accumulated interest, i.e.

  Total amount $A = P\left(1 + \dfrac{r}{100}\right)^n$, where $r\%$ is the interest rate per year and $n$ is the number of years.

  - Find out the interest rates of the savings accounts offered by different banks in Pakistan.

**8. Zakat and Ushr**

The amount of **zakat** and **ushr** payable are respectively calculated as a percentage of the yearly savings and agricultural output.

- Zakat: 2.5% of the yearly savings;
- Ushr: 5% of the agricultural output for artificially-irrigated land, and 10% for naturally-irrigated land.

**9. Income tax**

The income tax payable is calculated based on chargeable income, i.e.

chargeable income = total income − tax reliefs.

Depending on the chargeable income, different tax rates apply based on income tax slabs.

**10. Inheritance**

Under the Islamic Laws of Inheritance, the widow of a deceased man inherits $\frac{1}{8}$ of his assets and the remaining is inherited by his children such that a son inherits twice the share of a daughter.

**11. Partnership**

Unless otherwise agreed upon, the profit or returns of a business enterprise is divided among its business partners in the same ratio as the investment amounts.

CHAPTER 7

# Direct and Inverse Proportions


From as early as 2550 BC, human civilisation has been aware of a special relationship between the circumference of a circle and its diameter. No matter how big the circle may be, they realised that the circumference is about three times its diameter. This constant, which is actually slightly more than 3, has been given a special name — pi or π, which was obtained from the Greek word for "perimeter". Hence, we can say that the relationship between the circumference of a circle and its diameter can be expressed as

$$\frac{\text{circumference}}{\text{diameter}} = \pi\text{; or circumference} = \pi \times \text{diameter.}$$

In fact, there are many such relationships between two quantities in real life in which the two quantities change according to a fixed factor. For instance, suppose that you get 5 lucky draw attempts for every 10 tokens you win from the game stalls at a fun fair. How many lucky draw attempts will you have if you win 50 tokens? To obtain the answer, we see that the number of tokens won is related to the number of lucky draw attempts by a constant factor of 2.

In mathematics, we say that the number of lucky draw attempts is directly proportional to the number of tokens. In this chapter, we will examine the concept of **proportionality** by exploring quantities that are related by direct proportion or those that are related by inverse proportion.

## Learning Outcomes

What will we learn in this chapter?

- What direct and inverse proportions are
- How to explain the concept of proportion using tables, equations and graphs
- How to form a formula between variables which are related in direct and inverse proportions, and solve problems involving proportion

## Introductory Problem

A consignment of fodder can feed 1000 sheep for 20 days. Assuming that all the sheep consume the fodder at the same rate, how many consignments of fodder are needed to feed 550 sheep for 400 days?

In this chapter, we will learn about direct and inverse proportions and find out how to solve such problems.

# 7.1 Direct proportion

### Investigation — Direct proportion

In many public libraries, if we are late in returning borrowed books, we will incur a fine per day for each overdue book. Table 7.1 shows the fines for an overdue book.

| Number of days ($x$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Fine ($y$ cents) | 15 | 30 | 45 | 60 | 75 | 90 | 105 | 120 | 135 | 150 |

Table 7.1

1. If the number of days a book is overdue increases, will the fine increase or decrease?
2. If the number of days a book is overdue is doubled, how will the fine change?
   Hint: Compare the fines when a book is overdue for 3 days and for 6 days.
3. If the number of days a book is overdue is tripled, what will happen to the fine?
4. If the number of days a book is overdue is halved, how will the fine change?
   Hint: Compare the fines when a book is overdue for 10 days and for 5 days.
5. If the number of days a book is overdue is reduced to $\frac{1}{3}$ of the original number, what will happen to the fine?

From the above Investigation, we observe the following:

doubled $x = 3, y = 45$ doubled
$x = 6, y = 90$

and

halved $x = 10, y = 150$ halved
$x = 5, y = 75$

That is, when the number of days, $x$, a book is overdue changes, the fine, $y$ cents, changes *proportionally*.

This relationship is known as **direct proportion**. We say that the fine, $y$ cents, is *directly proportional* to the number of days, $x$, that a book is overdue.

**Big Idea**

**Proportionality**
In Book 1, we learnt two ideas about proportionality. The above Investigation illustrates the first idea. When one quantity is multiplied by a value, the other quantity is also multiplied by the same constant, e.g. when $x_1 = 1$ is doubled to $x_2 = 2$, $y_1 = 15$ is also doubled to $y_2 = 30$.

× 2

| $x$ | 1 | 2 |
|---|---|---|
| $y$ | 15 | 30 |

× 2

We can write this as
$\frac{x_2}{x_1} = 2 = \frac{y_2}{y_1}$, i.e. $\frac{y_2}{y_1} = \frac{x_2}{x_1}$.

Copy and complete Table 7.2.

| Number of days ($x$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Fine ($y$ cents) | 15 | 30 | 45 | 60 | 75 | 90 | 105 | 120 | 135 | 150 |
| Rate $\frac{y}{x}$ | $\frac{15}{1}=15$ | $\frac{30}{2}=15$ | $\frac{45}{3}=15$ | | | | | | | |

Table 7.2

What can we observe about the rate $\frac{y}{x}$?

What does $\frac{y}{x}$ represent? What does the constant '15' mean in this context?

In direct proportion, the rate $\frac{y}{x}$ is a *constant*.

To conclude, we have:

If $y$ is *directly proportional* to $x$, then

- $\frac{y_2}{y_1} = \frac{x_2}{x_1}$
- $\frac{y_1}{x_1} = \frac{y_2}{x_2}$

**Big Idea**

**Proportionality**
Table 7.2 illustrates the second idea about proportionality that we learnt in Book 1.
When two quantities are proportional to each other, the rate of change of one quantity with respect to the other quantity, i.e. $\frac{y}{x}$, is a constant.

**Worked Example 1**

### Simple problem involving direct proportion

If 6 kg of biscuits cost \$27, calculate the cost of 13 kg of biscuits.

**Problem-solving Tip**

Note that the cost of the biscuits is directly proportional to the mass of the biscuits.

**Solution**

**Method 1: Unitary method**

6 kg of biscuits cost \$27.

1 kg of biscuits costs $\frac{\$27}{6}$.

13 kg of biscuits cost $\frac{\$27}{6} \times 13 = \$58.50$.

**Information**

Method 1 is called the *unitary* method because it involves finding the cost of 1 kg (or *1 unit*) of biscuits first.

**Method 2: Proportion method**

Let the cost of 13 kg of biscuits be \$$a$.

**Method 2(a):**

$\frac{a}{13} = \frac{27}{6}$  $\quad \frac{y_2}{x_2} = \frac{y_1}{x_1}$

$a = \frac{27}{6} \times 13$

$= \$58.50$

$\therefore$ 13 kg of biscuits cost \$58.50.

**Method 2(b):**

$\frac{a}{27} = \frac{13}{6}$  $\quad \frac{y_2}{y_1} = \frac{x_2}{x_1}$

$a = \frac{13}{6} \times 27$

$= \$58.50$

**Reflection**

Do you see a similarity between Method 1 and Method 2(a)? Is there a similarity between Methods 2(a) and 2(b)? Which one of the 3 methods do you prefer? Why?

**Practise Now 1**

Similar and Further Questions
**Exercise 7A**
Questions 1, 2, 5(a), (b), 6

**(a)** If 50 g of sweets cost \$2.10, find the cost of 380 g of sweets, giving your answer correct to the nearest 5 cents.

**(b)** $\frac{3}{4}$ of a piece of metal has a mass of 15 kg. What is the mass of $\frac{2}{5}$ of the piece of metal?

# 7.2 Algebraic and graphical representations of direct proportion

In the example on overdue books in Section 7.1, we have found that $\frac{y}{x} = 15$, which is a constant. If we represent this constant by $k$, then $\frac{y}{x} = k$ or $y = kx$, where $k \neq 0$.

Hence, we have:

If $y$ is *directly proportional* to $x$, then $\frac{y}{x} = k$ or $y = kx$, where $k$ is a constant and $k \neq 0$.

In Chapter 1, Linear Functions and Graphs, we have learnt how to plot points and draw the graph of a linear function. Let us now see how the concept of direct proportion is linked to what we know about **functions**.

**Big Idea**

**Proportionality**
Earlier in Section 7.1, we observed that if $y$ is proportional to $x$, then $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. Any $x$- and corresponding $y$-value will give the same constant, so we can write $\frac{y}{x} = k$, where $k$ ($\neq 0$) is called the *constant of proportionality*. Expressing a proportional relationship as an equation allows us to study the relationship from another perspective, e.g. using graphs.

## Investigation — Graphical representation of direct proportion

Consider the example on overdue books in Section 7.1. Table 7.3 shows the fines, $y$ cents, for number of days, $x$, a book is overdue, where $\frac{y}{x} = 15$ or $y = 15x$. What does $y = 15x$ mean in this context?

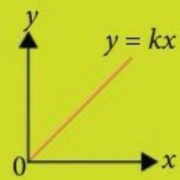
| **Number of days ($x$)** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Fine ($y$ cents)** | 0 | 15 | 30 | 45 | 60 | 75 | 90 | 105 | 120 | 135 | 150 |

Table 7.3

1. On a sheet of graph paper, using a scale of 1 cm to represent 1 day on the horizontal axis and 1 cm to represent 15 cents on the vertical axis, plot the graph of $y$ against $x$.
2. What is the shape of the graph you have drawn?
3. What is the $y$-intercept?
4. What does the gradient of the graph represent?

From the Investigation on page 210, we observe the following:

If $y$ is *directly proportional* to $x$, then the graph of $y$ against $x$ is a straight line that passes through the origin.




**Attention**

The equation of this straight lin will be given by $y = kx$, where th gradient $k$ is also the constant of proportionality in this direct proportion.

## Thinking Time

1. If $y$ is directly proportional to $x$, is $x$ directly proportional to $y$? Explain your answer.
2. Suppose $y$ is directly proportional to $x$. If we plot the graph of $x$ against $y$, will we get a straight line that passes through the origin? Explain your answer.
3. If $y$ is directly proportional to $x$, then the graph of $y$ against $x$ passes through the origin. If the graph of $y$ against $x$ does not pass through the origin, is $y$ directly proportional to $x$? Explain your answer.
4. As $x$ increases, $y$ also increases. Can we conclude that $y$ is directly proportional to $x$? Explain your answer.

**Worked Example 2**

### Forming equation of direct proportion

If $y$ is directly proportional to $x$ and $y = 12$ when $x = 4$, find

**(i)** an equation connecting $x$ and $y$,

**(ii)** the value of $y$ when $x = 8$,

**(iii)** the value of $x$ when $y = 21$.

***Solution**

**(i)** Since $y$ is directly proportional to $x$, then $y = kx$, where $k$ is a constant.

When $x = 4$, $y = 12$,

$12 = k \times 4$

$\therefore k = 3$

$\therefore y = 3x$

**Attention**

Since $y$ is directly proportional to $x$, then $\frac{y}{x} = k$ or $y = kx$, where $k$ is a constant and $k \neq 0$.

**(ii)** **Method 1:**

Substitute $x = 8$ into $y = 3x$:

$y = 3 \times 8$

$= 24$

**Method 2:**

When $x = 8$,    $x$ is doubled from $x = 4$

$y = 2 \times 12$    $y$ is doubled from $y = 12$

$= 24$

We can also use $\frac{y_2}{y_1} = \frac{x_2}{x_1}$,

i.e. $\frac{y}{12} = \frac{8}{4}$

$y = 2 \times 12$

$= 24$

**(iii)** Substitute $y = 21$ into $y = 3x$:

$$21 = 3x$$
$$\therefore x = \frac{21}{3}$$
$$= 7$$

**Reflection**

In **(iii)**, can we use $\frac{21}{x} = \frac{12}{4}$? Which method do you prefer? How are these methods related?

**Practise Now 2**

Similar and Further Questions
**Exercise 7A**
Questions 3, 4, 7, 8, 9(a), (b)

1. If $y$ is directly proportional to $x$ and $y = 10$ when $x = 2$, find
   **(i)** an equation connecting $x$ and $y$,
   **(ii)** the value of $y$ when $x = 10$,
   **(iii)** the value of $x$ when $y = 60$.
2. If $y$ is directly proportional to $x$ and $y = 5$ when $x = 2$, find the value of $y$ when $x = 7$.
3. Given that $q$ is directly proportional to $p$, copy and complete the table.

| $p$ | 4 | 5 | 7 | | |
|---|---|---|---|---|---|
| $q$ | | 30 | | 48 | 57 |

**Worked Example 3**

## Problem involving direct proportion

The cost, $\$C$, of catering food at a tea party is directly proportional to the number of guests, $N$. It costs \$1200 to cater food for 30 guests.

**(i)** Find an equation connecting $C$ and $N$.
**(ii)** Calculate the cost of catering food for 70 guests.
**(iii)** Draw the graph of $C$ against $N$.

**Solution**

**(i)** Since $C$ is directly proportional to $N$,
then $C = kN$, where $k$ is a constant.
When $N = 30$, $C = 1200$,
$1200 = k \times 30$
$\therefore k = 40$
$\therefore C = 40N$

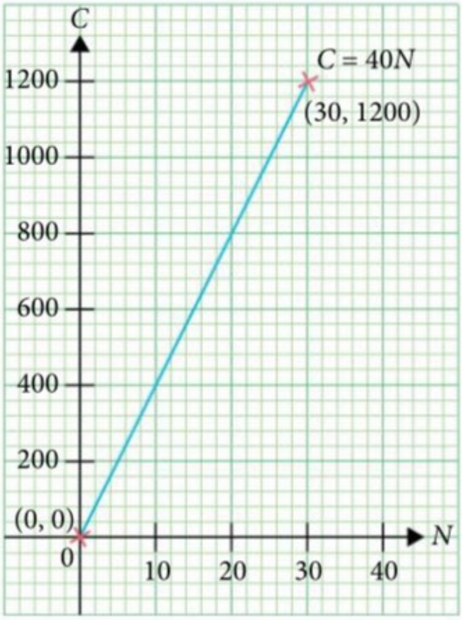
**(ii)** When $N = 70$,
$C = 40 \times 70$
$= 2800$
$\therefore$ it costs \$2800 to cater food for 70 guests.

**Attention**

An algebraic method is used in Worked Example 3. We can also use the unitary method or the proportion method (see Worked Example 1). However, the algebraic method is useful for some forms of direct proportion where the other methods will not work (see Worked Example 6).

(iii) $C = 40N$

Since $C$ is directly proportional to $N$, then the graph passes through the origin.
From part (i), when $N = 30$, $C = 1200$.




**Practise Now 3**

Similar and Further Questions
**Exercise 7A**
Questions 10–14

The cost, $\$C$, of transporting goods is directly proportional to the distance covered, $d$ km. The cost of transporting goods over a distance of 60 km is \$100.

**(i)** Find an equation connecting $C$ and $d$.

**(ii)** Find the cost of transporting goods over a distance of 45 km.

**(iii)** If the cost of transporting goods is \$120, calculate the distance covered.

**(iv)** Draw the graph of $C$ against $d$.

**Worked Example 4**

## Non-example of direct proportion

The total monthly charges, $\$C$, for a mobile plan consist of a fixed amount of \$20 and a variable amount which depends on the usage. For every minute used, \$0.20 is charged.

**(i)** If the duration of usage is 120 minutes, calculate the total monthly charges for the mobile plan.

**(ii)** If the total monthly charges for the mobile plan are \$50, find the duration of usage.

**(iii)** Write down a formula connecting $C$ and $n$, where $n$ is the number of minutes of usage.

**(iv)** Draw the graph of $C$ against $n$. Is $C$ directly proportional to $n$? Use your graph to explain your answer.

### Solution

**(i)** Total monthly charges for the mobile plan $= \$20 + 120 \times \$0.20$
$= \$44$

**(ii)** Variable amount $= \$50 - \$20$
$= \$30$

Duration of usage $= \dfrac{30}{0.20}$
$= 150$ minutes

(iii) Variable amount = $n \times \$0.20$

$= \$0.2n$

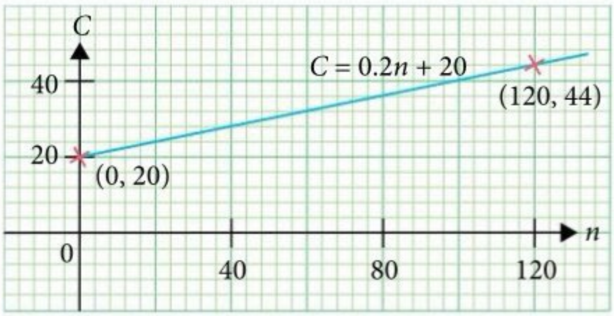
Total monthly charges = variable amount + fixed amount

$\therefore C = 0.2n + 20$

**(iv)** $C = 0.2n + 20$

When $n = 0$, $C = 20$.

From part **(i)**, when $n = 120$, $C = 44$.




$C$ is *not* directly proportional to $n$ because the line does not pass through the origin.

**Attention**

- Since $n$ cannot be negative, the line must start from $n = 0$.
- When $n = 120$, $C = 44$. When $n = 150$, $C = 50$. Since $\frac{44}{120} \neq \frac{50}{150}$, $\frac{C}{n}$ is not a constant. $\therefore$ $C$ is not directly proportional to $n$.

**Practise Now 4**

Similar and Further Questions
**Exercise 7A**
Questions 15–17

The total monthly cost, $\$C$, of running a kindergarten consists of a fixed amount of \$5000 and a variable amount which depends on the enrolment. For every child enrolled, the monthly cost increases by \$41.

**(i)** If the enrolment is 80, find the total monthly cost of running the kindergarten.

**(ii)** If the total monthly cost of running the kindergarten is \$7378, calculate the number of children enrolled in the kindergarten.

**(iii)** Write down a formula connecting $C$ and $n$, where $n$ is the number of children enrolled in the kindergarten.

**(iv)** Draw the graph of $C$ against $n$. Is $C$ directly proportional to $n$? Use your graph to explain your answer.

## Reflection

1. What do I already know about formulae and graphs of linear functions that could guide my learning in this section?
2. When two quantities are in direct proportion, can I
   **(a)** write the equation connecting the two quantities?
   **(b)** represent the relationship with a graph?
   **(c)** explain what the gradient of the graph represents?
   **Hint:** See Question 4 of the Investigation on page 210.

## Exercise 7A

1. 108 identical books have a mass of 30 kg. Find
   (i) the mass of 150 such books,
   (ii) the number of such books that have a mass of 20 kg.

2. In a bookstore, 60 identical books occupy a length of 1.5 m on a shelf. Find
   (i) the length occupied by 50 such books on the shelf,
   (ii) the number of such books needed to completely occupy a shelf that is 80 cm long.

3. If $x$ is directly proportional to $y$ and $x = 4.5$ when $y = 3$, find
   (i) an equation connecting $x$ and $y$,
   (ii) the value of $x$ when $y = 6$,
   (iii) the value of $y$ when $x = 12$.

4. If $Q$ is directly proportional to $P$ and $Q = 28$ when $P = 4$,
   (i) express $Q$ in terms of $P$,
   (ii) find the value of $Q$ when $P = 5$,
   (iii) calculate the value of $P$ when $Q = 42$.

5. Find the cost of
   (a) 10 kg of tea leaves when 3 kg of tea leaves cost \$18,
   (b) $a$ kg of sugar when $b$ kg of sugar cost \$$c$.

6. $\frac{5}{9}$ of a piece of metal has a mass of 7 kg. What is the mass of $\frac{2}{7}$ of the piece of metal?

7. If $z$ is directly proportional to $x$ and $z = 12$ when $x = 3$, find the value of $x$ when $z = 18$.

8. If $B$ is directly proportional to $A$ and $B = 3$ when $A = 18$, find the value of $B$ when $A = 24$.

9. For each of the following, $y$ is directly proportional to $x$. Copy and complete the tables.

(a)

| $x$ | 4 | 20 | 24 | | |
|---|---|---|---|---|---|
| $y$ | | | 6 | 9 | 11 |

(b)

| $x$ | 2 | 3 | 5.5 | | |
|---|---|---|---|---|---|
| $y$ | | 3.6 | | 9.6 | 11.4 |

10. If $y$ is directly proportional to $x$ and $y = 20$ when $x = 5$,
    (i) find an equation connecting $x$ and $y$,
    (ii) draw the graph of $y$ against $x$.

11. If $z$ is directly proportional to $y$ and $z = 48$ when $y = 6$,
    (i) find an equation connecting $y$ and $z$,
    (ii) draw the graph of $z$ against $y$.

12. The net force, $F$ newtons, needed to push a block along a horizontal surface is directly proportional to the mass, $m$ kg, of the block. When $m = 5$, $F = 49$.
    (i) Find an equation connecting $F$ and $m$.
    (ii) Find the value of $F$ when $m = 14$.
    (iii) Calculate the value of $m$ when $F = 215.6$.
    (iv) Draw the graph of $F$ against $m$.

13. The pressure, $P$ pascals, of a gas in a container is directly proportional to its temperature, $T$ kelvin. When $T = 10$, $P = 25$.
    (i) Find an equation connecting $P$ and $T$.
    (ii) Find the value of $P$ when $T = 24$.
    (iii) Calculate the value of $T$ when $P = 12$.
    (iv) Draw the graph of $P$ against $T$.

14. The amount of voltage, $V$ volts, needed to send a fixed amount of current through a wire, is directly proportional to its resistance, $R$ ohms. When $R = 6$, $V = 9$.
    (i) Find an equation connecting $V$ and $R$.
    (ii) Find the value of $V$ when $R = 15$.

Basic Intermediate Advanced

## Exercise 7A

(iii) Calculate the value of $R$ when $V = 15$.

(iv) Draw the graph of $V$ against $R$.

15. An ice-making machine needs to be switched on for 10 minutes *before* the production of ice begins. The mass, in tonnes, of ice produced is directly proportional to the number of hours of production. Given that 20 tonnes of ice are produced when the machine runs for half an hour, find the mass of ice manufactured when the machine runs for 1.75 hours.

16. The total monthly income, $\$D$, of a salesman who sells tyres consists of a basic salary of \$600 and a variable amount which depends on the number of tyres he sells. For each tyre he sells, he receives \$8.

(i) If the salesman sold 95 tyres in a particular month, find his total income for that month.

(ii) If the salesman's monthly income for a particular month was \$1680, calculate the number of tyres he sold in that month.

(iii) Write down a formula connecting $D$ and $n$, where $n$ is the number of tyres the salesman sells in a month.

(iv) Draw the graph of $D$ against $n$. Is $D$ directly proportional to $n$? Use your graph to explain your answer.

17. OPEN We often encounter different quantities involving direct proportion in our daily life. Describe two such quantities, and draw a graph to show their relationship. Can you find an equation connecting the quantities?

# 7.3 Other forms of direct proportion

In some cases, two quantities $x$ and $y$ may not be in direct proportion, but it may so happen that $x^m$ and $y^n$, where $m$ and $n$ are rational numbers, are in direct proportion.

### Investigation — Other forms of direct proportion

The variables $x$ and $y$ are connected by the equation $y = 3x^2$.

1. Some values of $x$, and the corresponding values of $y$ and $\frac{y}{x}$ are given in Table 7.4.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 3 | 12 | 27 | 48 |
| $\frac{y}{x}$ | 3 | 6 | 9 | 12 |

Table 7.4

**Attention**

In Chapter 4, we have learnt that a quadratic expression in one variable $x$ is of the form $ax^2 + bx + c$. Thus the equation $y = 3x^2$ is a *quadratic function*, and its graph in Fig. 7.1 is a *quadratic curve*.

We will learn more about quadratic functions and graphs in Book 3.

Fig. 7.1 shows the graph of $y$ against $x$.

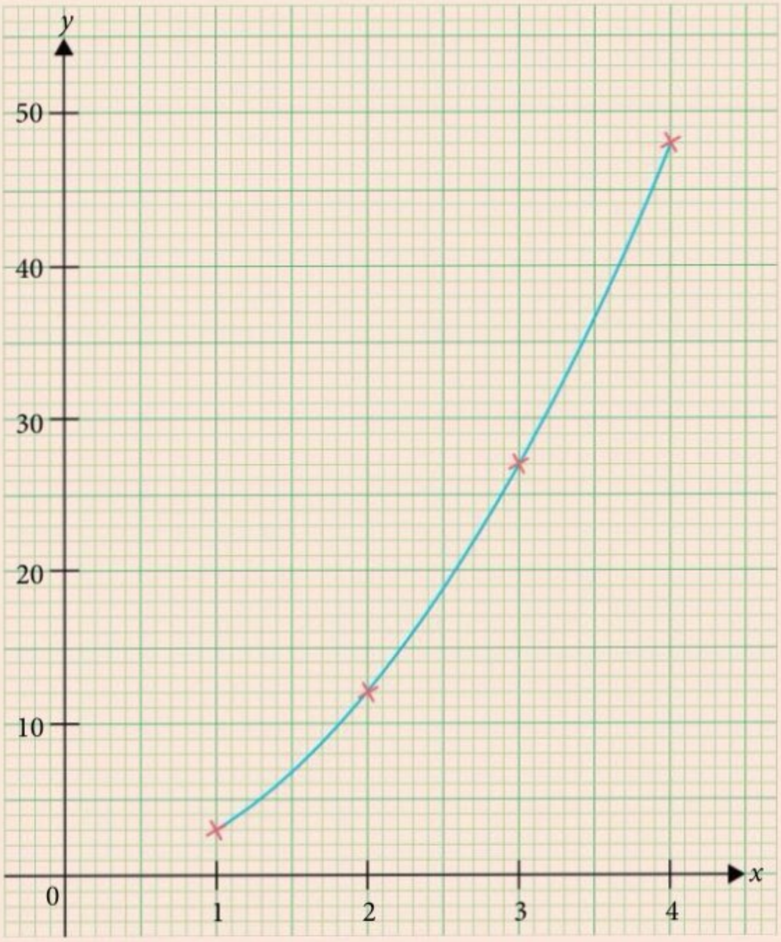



Fig. 7.1

Is $y$ directly proportional to $x$? Explain your answer.

2. Now, let us plot the graph of $y$ against $x^2$.

Some values of $x$, and the corresponding values of $x^2$, $y$ and $\frac{y}{x^2}$ are given in Table 7.5.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x^2$ | 1 | 4 | 9 | 16 |
| $y$ | 3 | 12 | 27 | 48 |
| $\frac{y}{x^2}$ | 3 | 3 | 3 | 3 |

Table 7.5

Plot the graph of $y$ against $x^2$ in Fig. 7.2. The first three points have been plotted for you.

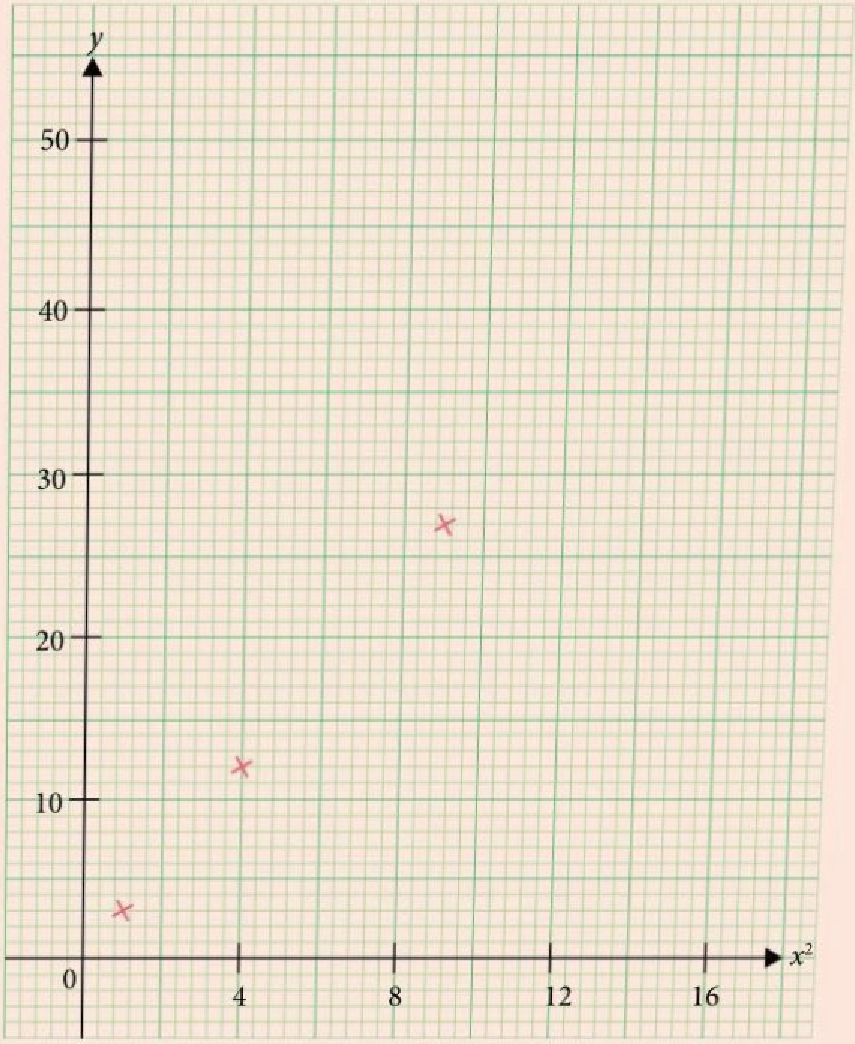



Fig. 7.2

Is $y$ directly proportional to $x^2$? Explain your answer.

**Attention**

Note that the horizontal axis is $x^2$ and not $x$.

From the above Investigation, when $y = 3x^2$, $y$ is not directly proportional to $x$ because $\frac{y}{x} = 3x$ is not a constant. However, $y$ is directly proportional to $x^2$ because $\frac{y}{x^2} = 3$ is a constant.

**Information**

Another way to look at this is to let $X = x^2$ such that $y = 3x^2$ becomes $y = 3X$, i.e. $\frac{y}{X} = 3$. Therefore, $y$ is directly proportional to $X$ $(= x^2)$.

**Worked Example 5**

### Identifying variables which are directly proportional to each other

For each of the following equations, state the two variables which are directly proportional to each other and explain your answer.

**(a)** $y = 5x^3$ **(b)** $y^2 = \sqrt{x}$

**Solution**

**(a)** Since $y = 5x^3$, i.e. $\frac{y}{x^3} = 5$ is a constant, then $y$ and $x^3$ are directly proportional to each other.

**(b)** Since $y^2 = \sqrt{x}$, i.e. $\frac{y^2}{\sqrt{x}} = 1$ is a constant, then $y^2$ and $\sqrt{x}$ are directly proportional to each other.

**Practise Now 5**

Similar and Further Questions
**Exercise 7B**
Questions 4(a)–(d)

For each of the following equations, state the two variables which are directly proportional to each other and explain your answer.

**(a)** $y = 6x^2$ **(b)** $\sqrt{y} = x^3$

**Worked Example 6**

### Equation of another form of direct proportion

If $y$ is directly proportional to $x^2$ and $y = 20$ when $x = 2$,

**(i)** find an equation connecting $x$ and $y$,

**(ii)** calculate the value of $y$ when $x = 3$,

**(iii)** find the values of $x$ when $y = 1.25$,

**(iv)** draw the graph of $y$ against $x^2$.

**Attention**

It is difficult to use the unitary method or the proportion method (see Worked Example 1) to solve direct proportion problems like in Worked Example 6. Thus we need to use the algebraic method (which we have learnt in Worked Example 3).

**Solution**

**(i)** Since $y$ is directly proportional to $x^2$,
then $y = kx^2$, where $k$ is a constant.
When $x = 2$, $y = 20$,

$$20 = k \times 2^2$$
$$20 = 4k$$
$$k = 5$$
$$\therefore y = 5x^2$$

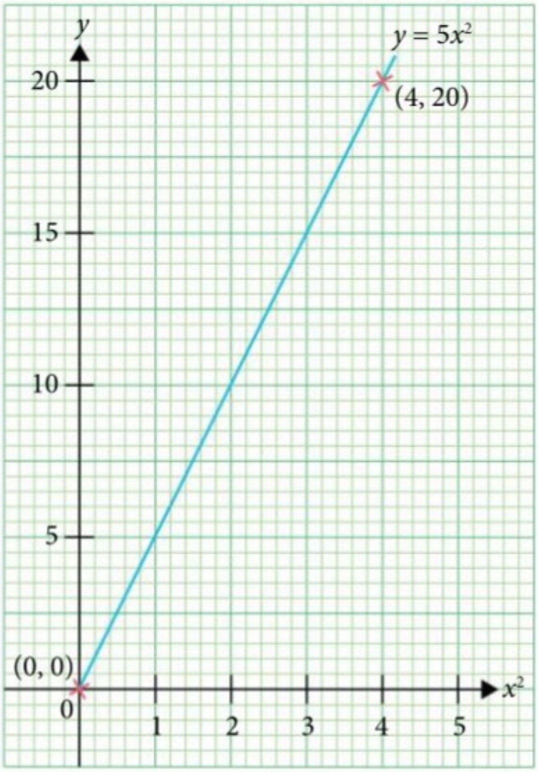
**(ii)** When $x = 3$,

$$y = 5 \times 3^2$$
$$= 45$$

**(iii)** When $y = 1.25$,

$$1.25 = 5x^2$$
$$x^2 = 0.25$$
$$\therefore x = \pm\sqrt{0.25} = \pm 0.5$$

**(iv)** Since $y$ is directly proportional to $x^2$, then the graph of $y$ against $x^2$ is a straight line that passes through the origin.
It was given that when $x = 2$, $y = 20$. When $x = 2$, $x^2 = 4$.
$\therefore$ the graph will pass through the point (4, 20).




**Attention**

Note that the horizontal axis is $x^2$ and not $x$. Also, as $x^2$ cannot be negative, the line must start from the origin where $x^2 = 0$.

**Practise Now 6**

Similar and Further Questions
**Exercise 7B**
Questions 1–3, 5–8, 11

1. If $y$ is directly proportional to $x^2$ and $y = 18$ when $x = 3$,
   **(i)** find an equation connecting $x$ and $y$,
   **(ii)** find the value of $y$ when $x = 5$,
   **(iii)** calculate the values of $x$ when $y = 32$,
   **(iv)** draw the graph of $y$ against $x^2$.
2. If $y$ is directly proportional to $x^2$ and $y = 21$ when $x = 2$, find the value of $y$ when $x = 4$.
3. Given that $k$ is directly proportional to $h^2$, where $h$ is a positive real number, copy and complete the table.

| $h$ | 2 | | 3 | 5 | |
|---|---|---|---|---|---|
| $k$ | | 56.25 | 81 | | 441 |

**Worked Example 7**

### Problem involving another form of direct proportion

The volume, $V$ cm³, of a solid is directly proportional to the cube of its radius, $r$ cm. When the radius of the solid is 6 cm, its volume is 905 cm³.

**(i)** Find an equation connecting $V$ and $r$.
**(ii)** Calculate the volume of the solid when its radius is 10 cm.

**Solution**

**(i)** Since $V$ is directly proportional to $r^3$, then $V = kr^3$, where $k$ is a constant.
When $r = 6$, $V = 905$,

$$905 = k \times 6^3$$

$$\therefore\ k = \frac{905}{216}$$

$$\therefore\ V = \frac{905}{216}r^3$$

(ii) When $r = 10$,

$$V = \frac{905}{216} \times 10^3$$
$$= 4190 \text{ (to 3 s.f.)}$$

$\therefore$ the volume of the solid is $4190\ \text{cm}^3$.

**Practise Now 7**

Similar and Further Questions
**Exercise 7B**
Questions 9, 10, 12, 13

The length, $l$ cm, of a simple pendulum is directly proportional to the square of its period (time taken to complete one oscillation), $T$ seconds. A pendulum with a length of 55.8 cm has a period of 1.5 seconds.

**(i)** Find an equation connecting $l$ and $T$.

**(ii)** Find the length of a pendulum which has a period of 0.8 seconds.

**(iii)** What is the period of a pendulum which has a length of 0.36 m?

## Reflection

From the above examples, how do I tell whether two variables are directly proportional to each other?

Advanced | Intermediate | Basic

## Exercise 7B

**1.** If $x$ is directly proportional to $y^3$ and $x = 32$ when $y = 2$,
**(i)** find an equation connecting $x$ and $y$,
**(ii)** find the value of $x$ when $y = 6$,
**(iii)** calculate the value of $y$ when $x = 108$,
**(iv)** draw the graph of $x$ against $y^3$.

**2.** If $z^2$ is directly proportional to $w$ and $z = 4$ when $w = 8$,
**(i)** find an equation connecting $w$ and $z$,
**(ii)** find the values of $z$ when $w = 18$,
**(iii)** calculate the value of $w$ when $z = 5$,
**(iv)** draw the graph of $z^2$ against $w$.

**3.** It is given that $y$ is directly proportional to $x^n$. Write down the value of $n$ when
**(i)** $y\ \text{m}^2$ is the area of a square of length $x$ m,
**(ii)** $y\ \text{cm}^3$ is the volume of a cube of length $x$ cm.

**4.** For each of the following equations, state the two variables which are directly proportional to each other and explain your answer.
**(a)** $y = 4x^2$  **(b)** $y = 3\sqrt{x}$
**(c)** $y^2 = 5x^3$  **(d)** $p^3 = q^2$

**5.** If $z^2$ is directly proportional to $x^3$ and $z = 8$ when $x = 4$, find the values of $z$ when $x = 9$.

**6.** If $q$ is directly proportional to $(p - 1)^2$ and $q = 20$ when $p = 3$, find the values of $p$ when $q = 80$.

**7.** Given that $y$ is directly proportional to $x^3$, copy and complete the table.

| $x$ | 3 | 4 | | 6 | |
|---|---|---|---|---|---|
| $y$ | | | 375 | 648 | 1029 |

## Exercise 7B

**8.** Given that the mass, $m$ g, of a sphere is directly proportional to the cube of its radius, $r$ cm, copy and complete the table.

| $r$ | 0.2 | | 0.7 | 1.5 | |
|---|---|---|---|---|---|
| $m$ | | 0.25 | | 6.75 | 11.664 |

**9.** During a certain period in the life of an earthworm, its length, $L$ cm, is directly proportional to the square root of $N$, where $N$ is the number of hours after its birth. The length of an earthworm one hour after its birth is 2.5 cm.
**(i)** Find an equation connecting $L$ and $N$.
**(ii)** Find the length of an earthworm 4 hours after its birth.
**(iii)** How long will it take for an earthworm to grow to a length of 15 cm?

**10.** If $y$ is directly proportional to $x^2$ for all positive values of $y$ and the difference in the values of $y$ when $x = 1$ and $x = 3$ is 32, find the value of $y$ when $x = -2$.

**11.** $y$ is directly proportional to $x^2$ and $y = a$ for a particular value of $x$. Find an expression for $y$ in terms of $a$, when this value of $x$ is doubled.

**12.** The braking distance of a vehicle is directly proportional to the square of its speed. When the speed of the vehicle is $b$ m/s, its braking distance is $d$ m. If the speed of the vehicle is increased by 200%, find the percentage increase in its braking distance.

**13.** The table shows a relationship between $x$ and $y$.

| $x$ | 2 | 4 | 5 | 8 | 10 |
|---|---|---|---|---|---|
| $y$ | 5.2 | 41.6 | 81.25 | 332.8 | 650 |

**(i)** Explain if $y$ is directly proportional to $x$ or $x^3$.
OPEN **(ii)** Give a real-life example of what the variables $x$ and $y$ can be.

# 7.4 Inverse proportion

## Investigation Inverse proportion

Table 7.6 shows the time taken for a car to travel a distance of 120 km at different speeds.

| **Speed ($x$ km/h)** | 10 | 20 | 30 | 40 | 60 | 120 |
|---|---|---|---|---|---|---|
| **Time taken ($y$ hours)** | 12 | 6 | 4 | 3 | 2 | 1 |

Table 7.6

1. If the speed of the car increases, will the time taken increase or decrease?
2. If the speed of the car is doubled, how will the time taken change?
   Hint: Compare the time taken when the speeds of the car are 20 km/h and 40 km/h.
3. If the speed of the car is tripled, what will happen to the time taken?
4. If the speed of the car is halved, how will the time taken change?
   Hint: Compare the time taken when the speeds of the car are 60 km/h and 30 km/h.
5. If the speed of the car is reduced to $\frac{1}{3}$ of its original speed, what will happen to the time taken?

From the Investigation on page 222, we observe the following:

$$\text{doubled}\left(\begin{array}{l} x = 20,\ y = 6 \\ x = 40,\ y = 3 \end{array}\right)\text{halved} \qquad \text{and} \qquad \text{halved}\left(\begin{array}{l} x = 60,\ y = 2 \\ x = 30,\ y = 4 \end{array}\right)\text{doubled}$$

That is, as the speed of the car, $x$ km/h, increases, the time taken, $y$ hours, decreases *proportionally*, and vice versa.

This relationship is known as **inverse proportion**. We say that the time taken, $y$ hours, is *inversely proportional* to the speed of the car, $x$ km/h.

## Class Discussion — Real-life examples of quantities in inverse proportion

1. Give a few more real-life examples of quantities that are in inverse proportion.
2. Explain why they are inversely proportional to each other.

Similar and Further Questions
**Exercise 7C**
Questions 1(a)–(e)

Copy and complete Table 7.7.

| **Speed ($x$ km/h)** | 10 | 20 | 30 | 40 | 60 | 120 |
|---|---|---|---|---|---|---|
| **Time taken ($y$ h)** | 12 | 6 | 4 | 3 | 2 | 1 |
| **Product ($xy$ km)** | $10 \times 12 = 120$ | $20 \times 6 = 120$ | | | | |

Table 7.7

What can we observe about the product $xy$?

In inverse proportion, the product $xy$ is a *constant*.

In this case, $xy = 120$ = distance travelled.

Let the speed of the car be $x_1 = 20$. Then the corresponding time taken is $y_1 = 6$.
Let the speed of the car be $x_2 = 40$. Then the corresponding time taken is $y_2 = 3$.
From Table 7.7, $x_1y_1 = 20 \times 6 = 120$ and $x_2y_2 = 40 \times 3 = 120$.

$\therefore\ x_1y_1 = x_2y_2 = 120$ (constant)

To conclude, we have:

> If $y$ is *inversely proportional* to $x$, then $x_2y_2 = x_1y_1$.

**Attention**

If two quantities $x$ and $y$ are in direct proportion, $\dfrac{y_2}{y_1} = \dfrac{x_2}{x_1}$.

For inverse proportion, $\dfrac{y_2}{y_1} = \dfrac{x_1}{x_2}$.

It is easier to rewrite this as $x_2y_2 = x_1y_1$ to solve problems, i.e. the product of the two variables is a constant. Does it matter whether we write the equality of product as $x_2y_2 = x_1y_1$ or $x_1y_1 = x_2y_2$? Explain.

**Worked Example 8**

### Problem involving inverse proportion

10 identical taps can fill a tank in 4 hours. Calculate the time taken for 8 such taps to fill the same tank.

**Solution**

First, we note that the time taken to fill the tank is inversely proportional to the number of taps used because as the number of taps increases, the time taken to fill the tank decreases.

**Method 1: Unitary method**

10 taps can fill the tank in 4 hours.

1 tap can fill the tank in $(10 \times 4)$ hours. fewer taps require more time

8 taps can fill the tank in $\frac{10 \times 4}{8} = 5$ hours. more taps require less time

**Method 2: Proportion method**

Let the time taken for 8 taps to fill the tank be $y$ hours.

Then $8y = 10 \times 4$. $x_2y_2 = x_1y_1$

$$y = \frac{10 \times 4}{8}$$
$$= 5$$

$\therefore$ 8 taps can fill the tank in 5 hours.

**Practise Now 8**

Similar and Further Questions
Exercise 7C
Questions 2, 5–7, 13

A tank can be filled by 4 identical taps in 70 minutes. Find the time taken for 7 such taps to fill the same tank.

**Worked Example 9**

## Problem involving direct and inverse proportions

In 3 days, 5 men can paint 2 identical houses. Assuming that all the men work at the same rate, how long will it take 6 men to paint 7 such houses?

### Solution

5 men can paint 2 houses in $3$ days.

1 man can paint 2 houses in $3 \times 5 = 15$ days.

6 men can paint 2 houses in $\frac{15}{6} = 2.5$ days.

6 men can paint 1 house in $\frac{2.5}{2} = 1.25$ days.

6 men can paint 7 houses in $1.25 \times 7 = 8.75$ days.

$\therefore$ 6 men will take 8.75 days to paint 7 such houses.

**Problem-solving Tip**

The 3 variables are the number of men, the number of houses and the number of days. At each stage, we have to keep 1 of the variables constant. E.g. at the start, we keep the number of houses constant at 2. Then the number of men is inversely proportional to the number of days. After we obtain 6 men, we keep the number of men constant. Then the number of houses is directly proportional to the number of days.

**Practise Now 9**

Similar and Further Questions
Exercise 7C
Questions 14, 15

**(a)** In 5 hours, 3 men can dig 2 identical trenches. Assuming that all the men work at the same rate, how long will it take 5 men to dig 7 such trenches?

**(b)** Given that 7 identical taps can fill 3 identical tanks in 45 minutes, how long will it take 5 of the taps to fill 1 such tank?

**Introductory Problem Revisited**

After learning about direct and inverse proportions, can you now solve the Introductory Problem, if you have not already done so?
If you have already solved it, was your method similar to, or more efficient than the method used in Worked Example 9?

## 7.5 Algebraic and graphical representations of inverse proportion

In the Investigation on page 222 of Section 7.4, we have found that $xy = 120$, which is a constant. If we represent this constant by $k$, then $xy = k$ or $y = \frac{k}{x}$, where $k \neq 0$.

Hence, we have:

> If $y$ is *inversely proportional* to $x$, then $xy = k$ or $y = \frac{k}{x}$, where $k$ is a constant and $k \neq 0$.

**Thinking Time**

If we substitute $k = 0$ into $y = \frac{k}{x}$, what can we say about the relationship between $x$ and $y$?

## Investigation Graphical representation of inverse proportion

Consider the example of the car in Section 7.4. Table 7.8 shows the time taken, $y$ hours, for the car to travel a distance of 120 km at different speeds, $x$ km/h, where $xy = 120$ or $y = \frac{120}{x}$.

| Speed ($x$ km/h) | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Time taken ($y$ hours) | 12 | 6 | 4 | 3 | 2.4 | 2 | 1.7 | 1.5 | 1.3 | 1.2 | 1.1 | 1 |

Table 7.8

Fig. 7.3 shows the graph of $y$ against $x$.

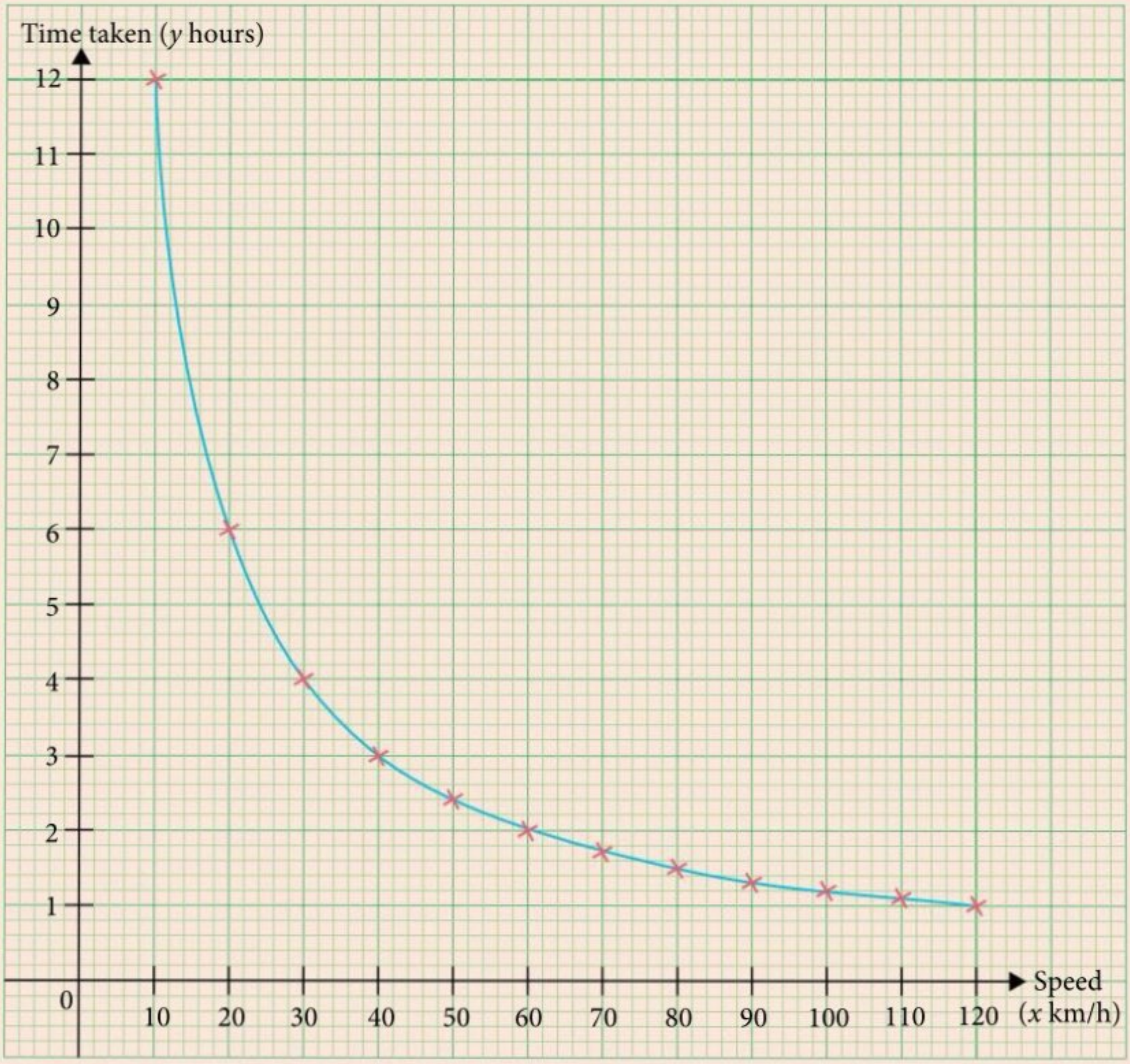



Fig. 7.3

1. What is the shape of the graph? Is it a straight line?
2. If the value of $x$ is doubled, how will the value of $y$ change?
   Illustrate this with an example by choosing two appropriate points on the graph in Fig. 7.3.

What if we plot the graph of $y$ against $\frac{1}{x}$ ? How would the graph look?

Let $X = \frac{1}{x}$ . Some values of $x$ and the corresponding values of $X$ and $y$ are given in Table 7.9.

3. (i) Copy and complete the table.

| **Speed ($x$ km/h)** | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X = \frac{1}{x}$ | 0.1 | 0.05 | 0.033 | | 0.02 | 0.017 | 0.014 | 0.013 | 0.011 | | 0.009 | 0.008 |
| **Time taken ($y$ hours)** | 12 | 6 | 4 | 3 | 2.4 | 2 | 1.7 | 1.5 | 1.3 | 1.2 | 1.1 | 1 |

Table 7.9

(ii) Plot the graph of $y$ against $X$ in Fig. 7.4.

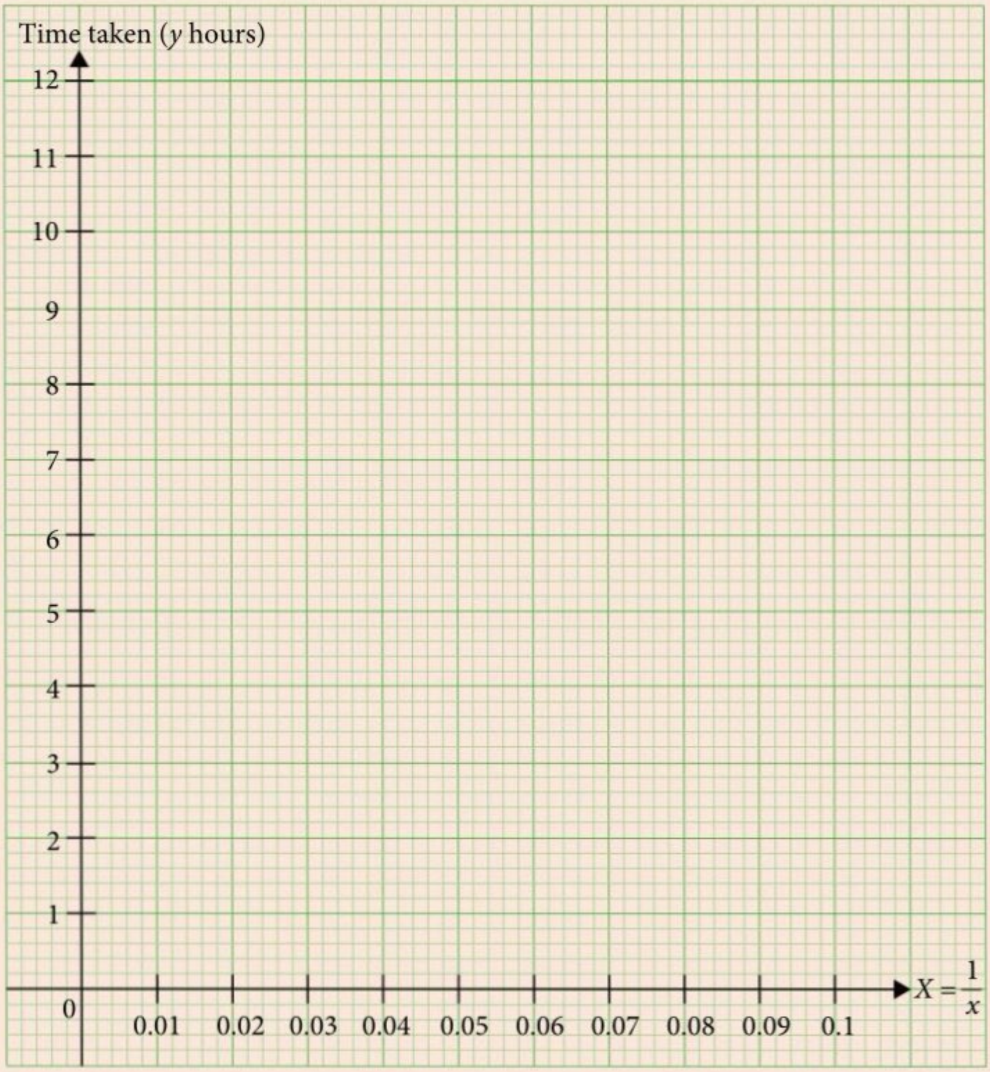


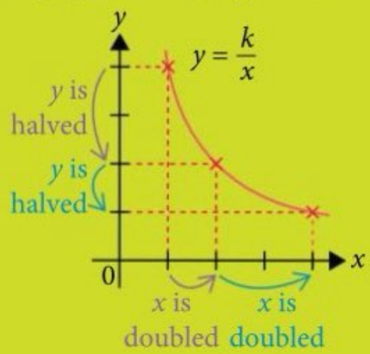

Fig. 7.4

4. Describe the graph obtained. What can we say about the relationship between $y$ and $X$?
5. Although $y$ is inversely proportional to $x$, what is the relationship between $y$ and $X$?
6. Write down an equation connecting $y$ and $X$. What does it tell you about the relationship between $y$ and $X$?

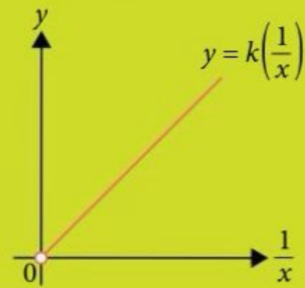
In general,

if $y$ is *inversely proportional* to $x$, then

- $xy = k$ or $y = \frac{k}{x}$, where $k$ is a constant and $k \neq 0$,
- the graph of $y$ against $x$ is a hyperbola,




- the graph of $y$ against $\frac{1}{x}$ is a straight line ($x \neq 0$).




**Attention**

If $y$ is inversely proportional to $x$, we can also say that $y$ is directly proportional to $\frac{1}{x}$.

We can write this as $y = k\left(\frac{1}{x}\right)$, i.e. $y = \frac{k}{x}$.

**Thinking Time**

If $y$ is inversely proportional to $x$, is $x$ inversely proportional to $y$? Explain your answer.

**Worked Example 10**

### Equation of inverse proportion

If $y$ is inversely proportional to $x$ and $y = 3$ when $x = 4$, find

**(i)** the value of $y$ when $x = 8$,

**(ii)** an equation connecting $x$ and $y$,

**(iii)** the value of $x$ when $y = 48$.

**Solution**

**(i)** When $x = 8$, — $x$ is doubled from $x = 4$

$y = \frac{3}{2}$ — $y$ is halved from $y = 3$

Alternatively,

$$x_2y_2 = x_1y_1$$
$$8 \times y = 4 \times 3$$
$$y = \frac{12}{8}$$
$$= \frac{3}{2}$$

**(ii)** Since $y$ is inversely proportional to $x$,

then $y = \frac{k}{x}$, where $k$ is a constant.

When $x = 4$, $y = 3$,

$$3 = \frac{k}{4}$$
$$\therefore k = 12$$
$$\therefore y = \frac{12}{x}$$

**(iii)** When $y = 48$,

$$48 = \frac{12}{x}$$
$$\therefore x = \frac{12}{48}$$
$$= \frac{1}{4}$$

**Attention**

**(ii)** Since $y$ is inversely proportional to $x$, then $xy = k$ or $y = \frac{k}{x}$, where $k$ is a constant and $k \neq 0$.

**Practise Now 10**

Similar and Further Questions
**Exercise 7C**
Questions 3, 4, 8, 9, 10(a), (b), 16

1. If $y$ is inversely proportional to $x$ and $y = 5$ when $x = 2$, find
   **(i)** the value of $y$ when $x = 8$,
   **(ii)** an equation connecting $x$ and $y$,
   **(iii)** the value of $x$ when $y = 10$.
2. If $y$ is inversely proportional to $x$ and $y = 9$ when $x = 2$, find the value of $y$ when $x = 3$.
3. Given that $n$ is inversely proportional to $m$, copy and complete the table.

| $m$ | 0.5 | | 2 | 3 | |
|---|---|---|---|---|---|
| $n$ | | 4 | 2 | | 0.8 |

Worked Example 11

## Problem involving inverse proportion

Boyle's Law states that the volume, $V$ dm$^3$, of a fixed mass of gas at constant temperature is inversely proportional to its pressure, $P$ pascals (Pa). The pressure of 1 dm$^3$ of a gas in an airtight container is 50 Pa. Assuming that the temperature in the container is constant, calculate the volume of the gas when its pressure is 1250 Pa.

**Information**

1 dm = 10 cm
1 dm$^3$ = 1000 cm$^3$
(dm$^3$ means 'cubic decimetre')

### Solution

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

Stage 1: **Understand the problem**

*What are the two variables involved? Does the question state whether the relationship between V and P is a direct or an inverse proportion?*

Stage 2: **Think of a plan**

*What have we learnt about inverse proportion that can help us to solve this problem? Can we use the idea of* $y = \frac{k}{x}$ *or the idea of* $xy$ = *constant to solve this problem?*

Method 1: **Using the idea of $y = \frac{k}{x}$**

Stage 3: **Carry out the plan**

Since $V$ is inversely proportional to $P$, then $V = \frac{k}{P}$, where $k$ is a constant.

$k = PV$

When $P = 50$, $V = 1$,

$$k = 50 \times 1$$
$$= 50$$
$$\therefore V = \frac{50}{P}$$

When $P = 1250$,

$$V = \frac{50}{1250}$$
$$= 0.04$$

$\therefore$ the volume of the gas is 0.04 dm$^3$.

Stage 4: **Look back**

*Since this is an inverse proportion, when the pressure increases from 50 Pa to 1250 Pa, should the volume of the gas increase or decrease from 1 dm$^3$? Hence, is the answer reasonable?*
*Is there another method to solve this problem? Will it be more efficient?*

Method 2: **Using the idea of $xy$ = constant**

Stage 3: **Carry out the plan**

$$V_2P_2 = V_1P_1$$
$$V_2 \times 1250 = 1 \times 50$$
$$V_2 = \frac{50}{1250}$$
$$= 0.04$$

$\therefore$ the volume of the gas is 0.04 dm$^3$.

**Stage 4: Look back**

*Is the answer obtained from the second method the same as that from the first method? Which method do you prefer? Why?*

**Practise Now 11**

Similar and Further Questions
Exercise 7C
Questions 11, 12

The current, $I$ amperes (A), flowing through a wire is inversely proportional to its resistance, $R$ ohms ($\Omega$). Given that the current flowing through a wire with a resistance of $0.5\ \Omega$ is 12 A, find

**(i)** the current flowing through the wire when its resistance is $3\ \Omega$,

**(ii)** the resistance of the wire when the current flowing through it is 3 A.

**Just For Fun**

Raju and Albert are going to compete in a race. Raju's average speed is twice that of Albert's. Albert wants to start 10 m in front of Raju. Albert says, 'After Raju runs 10 m, I will be 5 m in front of him. After Raju runs another 5 m, I will be 2.5 m in front of him. Thus Raju will never catch up with me.'
Is Albert correct?
This is a variation of one of Zeno's paradoxes. Zeno (490–430 BC) was a Greek philosopher.

Advanced | Intermediate | Basic

## Exercise 7C

**1.** Which of the following quantities are in inverse proportion? State the assumption made in each case.

**(a)** The number of pencils Ali buys and the total cost of the pencils.

**(b)** The number of taps filling a tank and the time taken to fill the tank.

**(c)** The number of men laying a road and the time taken to finish laying the road.

**(d)** The number of cattle to be fed and the amount of fodder.

**(e)** The number of cattle to be fed and the time taken to finish a certain amount of the fodder.

**2.** Given that 8 men can build a bridge in 12 days, find the time taken for 6 men to build the same bridge. State the assumption made.

**3.** If $x$ is inversely proportional to $y$ and $x = 40$ when $y = 5$, find

**(i)** the value of $x$ when $y = 25$,

**(ii)** an equation connecting $x$ and $y$,

**(iii)** the value of $y$ when $x = 400$.

**4.** If $Q$ is inversely proportional to $P$ and $Q = 0.25$ when $P = 2$,

**(i)** express $Q$ in terms of $P$,

**(ii)** find the value of $Q$ when $P = 5$,

**(iii)** calculate the value of $P$ when $Q = 0.2$.

**5.** 35 workers are employed to complete a construction project in 16 days. Before the project starts, the boss of the company is told that the project has to be completed in 14 days. Assuming that all the workers work at the same rate, how many more workers does he need to employ in order to complete the project on time?

Basic | Intermediate | Advanced

## Exercise 7C

6. A consignment of fodder can feed 1260 cattle for 50 days. Given that all the cattle consume the fodder at the same rate, find
   (i) the number of cattle an equal consignment of fodder can feed for 75 days,
   (ii) the number of days an equal consignment of fodder can last if it is used to feed 1575 cattle.

7. At a sports camp, there is sufficient food for 72 athletes to last 6 days. If 18 athletes are absent from the camp, how many more days can the food last for the other athletes? State the assumption made.

8. If $z$ is inversely proportional to $x$ and $z = 5$ when $x = 7$, find the value of $x$ when $z = 70$.

9. If $B$ is inversely proportional to $A$ and $B = 3.5$ when $A = 2$, find the value of $B$ when $A = 1.4$.

10. For each of the following, $y$ is inversely proportional to $x$. Copy and complete the tables.

(a)

| $x$ | | 2 | 2.5 | 3 | |
|---|---|---|---|---|---|
| $y$ | 24 | | | 4 | 1.5 |

(b)

| $x$ | 3 | 4 | | | 25 |
|---|---|---|---|---|---|
| $y$ | | 9 | 8 | 2.5 | |

11. The frequency, $f$ kilohertz (kHz), of a radio wave is inversely proportional to its wavelength, $\lambda$ m. The frequency of a radio wave that has a wavelength of 3000 m is 100 kHz. Find
   (i) the frequency of a radio wave that has a wavelength of 500 m,
   (ii) the wavelength of a radio wave that has a frequency of 800 kHz.

12. The time, $t$ hours, taken to complete a job is inversely proportional to $N$, where $N$ is the number of men working on it. Three men can complete the job in 8 hours.
   (i) Find an equation connecting $t$ and $N$.
   (ii) Find the number of hours 6 men will take to complete the job.
   (iii) How many men are required to complete the job in $\frac{3}{4}$ hour?

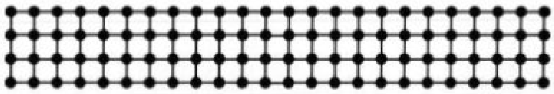
13. Tap A takes 6 minutes to fill a tank and Tap B takes 9 minutes to fill the same tank. Pipe C can empty the same tank in 15 minutes. How long will it take to fill up the tank if the pipe is in use when both taps are turned on?

14. 12 glassblowers can make 12 identical vases in 9 minutes. Assuming that all the glassblowers work at the same rate, how long will it take 8 glassblowers to make 32 such vases?

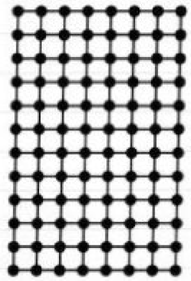
15. A contractor agrees to lay a road 3000 m long in 30 days. 50 men are employed and they work for 8 hours per day. After 20 working days, he finds that only 1200 m of the road is completed. How many more men does he need to employ in order to finish the project on time if each man now works 10 hours a day?

## Exercise 7C

16. 96 counters are arranged in different numbers of rows and columns so that they form a rectangular array. Some examples are shown below.



4 rows 24 columns



12 rows 8 columns

By creating a table for number of rows, $R$, and number of columns, $C$, plot the graph of $R$ against $C$. Hence deduce the relationship between $R$ and $C$ and express $R$ in terms of $C$.

# 7.6 Other forms of inverse proportion

We have learnt that if $y$ is inversely proportional to $x$, then $xy = k$ or $y = \dfrac{k}{x}$, where $k$ is a constant and $k \neq 0$.

Similarly, if $y$ is inversely proportional to $x^2$, then $x^2y = k$ or $y = \dfrac{k}{x^2}$, where $k$ is a constant and $k \neq 0$.

**Information**

Another way to look at this is to let $X = x^2$ such that

$x^2y = k$ becomes $Xy = k$

or $y = \dfrac{k}{x^2}$ becomes $y = \dfrac{k}{X}$,

where $k$ is a constant and $k \neq 0$. Therefore, $y$ is inversely proportional to $X\ (= x^2)$.

**Worked Example 12**

### Identifying variables which are inversely proportional to each other

For each of the following equations, state the two variables which are inversely proportional to each other and explain your answer.

**(a)** $y = \dfrac{2}{x^3}$ **(b)** $y = \dfrac{3}{\sqrt{x}}$

**Solution**

**(a)** Since $y = \dfrac{2}{x^3}$, i.e. $x^3y = 2$ is a constant, then $y$ and $x^3$ are inversely proportional to each other.

**(b)** Since $y = \dfrac{3}{\sqrt{x}}$, i.e. $y\sqrt{x} = 3$ is a constant, then $y$ and $\sqrt{x}$ are inversely proportional to each other.

**Practise Now 12**

Similar and Further Questions
**Exercise 7D**
Questions 3(a)–(e)

For each of the following equations, state the two variables which are inversely proportional to each other and explain your answer.

**(a)** $y = \frac{4}{x^2}$ **(b)** $y^2 = \frac{1}{\sqrt[3]{x}}$ **(c)** $y = \frac{5}{x+2}$

**Worked Example 13**

## Equation of another form of inverse proportion

If $y$ is inversely proportional to $\sqrt{x}$ and $y = 6$ when $x = 4$,

**(i)** calculate the value of $y$ when $x = 16$,
**(ii)** find an equation connecting $x$ and $y$,
**(iii)** find the value of $x$ when $y = 4$.

**Recall**

The square root sign $\sqrt{\ }$ is used to denote the *positive square root* only. Thus $\sqrt{4} = 2$ and $-\sqrt{4} = -2$. However, if $x^2 = 4$, then $x = \pm\sqrt{4} = \pm 2$. If we want both the positive and the negative square roots, we need to write $\pm\sqrt{\ }$.

**Solution**

**(i)** Since $y$ is inversely proportional to $\sqrt{x}$,
then $y\sqrt{x}$ is a constant.

$$\therefore y\sqrt{16} = 6\sqrt{4}$$
$$4y = 12$$
$$y = 3$$

**(ii)** Since $y$ is inversely proportional to $\sqrt{x}$, then $y = \frac{k}{\sqrt{x}}$, where $k$ is a constant.
When $x = 4$, $y = 6$,

$$6 = \frac{k}{\sqrt{4}}$$
$$6 = \frac{k}{2}$$
$$k = 12$$
$$\therefore y = \frac{12}{\sqrt{x}}$$

**(iii)** When $y = 4$,

$$4 = \frac{12}{\sqrt{x}}$$
$$\sqrt{x} = \frac{12}{4}$$
$$= 3$$
$$x = 9$$

**Practise Now 13**

Similar and Further Questions
**Exercise 7D**
Questions 1, 2, 4–6, 9, 10

1. If $y$ is inversely proportional to $x^2$ and $y = 2$ when $x = 4$,
   **(i)** find the value of $y$ when $x = 8$,
   **(ii)** find an equation connecting $x$ and $y$,
   **(iii)** calculate the values of $x$ when $y = 8$.
2. If $y$ is inversely proportional to $\sqrt{x}$ and $y = 6$ when $x = 9$, find the value of $y$ when $x = 25$.
3. Given that $b$ is inversely proportional to $\sqrt{a}$, copy and complete the table.

| | | | | | |
|---|---|---|---|---|---|
| ***a*** | | 1 | 4 | 16 | |
| ***b*** | 16 | 8 | | | $\frac{4}{3}$ |

**Worked Example 14**

### Solving problem involving another form of inverse proportion

In a computer simulation of an experiment, a drug is added to two identical flasks, each containing the same amount of a certain bacteria. The drug is allowed to react with the bacteria for various times in $t$ hours. It is found that the amount of bacteria left, $s$ units, is inversely proportional to $(t - 2)$ hours. In one flask, there are 6 units of bacteria left after 5 hours. Calculate the amount of bacteria left in the other flask after 7 hours.

**Solution**

Since $s$ is inversely proportional to $t - 2$, then $s(t - 2)$ is a constant.

$$\therefore s(7 - 2) = 6(5 - 2)$$
$$5s = 18$$
$$s = \frac{18}{5}$$
$$= 3.6$$

$\therefore$ the amount of bacteria left in the other flask after 7 hours is 3.6 units.

**Reflection**

Can we use the idea of $s = \frac{k}{t-2}$ to solve this problem? Which method do you prefer? Why?

**Practise Now 14**

Similar and Further Questions
**Exercise 7D**
Questions 7, 8, 11, 12

The force, $F$ newtons (N), between two particles is inversely proportional to the square of the distance, $d$ m, between them. When the particles are 2 m apart, the force between them is 10 N. Find

**(i)** the force between the particles when they are 5 m apart,

**(ii)** the distance between the particles when the force between them is 25 N.

## Performance Task

Work in groups to collect data of proportional quantities in real life.
Plot a graph to show how they are related and present your findings to the class.

Basic Intermediate Advanced

## Exercise 7D

**1.** If $x$ is inversely proportional to $y^3$ and $x = 50$ when $y = 2$,
**(i)** find the value of $x$ when $y = 4$,
**(ii)** find an equation connecting $x$ and $y$,
**(iii)** calculate the value of $y$ when $x = 3.2$.

**2.** If $z$ is inversely proportional to $\sqrt{w}$ and $z = 9$ when $w = 9$,
**(i)** find an equation connecting $w$ and $z$,
**(ii)** find the value of $z$ when $w = 16$,
**(iii)** calculate the value of $w$ when $z = 3$.

**3.** For each of the following equations, state the two variables which are inversely proportional to each other and explain your answer.

**(a)** $y = \frac{3}{x^2}$ **(b)** $y = \frac{1}{\sqrt{x}}$

**(c)** $y^2 = \frac{5}{x^3}$ **(d)** $n = \frac{7}{m-1}$

**(e)** $q = \frac{4}{(p+1)^2}$

**4.** If $z$ is inversely proportional to $\sqrt[3]{x}$ and $z = 5$ when $x = 64$, find the value of $z$ when $x = 216$.

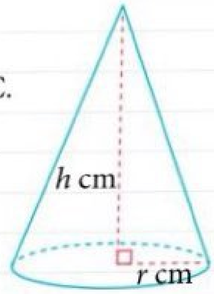
**5.** If $q^2$ is inversely proportional to $p + 3$ and $q = 5$ when $p = 2$, find the values of $q$ when $p = 17$.

**6.** Given that $t$ is inversely proportional to $s^3$, copy and complete the table.

| $s$ | 1 | 2 | 4 | | |
|---|---|---|---|---|---|
| $t$ | 80 | | | 0.08 | 0.01 |

**7.** The force of repulsion, $F$ newtons (N), between two particles is inversely proportional to the square of the distance, $d$ m, between the particles.
**(i)** Write down a formula connecting $F$ and $d$.
**(ii)** When the particles are a certain distance apart, the force of repulsion is 20 N. Find the force when the distance is halved.

**8.** For a fixed volume, the height, $h$ cm, of a cone is inversely proportional to the square of the base radius, $r$ cm. Cone A has a base radius of 6 cm and a height of 5 cm. The base radius of Cone B is 3 cm and the height of Cone C is 1.25 cm. If all the cones have the same volume, find
**(i)** the height of Cone B,
**(ii)** the base radius of Cone C.




**9.** If $y$ is inversely proportional to $2x + 1$ for all positive values of $y$ and the difference in the values of $y$ when $x = 0.5$ and $x = 2$ is 0.9, find the value of $y$ when $x = -0.25$.

$y$ is inversely proportional to $x^2$ and $y = b$ for a particular value of $x$. Find an expression in terms of $b$ for $y$ when this value of $x$ is tripled.

**11.** The force of attraction between two magnets is inversely proportional to the square of the distance between them. When the magnets are $r$ cm apart, the force of attraction between them is $F$ newtons (N). If the distance between the magnets is increased by 400%, the force of attraction between them becomes $cF$ N. Find the value of $c$.

**12.** Yasir has a few cylindrical containers. He pours the same amount of water into each container. He then takes note of the diameter of each container and the respective heights of the water level. He concludes that the diameter of the container and the height of the water level are inversely proportional to each other. Do you agree? Explain your answer.

## Looking Back

In this chapter, we have explored the idea of **proportionality**: how two quantities can be related by a constant factor. There are two ways in which the two quantities can be related proportionally — direct and inverse proportion. These two proportional relationships can be represented by a table of values (numerically), an equation involving a constant of proportionality, or as a graph. Proportional relationships are important because they can be found in many real-world situations. A simple example would be the circumference of a circle in relation to its diameter. That is, not only does the circumference increase when the diameter increases but more importantly, they are related by a constant factor, $\pi$!

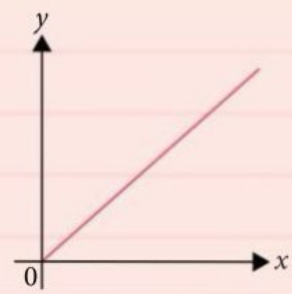
Other examples encountered in this chapter include the relationships between workload and the number of days needed to complete them given the same manpower (Worked Example 9) as well as the relationship between pressure and volume of gases at the same temperature (Worked Example 11). In these situations, proportionality forms the basis of writing functions that we can use to model relationships between the quantities involved. For instance, if we are told that the cost of data roaming ($C$) is directly proportional to the amount of data used ($A$), then we know that $C$ and $A$ are related by the equation

$$C = kA, \text{ where } k \text{ is a constant to be determined.}$$

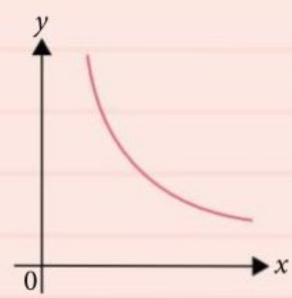
Likewise, if we can recognise an inverse proportion relationship between two quantities, we will be able to model that relationship by an appropriate function. Therefore, having a firm understanding of proportionality and being able to recognise proportionality in different contexts will be important as you continue to explore relationships between quantities using the language of mathematics.

## Summary

1. If $y$ is *directly proportional* to $x$, then
   - $\frac{y}{x} = k$ or $y = kx$, where $k$ is a constant and $k \neq 0$,
   - $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ or $\frac{y_2}{y_1} = \frac{x_2}{x_1}$,
   - the graph of $y$ against $x$ is a straight line that passes through the origin.

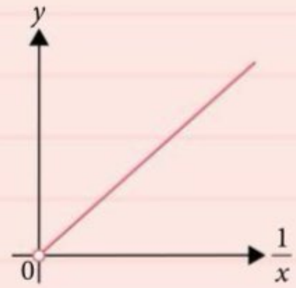



2. If $y$ is *inversely proportional* to $x$, then
   - $xy = k$ or $y = \frac{k}{x}$, where $k$ is a constant and $k \neq 0$,
   - $x_2y_2 = x_1y_1$,
   - the graph of $y$ against $x$ is a hyperbola,



Summary

- the graph of $y$ against $\frac{1}{x}$ is a straight line ($x \neq 0$).



# CHAPTER 8

# Congruence and Similarity

The photograph shows the Old Hill Street Police Station, which is a colonial landmark in Singapore. Can you identify the sets of windows with the same shape and size? This is an example of congruence. A photograph shows the image of a subject in a smaller size. This is an example of similarity.

Similarity between objects illustrates the idea of **proportionality** — that the distances in similar objects are related by a constant ratio. Congruence is considered a special case of similarity, in which that ratio is 1. What are some other examples of congruence and similarity that you can find around you?

Understanding congruence and similarity is useful in real life because we often work with smaller versions or replicas of objects before we work with the actual-sized objects. In this chapter, we will delve deeper into these ideas and explore how we can use them to solve problems involving similarity and congruence.

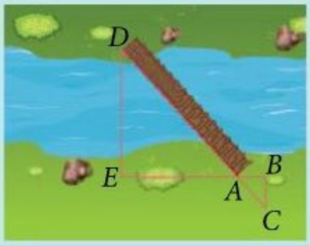
## Learning Outcomes

What will we learn in this chapter?

- How to examine whether two figures are congruent or similar
- What the properties of congruent and similar polygons are
- How to interpret scales on maps
- How to solve real-world problems involving congruence and similarity

**Introductory Problem**

The diagram below shows a river. A bridge is to be built directly across the river as shown.




The triangle $ABC$ is enlarged to become triangle $AED$ without distorting its shape. Explain how you can determine the length of the bridge, if you are only able to take measurements on the ground.

In this chapter, we will learn about congruent and similar figures, which can be used to solve such problems.

## 8.1 Congruent figures

Let us begin by looking at three types of transformations called translation, rotation and reflection.

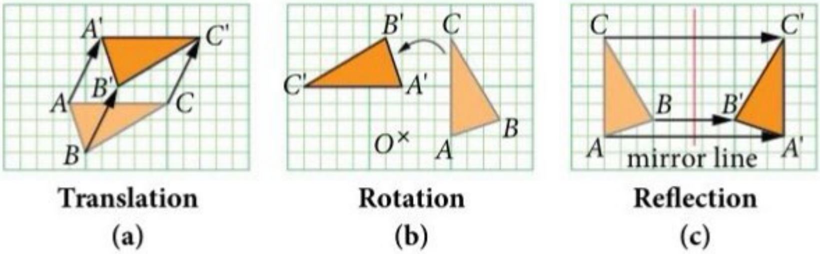



Fig. 8.1

Fig. 8.1(**a**) shows how $\triangle ABC$ is mapped onto $\triangle A'B'C'$ by a translation of 2 units to the right and 4 units upwards.
Fig. 8.1(**b**) shows how $\triangle ABC$ is mapped onto $\triangle A'B'C'$ by an anticlockwise rotation of 90° about the point $O$.
Fig. 8.1(**c**) shows how $\triangle ABC$ is mapped onto $\triangle A'B'C'$ by a reflection in the mirror line.

## Investigation Properties of congruent figures

Fig. 8.2 shows five pairs of scissors.

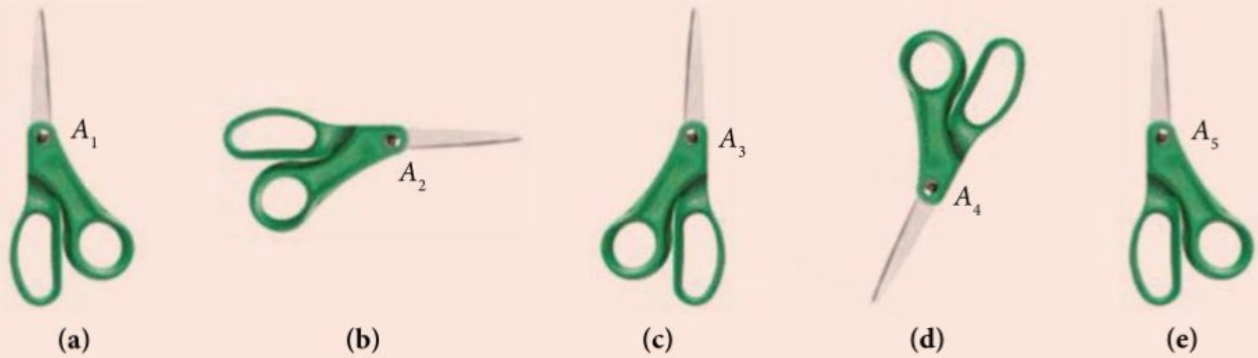



Fig. 8.2

1. What can we say about the shape, size, orientation and position of the pairs of scissors?
2. If we cut out the pairs of scissors and stack them up, what will we observe?
3. $A_1$ can be moved to $A_2$ by a translation to the right and a rotation of 90° about $A_2$. How can we move $A_1$ to $A_3$, $A_4$ and $A_5$?

From the above Investigation, we observe that:

- Two figures are **congruent** if they have exactly the **same shape** and the **same size**.
- They can be mapped onto each other under **translation**, **rotation** and **reflection**.

You can also investigate the effect of translation, rotation and reflection on a triangle or quadrilateral using the geometry software template 'Congruence' at www.sl-education.com/tmsoupp2/pg241 or scan the QR code on the right.

## Thinking Time

Fig. 8.3 shows two pairs of scissors of different colours.


Fig. 8.3

Are they congruent? Explain your answer.

Congruence is a property of *geometrical figures*. The two pairs of scissors in Fig. 8.3 are *congruent* because they have exactly the same shape and the same size.

Fig. 8.4 shows some patterns that are formed by congruent figures. These are known as tessellations, which can be found in many real-life objects.

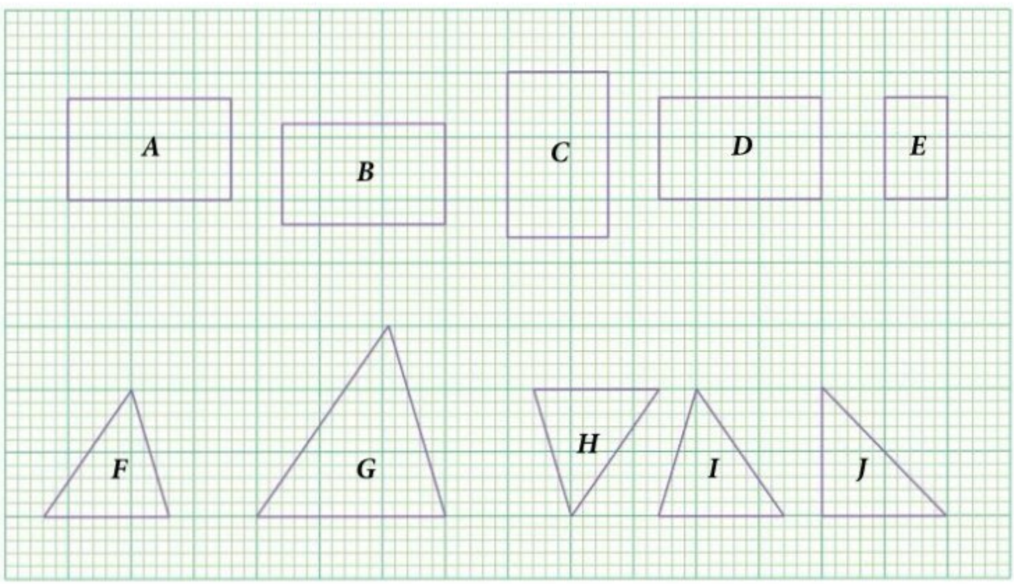
Fig. 8.4

Photocopying is another common real-life application of congruence when the photocopied document is of the same shape and size as the original document. The concept of congruence also plays an important role in the manufacturing sector. For example, the congruence of pen refills allows us to replace our pen refills when the ink runs dry.

**Worked Example 1**

## Identifying congruent shapes

Which shapes are congruent?




**Solution**

***A***, ***B***, ***C*** and ***D*** are congruent rectangles.

***F***, ***H*** and ***I*** are congruent triangles.

**Practise Now 1**

Similar and Further Questions
**Exercise 8A**
Question 1

Which shapes are congruent?

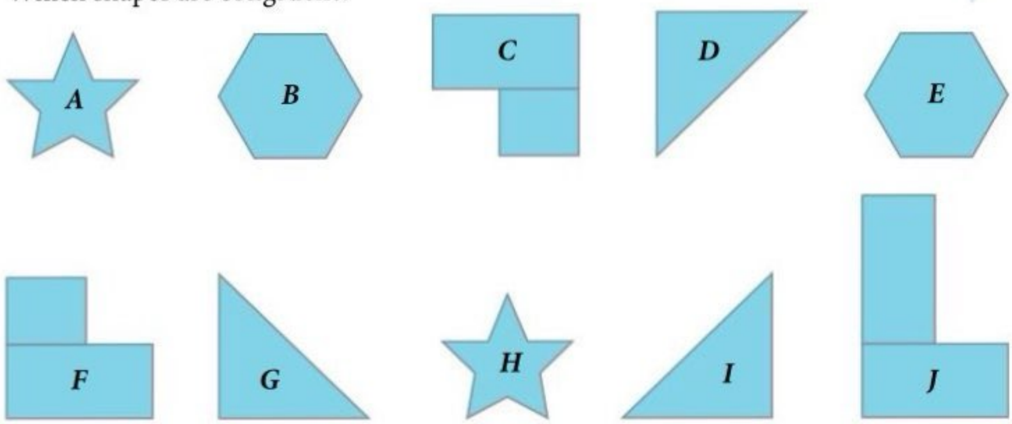



Fig. 8.5 shows a quadrilateral $ABCD$ being translated to the right to become quadrilateral $A'B'C'D'$. Therefore, the two quadrilaterals $ABCD$ and $A'B'C'D'$ are congruent.

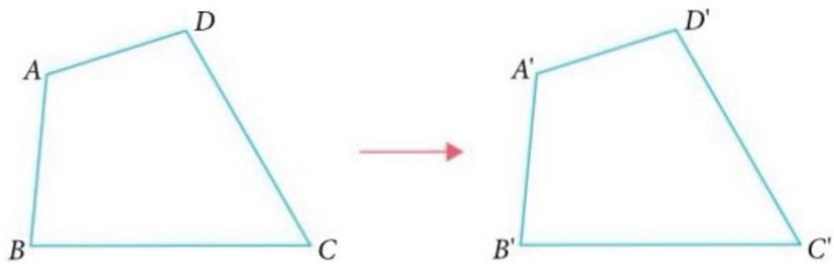
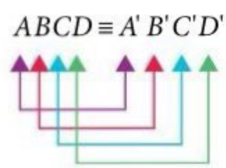



Fig. 8.5




We write $ABCD \equiv A'B'C'D'$. This is called the statement of congruence, where the symbol '$\equiv$' means 'is congruent to'.

Notice that the order in which the vertices of $A'B'C'D'$ are written must correspond to the order in which the vertices of $ABCD$ are written.

We can also write $BCDA \equiv B'C'D'A'$ because the corresponding vertices match.

Can we write $CDAB \equiv C'D'A'B'$? What about $DABC \equiv D'C'B'A'$?

In Fig. 8.5, we notice that when the two quadrilaterals $ABCD$ and $A'B'C'D'$ are congruent,

- all the corresponding angles are equal, i.e. $\angle DAB = \angle D'A'B'$, $\angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$ and $\angle CDA = \angle C'D'A'$; and
- all the corresponding sides are equal, i.e. $AB = A'B'$, $BC = B'C'$, $CD = C'D'$ and $DA = D'A'$.

**Big Idea**

**Notations**

The symbol '$\equiv$' expresses that two polygons are congruent in a concise and precise manner. What are some other rules or conventions about the vertices that we follow to make our mathematical statements clear and concise?

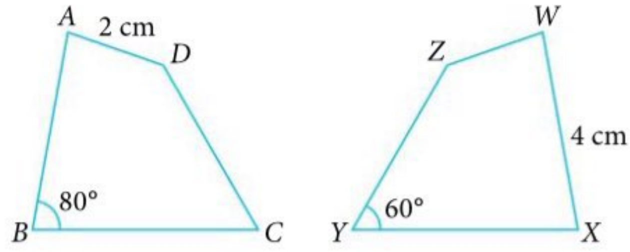
Since a quadrilateral is defined by its four angles and four sides, if all the corresponding angles of two quadrilaterals are equal and all the corresponding sides of the two quadrilaterals are equal, then the two quadrilaterals are congruent. In general, the following is true:

> Two polygons are congruent if and only if
> - all their corresponding angles are equal, *and*
> - all their corresponding sides are equal.

Worked Example 2

## Solving problem involving congruent figures

Given that $ABCD \equiv WXYZ$, copy and complete each of the following.




(i) $\angle ABC = \angle WXY =$ ____ °

(ii) ____ $= \angle XYZ =$ ____ °

(iii) $AD =$ ____ $=$ ____ cm

(iv) ____ $= WX =$ ____ cm

**Solution**

(i) $\angle ABC = \angle WXY = 80°$

(ii) $\angle BCD = \angle XYZ = 60°$

(iii) $AD = WZ = 2$ cm

(iv) $AB = WX = 4$ cm

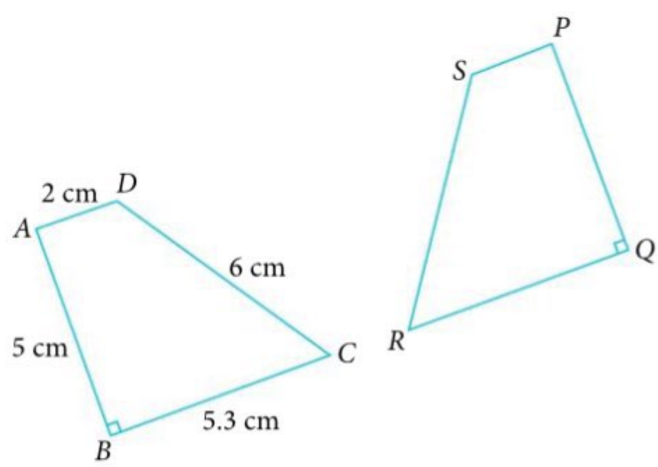
**Problem-solving Tip**

Since $ABCD \equiv WXYZ$, then the corresponding vertices match: $A \leftrightarrow W$, $B \leftrightarrow X$, $C \leftrightarrow Y$, $D \leftrightarrow Z$.

**Practise Now 2**

Similar and Further Questions
Exercise 8A
Questions 2, 3

Given that $ABCD \equiv PQRS$, copy and complete each of the following.




(i) $PQ = AB =$ ____ cm

(ii) $SR =$ ____ $= 6$ cm

(iii) $PS =$ ____ $=$ ____ cm

(iv) $QR =$ ____ $=$ ____ cm

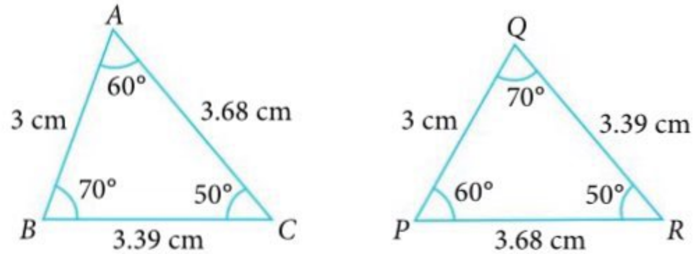
(v) $\angle PQR =$ ____ $=$ ____ °

Worked Example 3

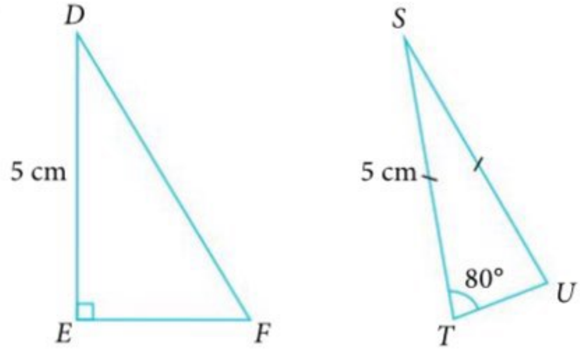
## Identifying congruent triangles and writing statement of congruence

Are the following pairs of triangles congruent? If so, explain your answer and write down the statement of congruence. If not, explain your answer.

**(a)**




**(b)**




**Solution**

**(a)** $\angle BAC = \angle QPR = 60°$ the corresponding vertices match

$\angle ABC = \angle PQR = 70°$

$\angle ACB = \angle PRQ = 50°$

$AB = PQ = 3$ cm the corresponding vertices match

$BC = QR = 3.39$ cm

$AC = PR = 3.68$ cm

$\therefore \triangle ABC \equiv \triangle PQR.$

**Problem-solving Tip**

**Step 1:** Identify the corresponding vertices by comparing the size of the angles: $A \leftrightarrow P$, $B \leftrightarrow Q$, $C \leftrightarrow R$.

**Step 2:** Write proper statements using the corresponding vertices identified in **Step 1**.

**(b)** In $\triangle STU$,

$\angle T = \angle U = 80°$ (base $\angle$s of isos. $\triangle STU$)

$\angle S = 180° - 80° - 80°$ ($\angle$ sum of $\triangle STU$)

$= 20°$

$\triangle STU$ does not have any right angle that corresponds to that in $\triangle DEF$.

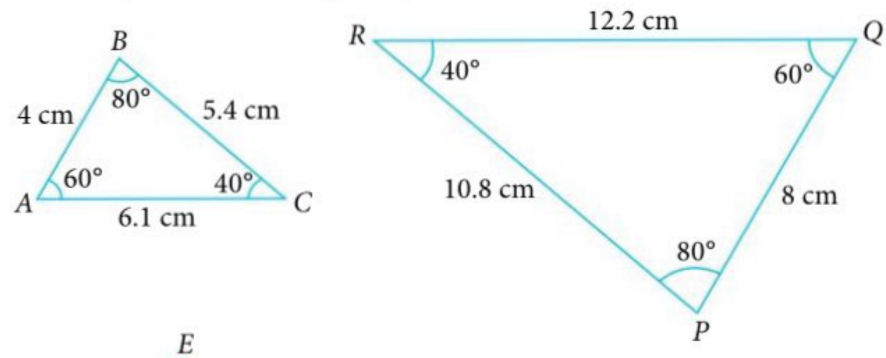
$\therefore \triangle STU$ is not congruent to $\triangle DEF$.

**Practise Now 3**

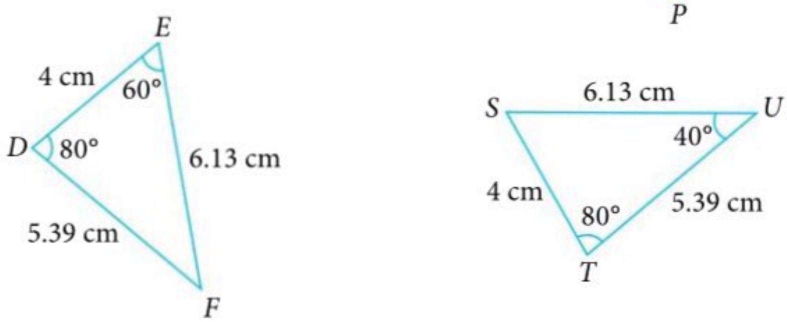
Similar and Further Questions
**Exercise 8A**
Questions 4(a)–(c)

Are the following pairs of triangles congruent? If so, explain your answer and write down the statement of congruence. If not, explain your answer.

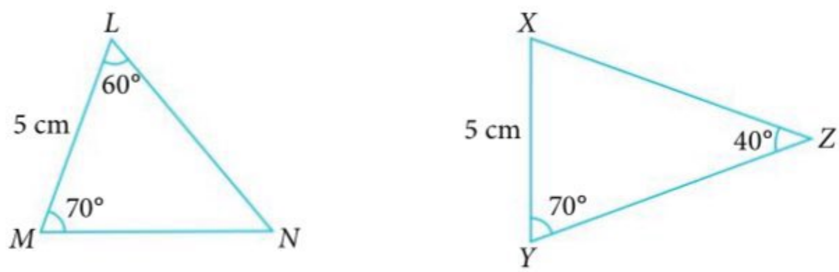
**(a)**




**(b)**




**(c)**




**Worked Example 4**

### Solving problem involving congruent triangles

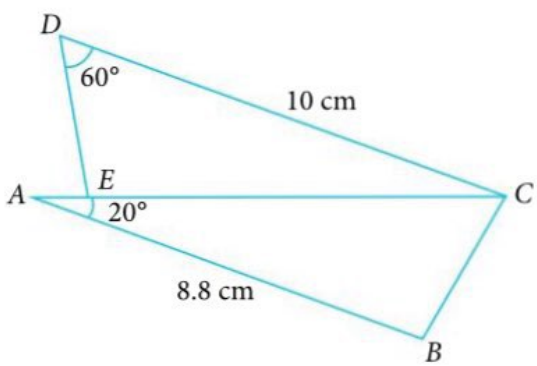
In the figure, $\triangle ABC \equiv \triangle CED$.

**(a)** Given that $\angle BAC = 20°$, $\angle CDE = 60°$, $AB = 8.8$ cm and $CD = 10$ cm, find

**(i)** $\angle ECD$,  **(ii)** $\angle ECB$,  **(iii)** $\angle ABC$,

**(iv)** the length of $AC$,  **(v)** the length of $AE$.

**(b)** What can we say about the lines $AB$ and $DC$?




**Problem-solving Tip**

Since $\triangle ABC \equiv \triangle CED$, then the corresponding vertices match: $A \leftrightarrow C$, $B \leftrightarrow E$, $C \leftrightarrow D$.

**Solution**

**(a) (i)** $\angle ECD = \angle BAC$

$= 20°$

**(ii)** $\angle ECB = \angle ACB$ from the figure, $\angle ECB$ belongs to $\triangle ABC$
$= \angle CDE$
$= 60°$

**(iii)** $\angle ABC = 180° - 20° - 60°$ ($\angle$ sum of $\triangle ABC$)
$= 100°$

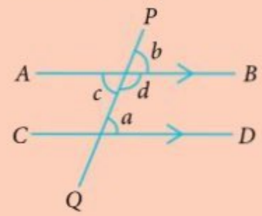
**(iv)** Length of $AC$ = length of $CD$
$= 10$ cm

**(v)** Length of $EC$ = length of $BA$
$= 8.8$ cm

$\therefore$ length of $AE$ = length of $AC$ – length of $EC$
$= 10 - 8.8$
$= 1.2$ cm

**(b)** Since $\angle BAC = \angle ECD$ (= 20°), then $AB$ // $DC$ (converse of alt. $\angle$s).

**Recall**




When two lines $AB$ and $CD$ are cut by a transversal $PQ$, and
- if $\angle a = \angle b$, then $AB$ // $CD$ (converse of corr. $\angle$s);
- if $\angle a = \angle c$, then $AB$ // $CD$ (converse of alt. $\angle$s);
- if $\angle a + \angle d = 180°$, then $AB$ // $CD$ (converse of int. $\angle$s).

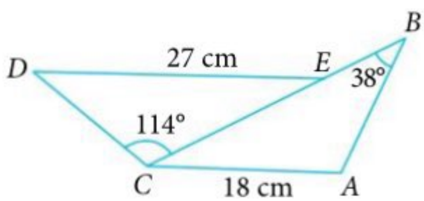
**Practise Now 4**

Similar and Further Questions
**Exercise 8A**
Questions 5–7

In the figure, $\triangle ABC \equiv \triangle CDE$.

**(a)** Given that $\angle ABC = 38°$, $\angle DCE = 114°$, $AC = 18$ cm and $DE = 27$ cm, find
**(i)** $\angle CDE$, **(ii)** $\angle CED$, **(iii)** $\angle ACB$,
**(iv)** the length of $BC$, **(v)** the length of $BE$.

**(b)** What can we say about the lines $AC$ and $ED$?

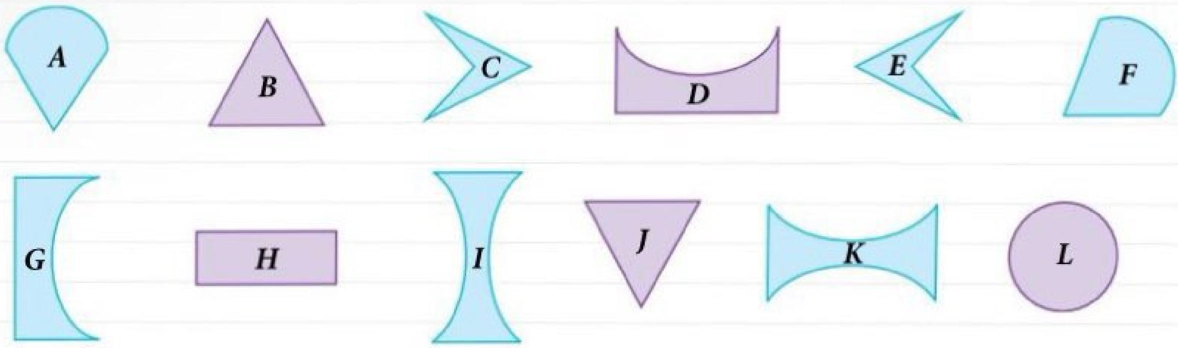



## Reflection

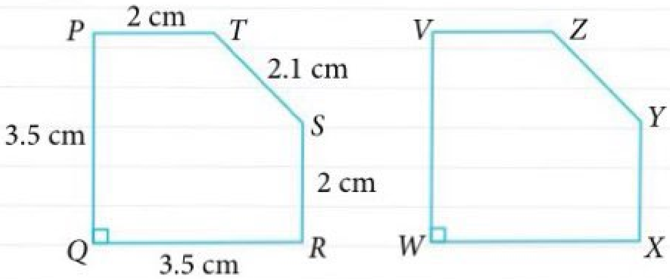
When are two figures or polygons congruent?

## Exercise 8A

**1.** Which pairs of shapes are congruent?




**2.** Given that $PQRST \equiv VWXYZ$, copy and complete each of the following.




(i) $PQ = VW =$ ______ cm

(ii) $PT =$ ______ $= 2$ cm

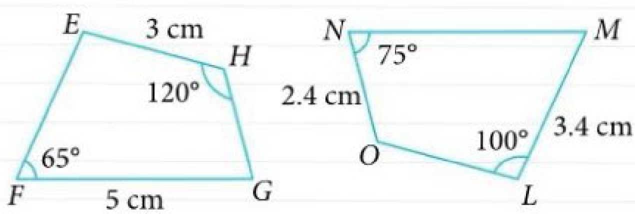
(iii) $QR =$ ______ $=$ ______ cm

(iv) $TS =$ ______ $=$ ______ cm

(v) $SR =$ ______ $=$ ______ cm

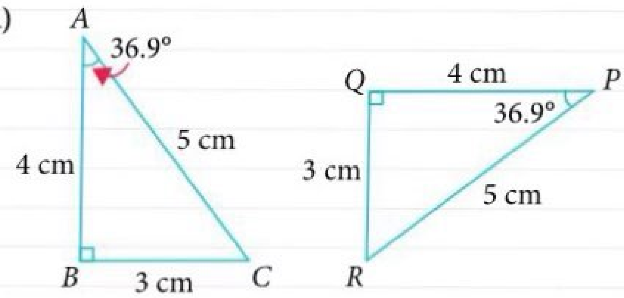
(vi) $\angle PQR =$ ______ $=$ ______ °

**3.** Given that $EFGH \equiv LMNO$, write down all the missing measurements.

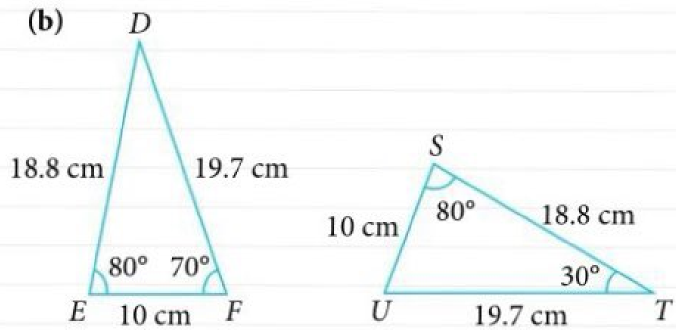



**4.** Are the following pairs of triangles congruent? If so, explain your answer and write down the statement of congruence. If not, explain your answer.

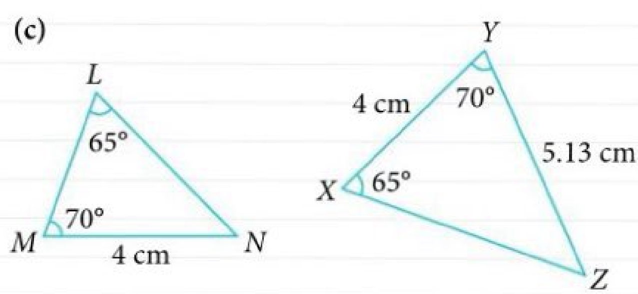
(a)




(b)




(c)

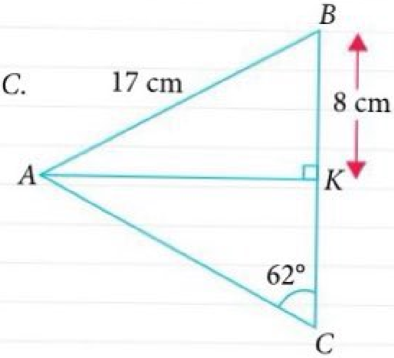


## Exercise 8A

**5.** In the figure, $\triangle ABK \equiv \triangle ACK$. Given that $\angle AKB = 90°$, $\angle ACK = 62°$, $AB = 17$ cm and $BK = 8$ cm, find

**(i)** $\angle BAC$,

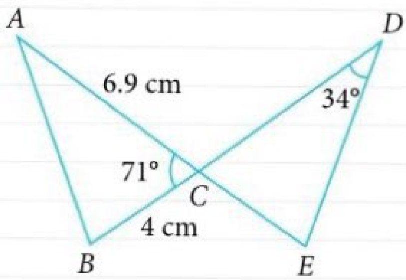
**(ii)** the length of $BC$.




**6.** In the figure, $\triangle ABC \equiv \triangle DEC$ and $ACE$ and $BCD$ are straight lines. Given that $\angle ACB = 71°$, $\angle CDE = 34°$, $AC = 6.9$ cm and $BC = 4$ cm, find

**(i)** $\angle ABC$,

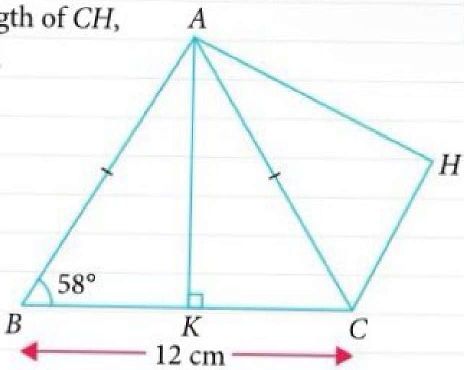
**(ii)** the length of $BD$.




**7.** In the figure, $\triangle ABC$ is an isosceles triangle where $AB = AC$, $BC = 12$ cm and $\angle ABK = 58°$. Given that $\triangle ABK \equiv \triangle ACH$ and $\angle AKC = 90°$, find

**(i)** the length of $CH$,

**(ii)** $\angle BAH$.




# 8.2 Similar figures

Fig. 8.6 shows three cups which look alike but are of different sizes.

Fig. 8.6

Two figures are **similar** if they have exactly the same shape but *not* necessarily the same size.

If two similar figures also have exactly the same size, then they are congruent. In other words, congruence is a *special case* of similarity.

## Investigation Similar polygons

The triangle $ABC$, with dimensions 4 units, 5 units and 7 units, is enlarged without distorting its shape to become $A'B'C'$.

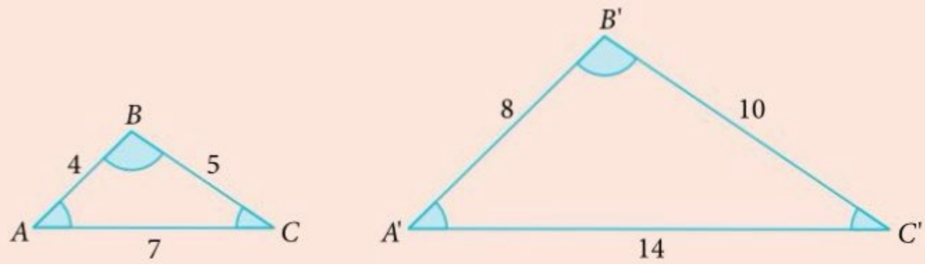



Fig. 8.7

1. Measure and record each of the angles.
   (a) $\angle A =$ ____ °, $\angle A' =$ ____ °
   (b) $\angle B =$ ____ °, $\angle B' =$ ____ °
   (c) $\angle C =$ ____ °, $\angle C' =$ ____ °

   What do you notice about the size of each of the above pairs of corresponding angles?

2. Find the value of each of the following ratios using the lengths indicated in Fig. 8.7.
   (a) $\frac{A'B'}{AB} =$ ____ (b) $\frac{B'C'}{BC} =$ ____ (c) $\frac{A'C'}{AC} =$ ____

   What do you notice about the values of the ratios of the lengths of the corresponding sides?

From the Investigation on page 250, we observe that if two triangles are similar, then

- all the corresponding angles are equal,
  i.e. $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, *and*
- the length of each side of a triangle is increased by the same factor,
  i.e. $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = k$, where $k$ is a constant.

On the other hand, if all the corresponding angles of two triangles are equal, and all the ratios of the lengths of their corresponding sides are equal, then the two triangles are similar.

**Big Idea**

**Proportionality**

Since $\frac{A'B'}{AB} = k$, then $A'B' = kAB$. The same applies to all the other corresponding sides. In other words, the corresponding sides of two similar triangles are directly proportional to each other. In addition, the ratios of pairs of sides within a triangle in a family of similar triangles are also equal, e.g. $\frac{AB}{BC} = \frac{A'B'}{B'C'}$, $\frac{AB}{AC} = \frac{A'B'}{A'C'}$ and $\frac{AC}{BC} = \frac{A'C'}{B'C'}$.

## Thinking Time

1. Fig. 8.8 shows two rectangles.

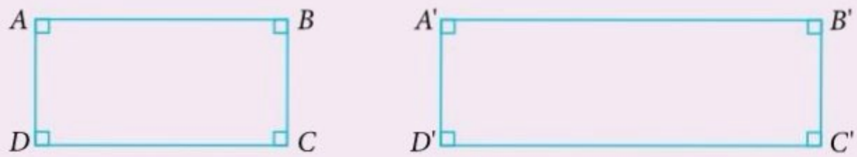



Fig. 8.8

Measure the sides of each rectangle.

**(i)** Are all the ratios of the corresponding sides $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$, $\frac{C'D'}{CD}$ and $\frac{D'A'}{DA}$ equal?

**(ii)** Are all the corresponding angles $\angle DAB$ and $\angle D'A'B'$, $\angle ABC$ and $\angle A'B'C'$, $\angle BCD$ and $\angle B'C'D'$, as well as $\angle CDA$ and $\angle C'D'A'$ equal?

**(iii)** Are the two rectangles similar?

2. Fig. 8.9 shows a square and a rhombus.

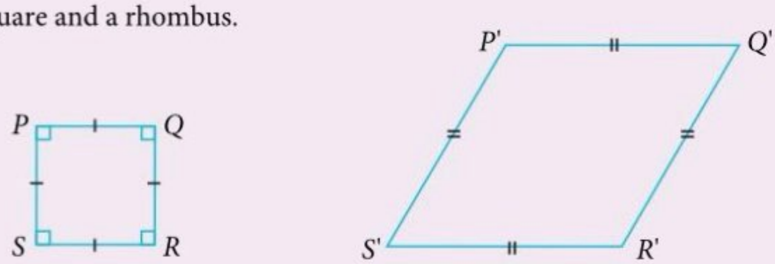



Fig. 8.9

Measure the length of the square and the length of the rhombus.

**(i)** Are all the ratios of the corresponding sides $\frac{P'Q'}{PQ}$, $\frac{Q'R'}{QR}$, $\frac{R'S'}{RS}$ and $\frac{S'P'}{SP}$ equal?

**(ii)** Are all the corresponding angles $\angle SPQ$ and $\angle S'P'Q'$, $\angle PQR$ and $\angle P'Q'R'$, $\angle QRS$ and $\angle Q'R'S'$, as well as $\angle RSP$ and $\angle R'S'P'$ equal?

**(iii)** Are the two quadrilaterals similar?

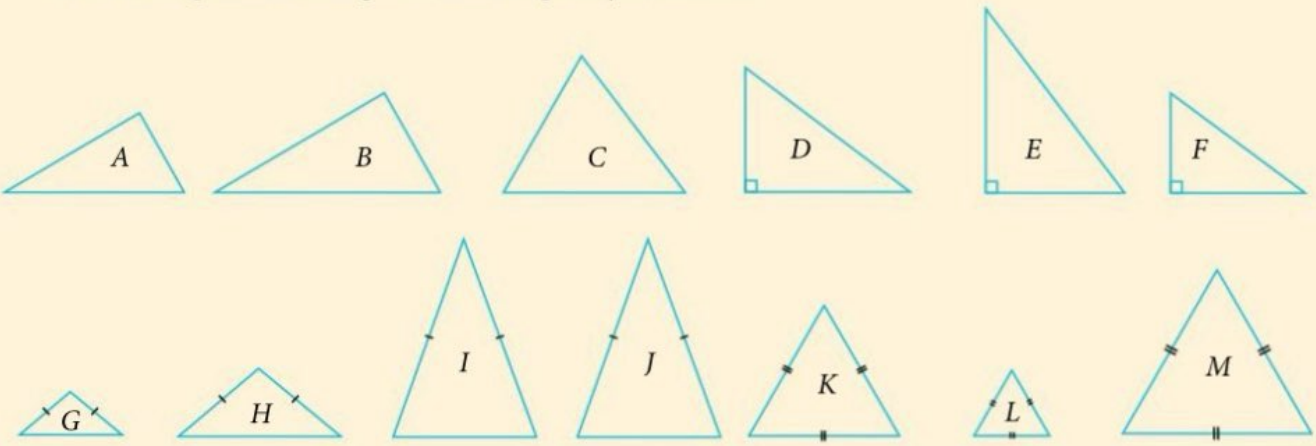
In general, the following is true:

Two polygons are similar if and only if
- all their corresponding angles are equal, *and*
- all the ratios of the lengths of their corresponding sides are equal.

Class Discussion

## Identifying similar triangles

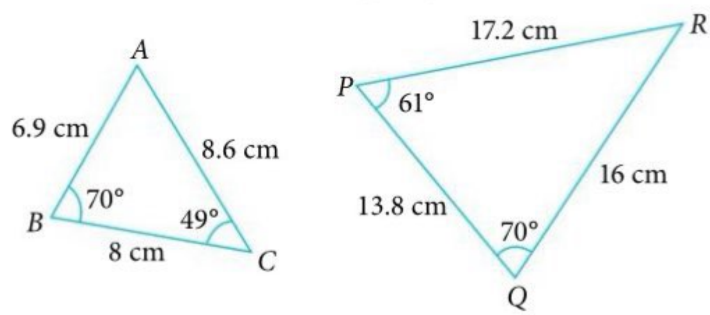
1. Photocopy the following triangles and cut them out.
2. Which triangles are similar? Explain your answer.
3. (a) Are all right-angled triangles similar? Explain your answer.
   (b) Are all isosceles triangles similar? Explain your answer.
   (c) Are all equilateral triangles similar? Explain your answer.



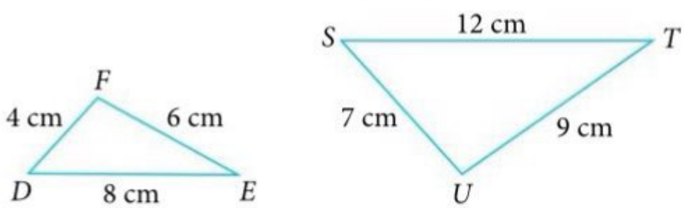

Worked Example 5

## Identifying similar triangles

(a) Is $\triangle ABC$ similar to $\triangle PQR$? Explain your answer.




(b) Is $\triangle DEF$ similar to $\triangle STU$? Explain your answer.



**Solution**

**(a)** $\angle A = 180° - 70° - 49°$ ($\angle$ sum of $\triangle ABC$)

$= 61°$

$\angle R = 180° - 61° - 70°$ ($\angle$ sum of $\triangle PQR$)

$= 49°$

$\angle A = \angle P = 61°$

$\angle B = \angle Q = 70°$

$\angle C = \angle R = 49°$

$$\frac{PQ}{AB} = \frac{13.8}{6.9} = 2$$

$$\frac{QR}{BC} = \frac{16}{8} = 2$$

$$\frac{PR}{AC} = \frac{17.2}{8.6} = 2$$

Since all the corresponding angles are equal and all the ratios of the lengths of the corresponding sides are equal, then $\triangle ABC$ is similar to $\triangle PQR$.

**Problem-solving Tip**

$A \leftrightarrow P, B \leftrightarrow Q, C \leftrightarrow R$

**Attention**

There is no standard notation for similarity. We just write '$\triangle ABC$ is similar to $\triangle PQR$'.

**(b)** $$\frac{ST}{DE} = \frac{12}{8} = 1.5$$

$$\frac{TU}{EF} = \frac{9}{6} = 1.5$$

$$\frac{SU}{DF} = \frac{7}{4} = 1.75$$

Since not all the ratios of the corresponding sides are equal, then $\triangle DEF$ is not similar to $\triangle STU$.

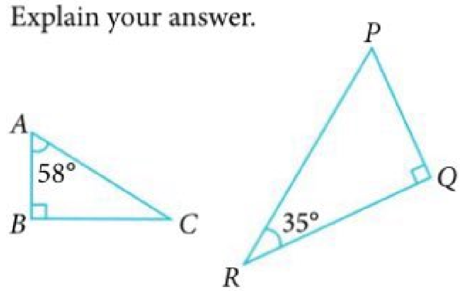
**Problem-solving Tip**

If $\triangle DEF$ is similar to $\triangle STU$, then the longest side of $\triangle DEF$ will correspond to the longest side of $\triangle STU$, the second longest side of $\triangle DEF$ will correspond to the second longest side of $\triangle STU$, and the shortest side of $\triangle DEF$ will correspond to the shortest side of $\triangle STU$.
Thus, we compare the ratios of these corresponding sides.

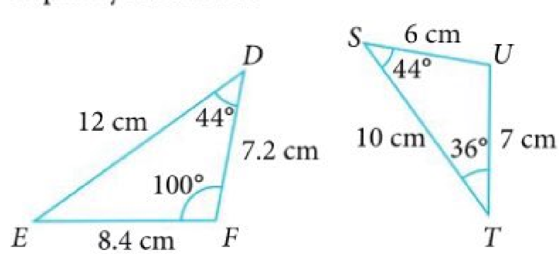
**Practise Now 5**

Similar and Further Questions
**Exercise 8B**
Questions 2(a), (b)

**(a)** Is $\triangle ABC$ similar to $\triangle PQR$?
Explain your answer.



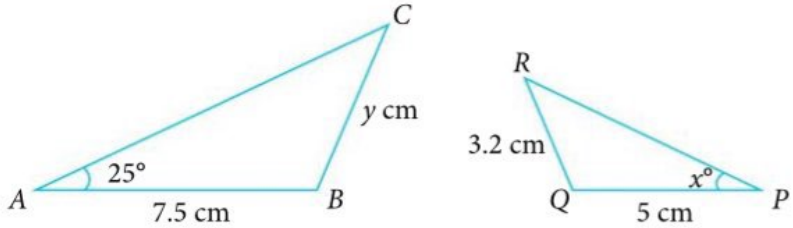

**(b)** Is $\triangle DEF$ similar to $\triangle STU$?
Explain your answer.



Worked Example 6

### Solving problem involving similar triangles

Given that $\triangle ABC$ is similar to $\triangle PQR$, calculate the values of the unknowns in the triangles.




**Solution**

Since $\triangle ABC$ is similar to $\triangle PQR$, then all the corresponding angles are equal.

$$\therefore x° = \angle QPR$$
$$= \angle BAC$$
$$= 25°$$
$$\therefore x = 25$$

Since $\triangle ABC$ is similar to $\triangle PQR$, then all the ratios of the corresponding sides are equal.

$$\therefore \frac{BC}{QR} = \frac{AB}{PQ}$$

i.e. $\dfrac{y}{3.2} = \dfrac{7.5}{5}$

$$\therefore y = \frac{7.5}{5} \times 3.2$$
$$= 4.8$$

**Problem-solving Tip**

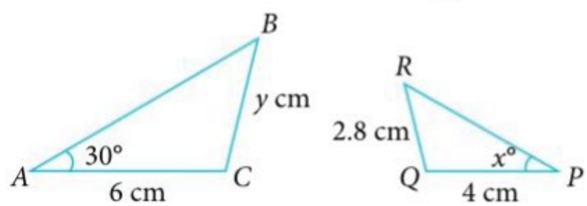
Since $\triangle ABC$ is similar to $\triangle PQR$, then the corresponding vertices match:

$A \leftrightarrow P, B \leftrightarrow Q, C \leftrightarrow R.$

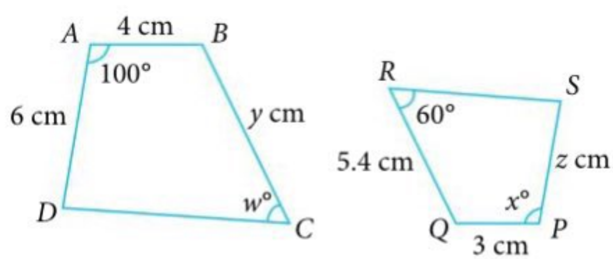
**Practise Now 6**

Similar and Further Questions
Exercise 8B
Questions 1(a)–(d), 3(a), (b), 4, 5

1. Given that $\triangle ABC$ is similar to $\triangle PRQ$, find the values of the unknowns in the triangles.



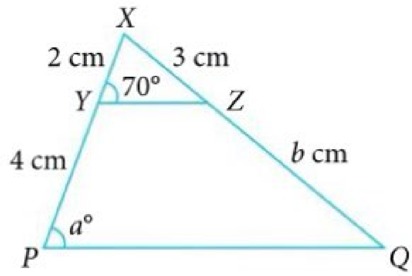

2. Given that the quadrilateral $ABCD$ is similar to the quadrilateral $PQRS$, find the values of the unknowns in the quadrilaterals.



Worked Example 7

## Solving another type of problem involving similar triangles

Given that $\triangle XYZ$ is similar to $\triangle XPQ$, calculate the values of the unknowns in the figure.




**Solution**

Since $\triangle XYZ$ is similar to $\triangle XPQ$, then all the corresponding angles are equal.

$$\therefore a° = \angle XPQ$$
$$= \angle XYZ$$
$$= 70°$$
$$\therefore a = 70$$

> **Problem-solving Tip**
>
> Since $\triangle XYZ$ is similar to $\triangle XPQ$, then the corresponding vertices match:
> $X \leftrightarrow X$, $Y \leftrightarrow P$, $Z \leftrightarrow Q$.

Since $\triangle XYZ$ is similar to $\triangle XPQ$, then all the ratios of the corresponding sides are equal.

$$\therefore \frac{XQ}{XZ} = \frac{XP}{XY}$$

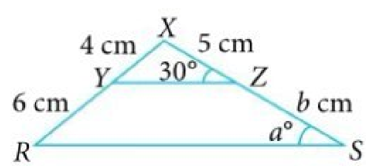
i.e. $$\frac{XQ}{3} = \frac{2+4}{2}$$

$$XQ = \frac{6}{2} \times 3$$
$$= 9 \text{ cm}$$
$$\therefore b = 9 - 3$$
$$= 6$$

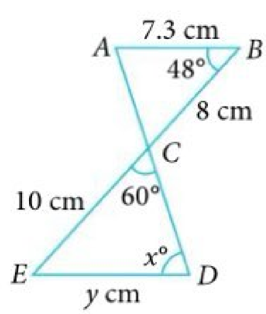
**Practise Now 7**

Similar and Further Questions
**Exercise 8B**
Questions 6–8, 10

1. Given that $\triangle XYZ$ is similar to $\triangle XRS$, find the values of the unknowns in the figure.



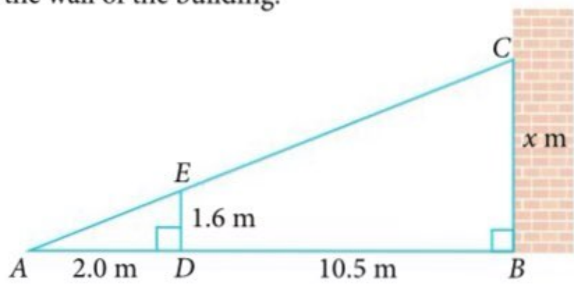

2. Given that $\triangle ABC$ is similar to $\triangle DEC$, find the values of the unknowns in the figure.



Worked Example 8

### Similar triangles in real-world context

Nadia stands at point $D$, 2.0 m in front of a spotlight at point $A$. She is 1.6 m tall and is facing the wall of a building which is 10.5 m away from her. Given that $\triangle ADE$ and $\triangle ABC$ are similar, find the height of her shadow $BC$ on the wall of the building.




**Solution**

Let the height of her shadow, $BC$, be $x$ m.

Since $\triangle ABC$ and $\triangle ADE$ are similar, then all the ratios of the corresponding sides are equal.

$$\frac{BC}{DE} = \frac{AB}{AD}$$

$$\frac{x}{1.6} = \frac{2+10.5}{2}$$

$$x = \frac{1.6(12.5)}{2}$$

$$= \frac{20}{2}$$

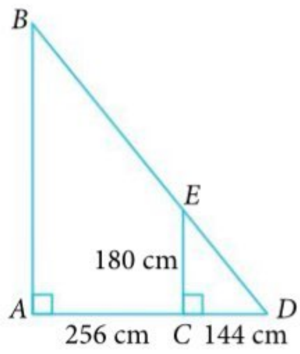
$$= 10$$

$\therefore$ the height of her shadow on the wall is 10 m.

**Practise Now 8**

Similar and Further Questions
Exercise 8B
Questions 9, 11, 12

Yasir stands at point $C$, 256 cm away from a lamp post $AB$. He is 180 cm tall and his shadow $CD$ from the lamp is 144 cm long. Given that $\triangle ABD$ and $\triangle CED$ are similar, find the height of the lamp post, $AB$, in metres.

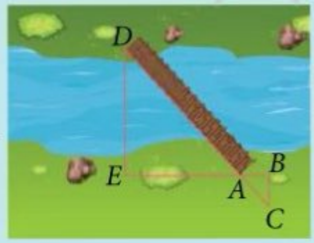



## Reflection

1. When are two figures or polygons similar?
2. What are the differences in the conditions for two figures to be congruent and for two figures to be similar?

## Introductory Problem Revisited

In the Introductory Problem, you were asked to determine the length of the bridge using measurements on the ground. After learning about similar figures, do you know how to solve the problem? Discuss with your classmates.

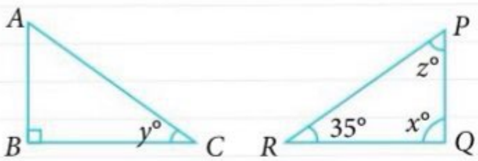



Advanced Intermediate Basic

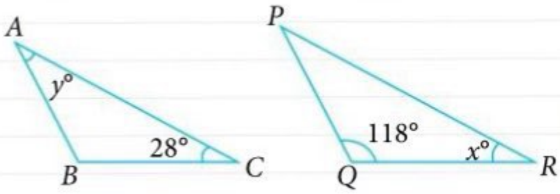
## Exercise 8B

**1.** Given that $\triangle ABC$ is similar to $\triangle PQR$, find the values of the unknowns in each of the following pairs of triangles.

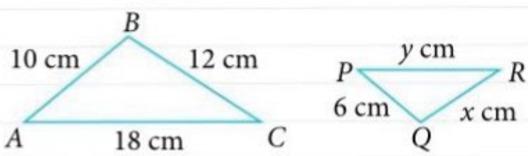
**(a)**



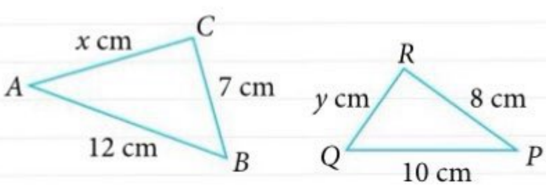

**(b)**



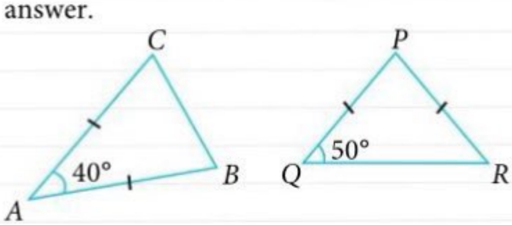

**(c)**



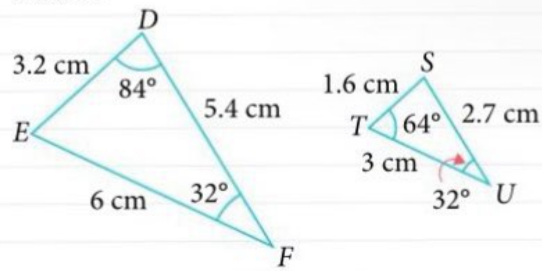

**(d)**




**2.** **(a)** Is $\triangle ABC$ similar to $\triangle PQR$? Explain your answer.

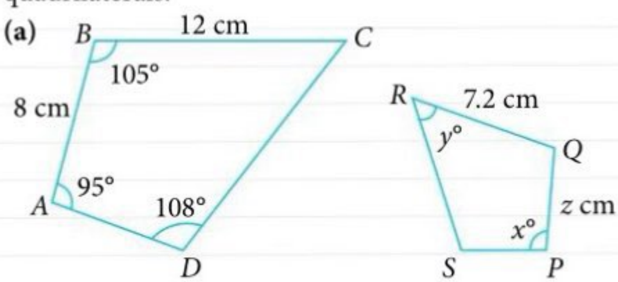



**(b)** Is $\triangle DEF$ similar to $\triangle STU$? Explain your answer.

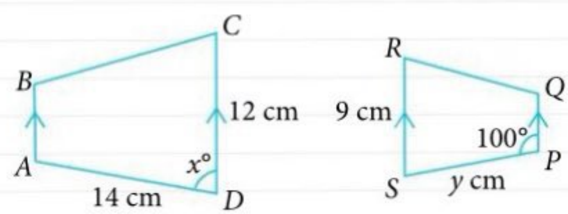



**3.** Given that the quadrilateral $ABCD$ is similar to the quadrilateral $PQRS$, find the values of the unknowns in each of the following pairs of quadrilaterals.

**(a)**

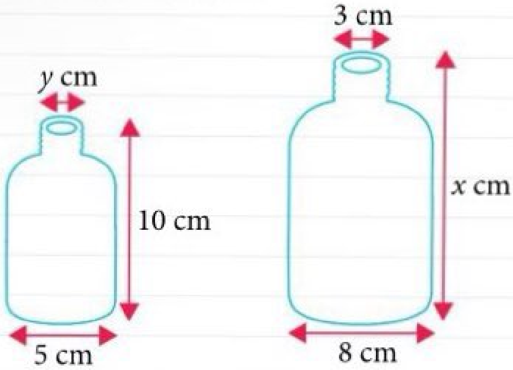



**(b)**

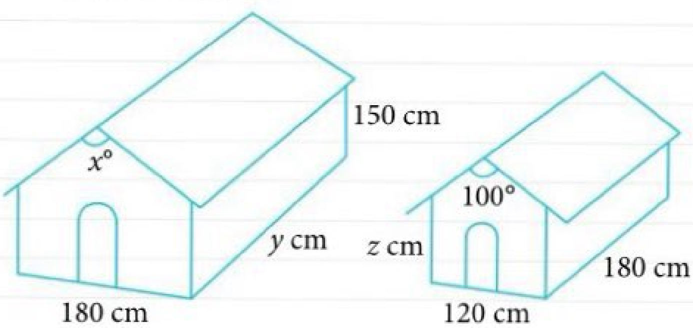


## Exercise 8B

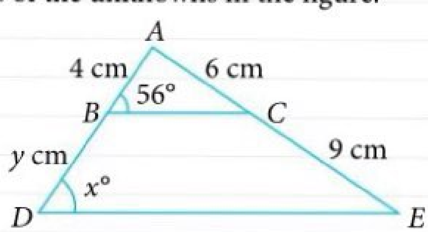
**4.** Two similar water bottles are as shown. Find the values of the unknowns.



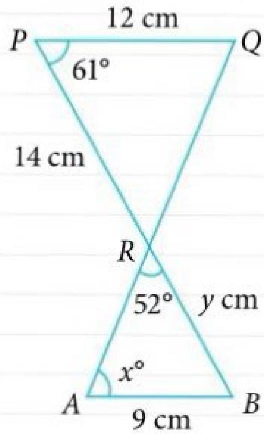

**5.** Two similar toy houses are as shown. Find the values of the unknowns.



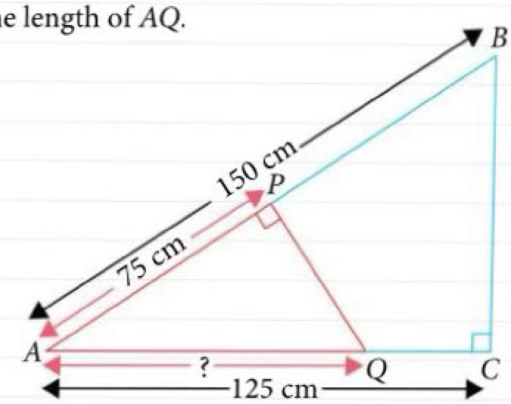

**6.** Given that $\triangle ABC$ is similar to $\triangle ADE$, find the values of the unknowns in the figure.



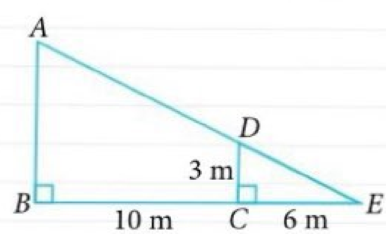

**7.** Given that $\triangle PQR$ is similar to $\triangle BAR$, find the values of the unknowns in the figure.




**8.** Given that triangles $AQP$ and $ABC$ are similar, find the length of $AQ$.

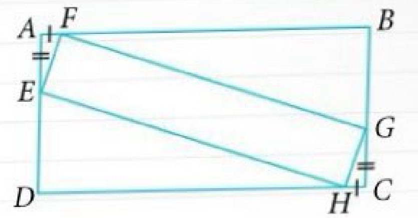



**9.** A pole $CD$ of height 3 m is placed in front of a standing lamp $AB$ 10 m away. The length of the pole's shadow $CE$, that is cast as a result of the light from the lamp, is 6 m. Given that $\triangle ABE$ and $\triangle DCE$ are similar, find the height of the lamp $AB$.



## Exercise 8B

10. The diagram shows a rectangle $EFGH$ drawn inside another rectangle $ABCD$, such that $AF = 3$ cm, $AE = 4$ cm, $BG : BC = 3 : 5$. Given that triangles $AEF$ and $BFG$ are similar, find the area of rectangle $EFGH$.

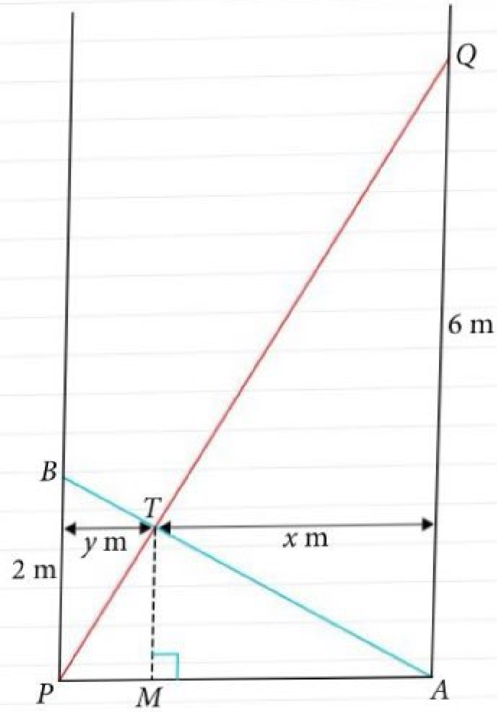



11. Two ladders $AB$ and $PQ$ are resting against opposite walls of an alley. The ladders $AB$ and $PQ$ are 2 m and 6 m above the ground respectively and $T$ is the point where the 2 ladders meet.

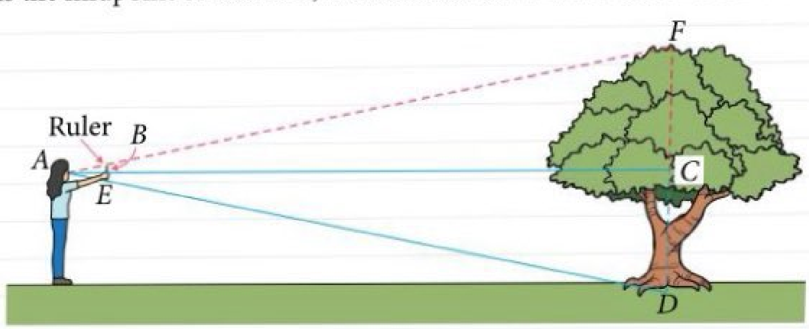
**(a)** **(i)** Given that $\triangle TBP$ is similar to $\triangle TAQ$, find an expression, in terms of $y$, for the length of $PA$.

**(ii)** Given that $\triangle PTM$ is similar to $\triangle PQA$, find the length of $TM$.

**(b)** If the points $B$ and $Q$ are $h$ m and $k$ m above the ground respectively, show how you can express $TM$ in terms of $h$ and $k$.




12. Joyce wants to estimate the height of a tree, $DF$. She stands 8 m from the tree and holds a 30 cm-long ruler vertically in front of her as shown in the diagram. It is given that the horizontal distance from her eyes to the ruler $AB$, where $B$ is the midpoint of the ruler, is 60 cm and that $\triangle ABE$ and $\triangle ACD$ are similar.



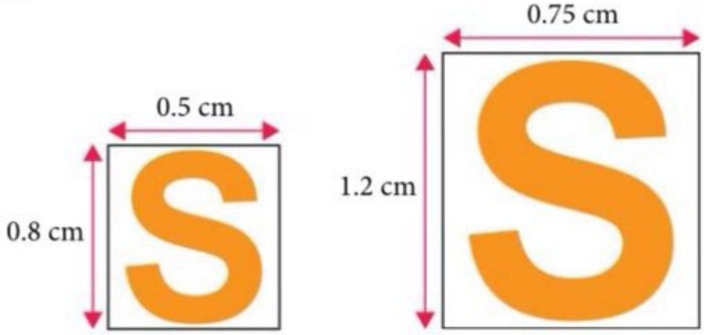

**(i)** If $C$ is the midpoint of the tree, find the estimated height of the tree, $DF$, in metres.
**(ii)** Explain why this method may not be suitable to estimate the height of a very tall building.

# 8.3 Similarity and enlargement

## A. Similarity and enlargement

In the previous section, we have learnt that two figures are similar if they have exactly the same shape but not necessarily the same size. Their dimensions are in proportion.

Look at Fig. 8.10(**a**). A letter 'S' may appear small on a book. If we use a magnifying glass to enlarge the letter for a clearer view (see Fig. 8.10(**b**)), the letter would appear larger.




The letter as it appears on a book
**(a)**

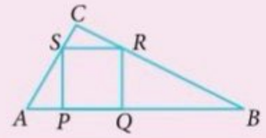
The letter as seen through a magnifying glass
**(b)**

Scale factor
$= \frac{0.75}{0.5}$
$= 1.5$

Fig. 8.10

**Just For Fun**

Draw a triangle $ABC$ where $AB = 8$ cm, $BC = 6$ cm and $AC = 4$ cm. Using the concept of enlargement, construct a square $PQRS$ inside the triangle such that $PQ$ is on $AB$, $R$ is on $BC$ and $S$ is on $AC$.




The larger letter is an enlargement of the original letter 'S' and the two letters are similar to each other. So, the corresponding lengths of the letters are in proportion. The ratio of the length of the enlarged letter to the corresponding length of the original letter is known as the **scale factor**.

In Fig. 8.11, $\triangle ABC$ is similar to $\triangle A'B'C'$. We say that $\triangle A'B'C'$ is an enlargement of $\triangle ABC$ with a scale factor of $k = \frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$.

**Information**

An enlargement with a scale factor between 0 and 1, e.g. $\frac{1}{2}$, indicates a reduction. A smaller image, with lengths $\frac{1}{2}$ that of the corresponding lengths of the original, will be produced.

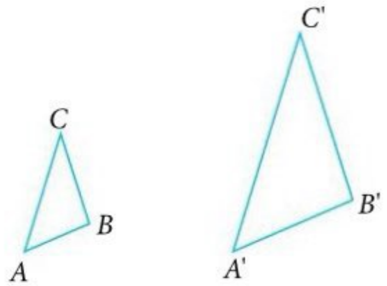
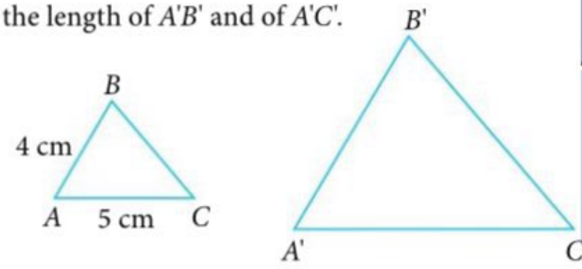



Fig. 8.11

**Worked Example 9**

### Solving problem involving enlargement of figures

In the figure, $\triangle A'B'C'$ is an enlargement of $\triangle ABC$ with a scale factor of 2. Given that $AB = 4$ cm and $AC = 5$ cm, find the length of $A'B'$ and of $A'C'$.




**Solution**

$\triangle ABC$ is similar to $\triangle A'B'C'$ under enlargement.

$\therefore \dfrac{A'B'}{AB} = \dfrac{A'C'}{AC} = 2$ scale factor

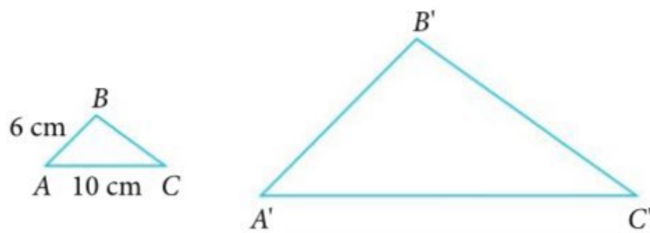
i.e. $\dfrac{A'B'}{4} = 2$ and $\dfrac{A'C'}{5} = 2$

$\therefore A'B' = 8$ cm and $A'C' = 10$ cm

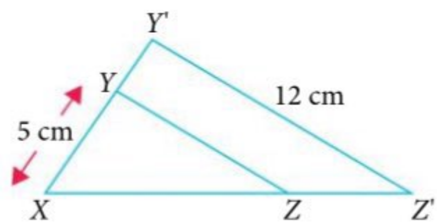
**Practise Now 9**

Similar and Further Questions
**Exercise 8C**
Questions 1, 2, 8, 9

1. In the figure, $\triangle A'B'C'$ is an enlargement of $\triangle ABC$ with a scale factor of 3. Given that $AB = 6$ cm and $AC = 10$ cm, find the length of $A'B'$ and of $A'C'$.




2. In the figure, $\triangle XY'Z'$ is an enlargement of $\triangle XYZ$ with a scale factor of 1.5. Given that $XY = 5$ cm and $Y'Z' = 12$ cm, find the length of $XY'$ and of $YZ$.




3. A photograph shows Raju, who is 180 cm tall, standing in front of his terrace house. In the photograph, the height of Raju is 9 cm and that of his house is 22.5 cm. Find the actual height of the house, giving your answer in metres.

## B. Similarity and floor plans

In our daily activities, we sometimes need to enlarge or reduce pictures or drawings of actual objects. For example, a coach might draw a plan of a badminton court to explain the rules of the game, and would need to make a much smaller drawing on paper or on a whiteboard. If we wish to show a diagram of the apparatus used for a science experiment, we can also enlarge the diagram on a screen using a visualiser.

Fig. 8.12 shows the floor plan of a house. It is similar to the actual floor of the house and hence, the dimensions of the floor plan are **proportional** to the actual dimensions of the house. Fig. 8.12 has been drawn to a scale of 1 cm to 2 m, i.e. 1 cm on the plan represents 2 m on actual ground.

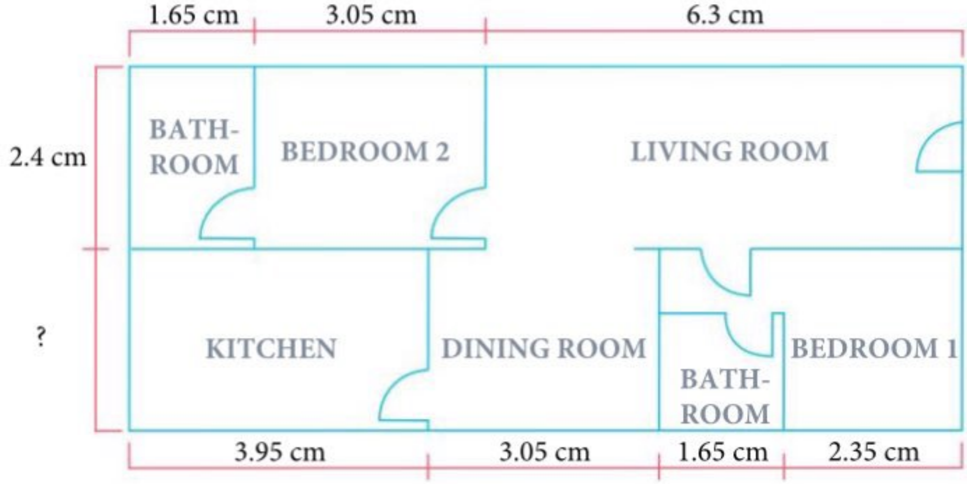



Fig. 8.12

From Fig. 8.12, we have:

**(i)** Length of the living room $= (6.3 \times 2)$ m
$= 12.6$ m

Width of the living room $= (2.4 \times 2)$ m
$= 4.8$ m

**(ii)** Area of Bedroom 1 on the plan $= 2.4$ cm $\times$ $2.35$ cm

Actual area of Bedroom 1 $= (2.4 \times 2)$ m $\times$ $(2.35 \times 2)$ m
$= 22.56$ m$^2$

**(iii)** If we are given that the actual width of the kitchen is 4.8 m, we can also determine the width on the plan using the scale:

| **Actual** | | **Plan** |
|---|---|---|
| 2 m | is represented by | 1 cm |
| 1 m | is represented by | $\frac{1}{2} = 0.5$ cm |
| 4.8 m | is represented by | $(4.8 \times 0.5)$ cm |
| | | $= 2.4$ cm |

$\therefore$ the width of the kitchen on the plan is 2.4 cm.

**Attention**

For ease of manipulation, we write the unknown that we want to find on the right-hand side. In **(iii)**, since we want to find the length on the plan, we write 'Plan' on the right-hand side.

**Practise Now 10A**

Similar and Further Questions
**Exercise 8C**
Questions 3, 10, 11

1. The scale of the floor plan of a house is 1 cm to 2.5 m. Find
   (i) the actual length of the dining room if it is represented by a length of 1.25 cm on the plan,
   (ii) the length on the plan that represents the width of a bedroom if its actual width is 3.4 m.

2. A model of a cruise liner is made to a scale of 1 cm to 4 m. The length of the model cruise liner is 67 cm. Find
   (i) the actual length of the cruise liner,
   (ii) the length of the model cruise liner if it is made to a scale of 1 cm to 10 m.

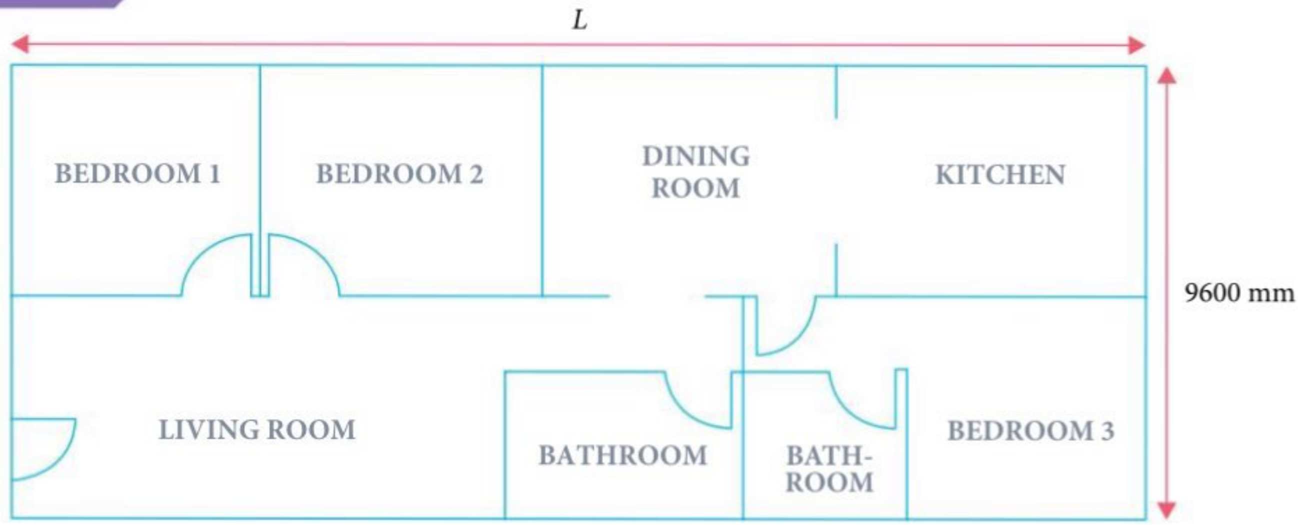
**Worked Example 10**

### Finding the scale of a drawing by measurements

The diagram shows a scale drawing of an apartment.

(i) By measuring the scale drawing, find its scale.

(ii) Find the actual length $L$, in metres, of the apartment.




**Solution**

(i) By measuring the length of the apartment that represents 9600 mm, we obtain 6 cm = 60 mm.

| **Plan** | | **Actual** |
|---|---|---|
| 60 mm | represents | 9600 mm |
| 1 mm | represents | $(9600 \div 60)$ mm = 160 mm |

$\therefore$ the scale is 1 : 160.

(ii)

| **Plan** | | **Actual** |
|---|---|---|
| 1 mm | represents | 160 mm |
| 150 mm | represents | $(150 \times 160)$ mm = 24 000 mm |
| | | = 24 m |

$\therefore$ the actual length $L$ of the apartment is 24 m.

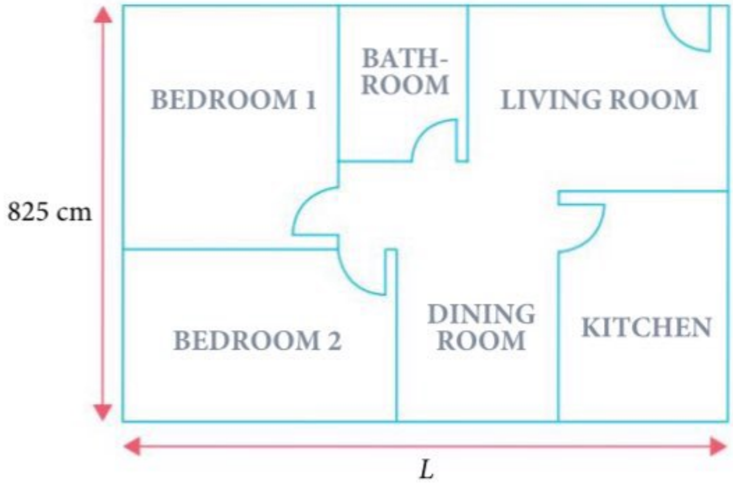
**Practise Now 10B**

Similar and Further Questions
**Exercise 8C** Question 4

The diagram shows a scale drawing of another apartment.

**(i)** By measuring the scale drawing, find its scale.

**(ii)** Find the actual length $L$, in metres, of the apartment.




## C. Similarity and map scales

Maps are drawings of actual land. Since a map is similar to the actual land, the distance between two points on a map is **proportional** to the distance between the same two points on the ground. The **linear scale** of a map is usually given at a corner of the map. There are several ways to represent the scale of a map. For example, on a map of Asia, the scale as shown in Fig. 8.13 may be given.

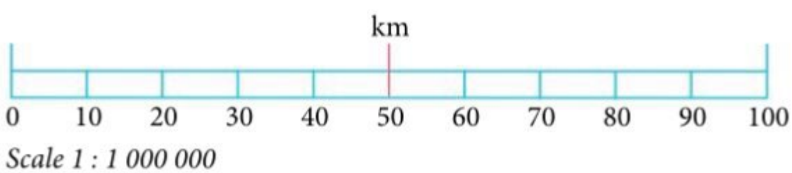



Fig. 8.13

There are two ways to read the scale. If we use a ruler to measure the length from 0 to 10 km on the scale, we will find that it is 1 cm. Thus 1 cm represents 10 km. This is the same as the scale 1 : 1 000 000. When a scale is given in this form, it means that we have to use the same units on both sides, i.e. 1 m : 1 000 000 m, 1 km : 1 000 000 km or 1 cm : 1 000 000 cm = 10 km, so 1 cm represents 10 km.

The scale of 1 : 1 000 000 can also be represented as a **representative fraction** (R.F.) of $\frac{1}{1\ 000\ 000}$. For example, if the R.F. is $\frac{1}{200}$, the scale is 1 : 200. When we use R.F., the numerator must always be 1.

**Worked Example 11**

## Solving problem involving map scale

A map has a scale of 1 cm to 3 km.

**(i)** If a road has a length of 3 cm on the map, calculate its actual length.

**(ii)** If the distance between two stadiums is 8.5 km, find the corresponding distance on the map.

**(iii)** Express the scale of the map in the form $\frac{1}{n}$, where $n$ is an integer.

### Solution

**(i)**

| **Map** | | **Actual** | |
|---|---|---|---|
| 1 cm | represents | 3 km | scale |
| 3 cm | represents | $(3 \times 3)$ km = 9 km | |

$\therefore$ the actual length of the road is 9 km.

**(ii)**

| **Actual** | | **Map** | |
|---|---|---|---|
| 3 km | is represented by | 1 cm | scale |
| 1 km | is represented by | $\frac{1}{3}$ cm | |
| 8.5 km | is represented by | $\left(7.5 \times \frac{1}{3}\right)$ cm = 2.5 cm | |

$\therefore$ the distance between the two stadiums on the map is 2.5 cm.

> **Attention**
> **(ii)** To find the length on the map, we write 'Map' on the right-hand side.

**(iii)** 3 km = 300 000 cm

i.e. the scale of the map is $\frac{1}{300\ 000}$.

**Practise Now 11**

Similar and Further Questions
**Exercise 8C**
Questions 5, 6, 12, 13, 16

1. A map has a scale of 1 cm to 5 km.
   **(i)** If a road has a length of 6.5 cm on the map, find its actual length.
   **(ii)** If the distance between two towns is 25 km, calculate the corresponding distance on the map.
   **(iii)** Express the scale of the map in the form $\frac{1}{n}$, where $n$ is an integer.

2. A map is drawn to a scale of 1 : 50 000.
   **(i)** Find the actual length that is represented by 2 cm on the map, giving your answer in kilometres.
   **(ii)** Calculate the length on the map that represents an actual length of 14.5 km.

> **Problem-solving Tip**
> **2.** '1 : 50 000' means 1 cm on the map represents 50 000 cm on actual ground.

We can also find the actual area of a site from its area on the map. For example, if the scale of a map is 1 cm to 2 km, then 1 cm$^2$ on the map represents an actual area of $(2 \text{ km})^2 = 4 \text{ km}^2$ (see Fig. 8.14). Therefore, the **area scale** of the map is 1 cm$^2$ to 4 km$^2$.

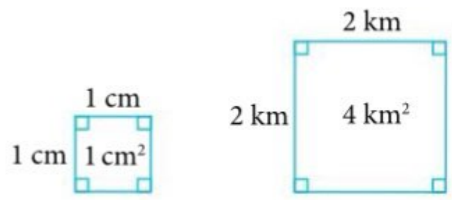



Fig. 8.14

Worked Example 12

## Solving problem involving area scale

A scale of 1 cm to 0.5 km is used for a map.

**(i)** If a plot of land has an area of 8 cm² on the map, calculate its actual area.

**(ii)** If the actual area of a pond is 50 000 m², find its area on the map.

### Solution

**(i)** **Map** | **Actual**

| Map | | Actual | |
|---|---|---|---|
| 1 cm | represents | $0.5 \text{ km} = \frac{1}{2} \text{ km}$ | scale |
| $1 \text{ cm}^2$ | represents | $\left(\frac{1}{2} \text{ km}\right)^2 = \frac{1}{4} \text{ km}^2$ | |
| $8 \text{ cm}^2$ | represents | $\left(8 \times \frac{1}{4}\right) \text{ km}^2 = 2 \text{ km}^2$ | |

$\therefore$ the actual area of the plot of land is $2 \text{ km}^2$.

**(ii)**

| Actual | | Map |
|---|---|---|
| 0.5 km | is represented by | 1 cm |
| i.e. 500 m | is represented by | 1 cm |
| 1 m | is represented by | $\frac{1}{500}$ cm |
| $1 \text{ m}^2$ | is represented by | $\left(\frac{1}{500} \text{ cm}\right)^2 = \frac{1}{250\ 000} \text{ cm}^2$ |
| $50\ 000 \text{ m}^2$ | is represented by | $\left(50\ 000 \times \frac{1}{250\ 000}\right) \text{ cm}^2$ |
| | | $= 0.2 \text{ cm}^2$ |

$\therefore$ the area of the pond on the map is $0.2 \text{ cm}^2$.

**Problem-solving Tip**

**(ii)** Since the actual area of the pond is given in m², it will be easier to convert the linear scale from km to m before finding the area scale.

**Practise Now 12**

Similar and Further Questions
Exercise 8C
Questions 7, 14, 15, 17

**1.** A scale of 1 cm to 2 km is used for a map.

**(i)** If a plot of land has an area of 3 cm² on the map, find its actual area.

**(ii)** If the actual area of a lake is 18 000 000 m², calculate its area on the map.

**2.** A map has a scale of 5 cm to 1 km.

**(i)** Write this scale in the form $1 : n$, where $n$ is an integer.

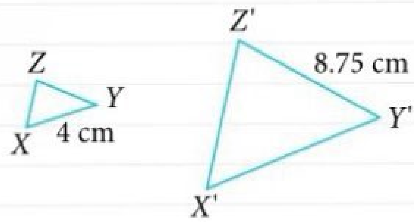
**(ii)** A plot of land is represented by an area of 14 cm² on the map. Calculate the actual area of the plot of land in square kilometres.

## Reflection

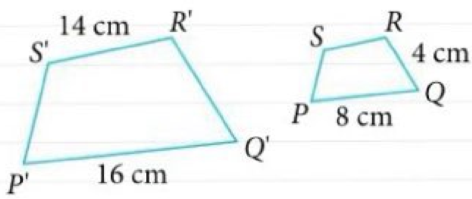
1. What have I learnt about similar figures that could help me with solving problems involving enlargement, floor plans and map scales?
2. How do enlargement, floor plans and map scales make use of proportionality?

## Exercise 8C

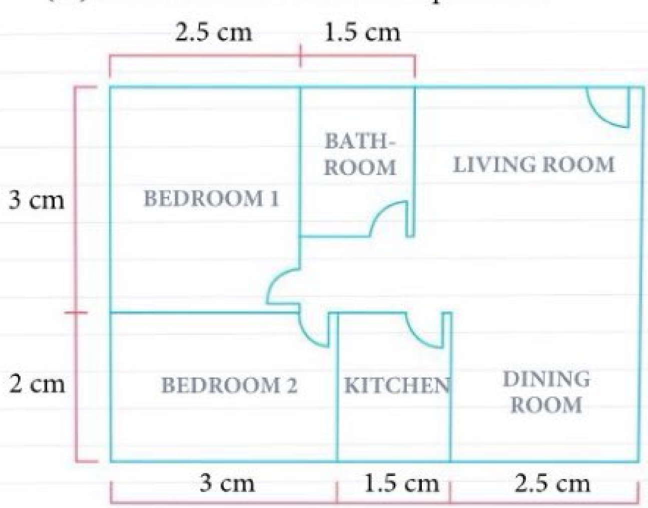
1. In the figure, $\triangle X'Y'Z'$ is an enlargement of $\triangle XYZ$ with a scale factor of 2.5. Given that $XY = 4$ cm and $Y'Z' = 8.75$ cm, find the lengths of $X'Y'$ and $YZ$.



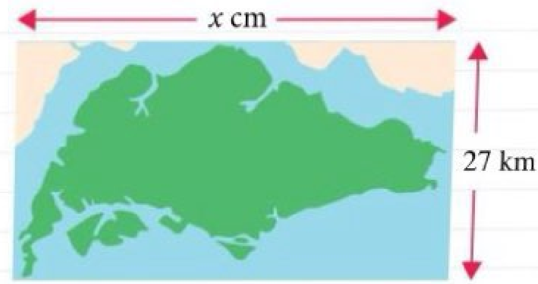

2. In the figure, $P'Q'R'S'$ is an enlargement of $PQRS$ with a scale factor of $k$.
   (i) Given that $PQ = 8$ cm and $P'Q' = 16$ cm, find the value of $k$.
   (ii) Given that $QR = 4$ cm and $S'R' = 14$ cm, calculate the length of $Q'R'$ and of $SR$.




3. The figure shows the floor plan of an apartment which has been drawn to a scale of 1 cm to 1.5 m. Find
   (i) the actual dimensions of Bedroom 1,
   (ii) the actual area of the kitchen,
   (iii) the actual total area of the apartment.




4. The figure shows a map of Singapore. The actual length of Singapore from the North to the South is 27 km.
   (i) By taking measurements, find the scale of the map.
   (ii) What is the actual distance between the East and West of Singapore which is represented by $x$ cm on the map?



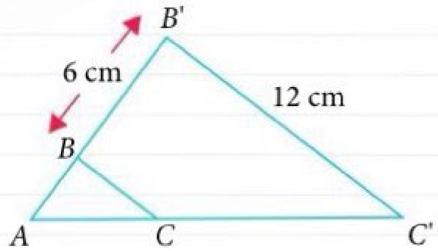

5. A map of Pakistan has a scale of 1 cm to 50 km.
   (i) Given that the Karakoram Highway has a length of 26 cm on the map, find its actual length.
   (ii) Express the scale of the map in the form $\frac{1}{n}$, where $n$ is an integer.

6. A map is drawn to a scale of 1 : 20 000.
   (i) Find the actual length that is represented by $5\frac{1}{2}$ cm on the map, giving your answer in kilometres.
   (ii) Calculate the length on the map that represents an actual length of 100 m.

7. A scale of 1 cm to 8 km is used for a map.
   (i) If a forest has an area of 5 cm$^2$ on the map, find its actual area.
   (ii) If the actual area of a park is 128 km$^2$, calculate its area on the map.

## Exercise 8C

**8.** In the figure, $\triangle AB'C'$ is an enlargement of $\triangle ABC$ with a scale factor of 3. Given that $B'C' = 12$ cm and $BB' = 6$ cm, find the length of $BC$ and of $AB'$.




**9.** In a television commercial, a toddler of height 75 cm is standing next to a tin of milk of height 24 cm. If the height of the toddler is 25 cm on a television screen, find the height of the tin of milk on the screen.

**10.** On the floor plan of an apartment, the length of a bedroom is 12 cm. The actual length of the bedroom is 3 m.

**(i)** What is the scale used?

**(ii)** Find the width of the living room on the floor plan if its actual width is 425 cm.

**11.** A model of the Bahria Town Icon is made to a scale of 1 cm to 15 m. The height of the model tower is 18.2 cm. Find

**(i)** the actual height of the tower,

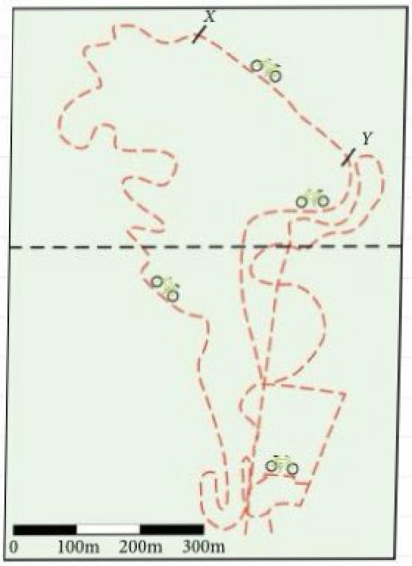
**(ii)** the height of the model tower, to the nearest 0.1 cm, if it is made to a scale of 1 cm to 12 m.

**12.** A map of Islamabad has a scale of 4 cm to 5 km. The distance between Centaurus Mall and Giga Mall on the map is 21.04 cm. Find

**(i)** the actual distance between the two malls,

**(ii)** the distance between the two malls on another map of Islamabad that is drawn to a scale of 1 : 175 000.

**13.** The figure shows a map of a mountain bike trail. The scale is given in the form of a bar at the bottom of the map, showing 0 m to 300 m.




**(i)** Express the scale of the map in the form $1 : n$, where $n$ is an integer.

**(ii)** Estimate the actual distance $XY$ of the biking trail.

**(iii)** Albert cycles from $X$ to $Y$, but he discovers that the actual distance $XY$ is about 350 m. Suggest a reason why the actual distance $XY$ is different from your estimate in part **(ii)**.

**14.** A map is drawn to a scale of 1 : 240 000.

**(i)** If a seawater lake has an area of 3.8 cm$^2$ on the map, find its actual area in square kilometres.

**(ii)** If the actual area of a plot of land is 2 908 800 m$^2$, calculate its area on the map.

**15.** A map has a scale of 1 cm to 500 m.

**(i)** Express the scale of the map in the form $1 : n$, where $n$ is an integer.

**(ii)** If the distance between two districts is 28 km, find the corresponding distance on the map.

**(iii)** If a jungle has an area of 12 cm$^2$ on the map, calculate its actual area in square kilometres.

## Exercise 8C

**16.** The map below shows Singapore and the central and southern regions of Peninsula Malaysia. Use the map to answer the following questions. You may use a ruler to measure the approximate distances between any two places before finding the actual distances from the given scale.


**(i)** Express the scale of the map in the form $\frac{1}{n}$, where $n$ is an integer.

**(ii)** Estimate the actual straight-line distance between Singapore and Kuantan.

**(iii)** How much would it cost to hire a taxi to travel from Melaka to Kuala Lumpur if the taxi fare is 60 cents per kilometre?

**(iv)** Calculate the time taken for a car to travel from Batu Pahat to Port Dickson if its average speed is 60 km/h, giving your answer in hours and minutes.

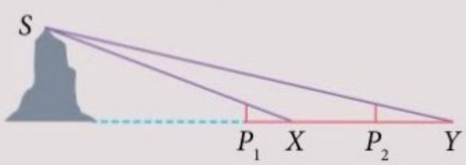
**(v)** A train takes 4 hours to travel from Johor Bahru to Segamat. Find its average speed, giving your answer in km/h.

**17.** A map is drawn to a scale of 1 cm to $x$ m. A plot of land with an area of 28 hectares is represented by an area of 112 cm². Find the value of $x$.
(1 hectare = 10 000 m²)

## Looking Back

Congruent and similar triangles are examples of how **proportionality** is applied in real-life situations. Many of the problems in the past revolved around these ideas. For example, Chinese Mathematician Liu Hui (220–280 AD), published the following problem, translated to English below, in a renowned mathematical book *Hai Dao Suan Jing (Sea Island Mathematical Manual)*..

> $P_1$ and $P_2$ are poles 5 pu* high and 1000 pu apart. When viewed from $X$ at ground level, 123 pu behind $P_1$, the summit $S$ of the island is in line with the top of $P_1$. Similarly, when viewed from $Y$ at ground level, 127 pu behind $P_2$, the top of the island is in line with the top of $P_2$. Calculate the height of the island and its distance from $P_1$.




* 'pu' is a measure of length; 1 m ≈ 0.56 pu.

This problem and other related problems such as finding the height of a tree on the side of a mountain, the depth of a valley, or the height of a tower on a hill, can be modelled using similar triangles. We can then use ideas of proportionality to solve them. In this chapter, we begin an exciting journey into the world of similarity and congruence, which will be important in other mathematical topics such as trigonometry.

## Summary

1. Two figures are **congruent** if they have exactly the same shape and size.
   They can be mapped onto each other under **translation**, **rotation** and **reflection**.
   - Draw two congruent figures.

2. Two polygons are congruent if and only if
   - all their corresponding angles are equal, *and*
   - all their corresponding sides are equal.
     - Draw two congruent polygons.

3. Two figures are **similar** if they have exactly the same shape but *not* necessarily the same size.
   If two similar figures also have exactly the same size, then they are congruent. In other words, congruence is a *special case* of similarity.
   - Draw two similar (but not congruent) figures.

4. Two polygons are similar if and only if
   - all their corresponding angles are equal, *and*
   - all the ratios of the lengths of their corresponding sides are equal.
     - Draw two similar (but not congruent) polygons.

5. A figure and its image under an *enlargement* are *similar*.

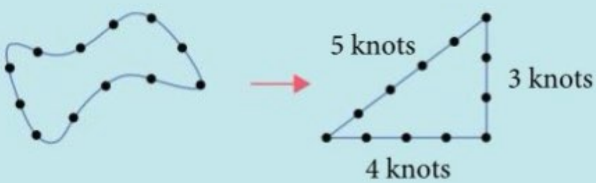
| Scale factor, $k$ | Image |
|---|---|
| $k > 1$ | Bigger than original figure |
| $0 < k < 1$ | Smaller than original figure |
| $k = 1$ | Congruent with original figure |

6. If the **linear scale** of a map is $1 : x$, it means that 1 unit on the map represents $x$ units on the actual ground, and the **area scale** of the map is $1 : x^2$.

# CHAPTER 9

# Pythagoras' Theorem

The ancient Egyptians made use of knotted ropes to form right-angled triangles for the construction of various structures such as the pyramids. These people, known as 'rope-stretchers', used a rope with 12 evenly-spaced knots tied in a circle to form a right-angled triangle such that the length of each side is a whole number unit.




Will a rope with 13 evenly-spaced knots work as well? Why? We shall discover the secret of these 'rope-stretchers' in this chapter.

## Learning Outcomes

What will we learn in this chapter?

- What Pythagoras' Theorem is
- Why Pythagoras' Theorem applies only to right-angled triangles
- How to determine whether a triangle is a right-angled triangle given the lengths of its three sides
- Why Pythagoras' Theorem has useful applications in real life

Introductory Problem

A rectangular living room is 5 m long, 5 m wide and 2.5 m high. A lizard is positioned in the middle of one wall while a fly rests on the adjacent wall, 2 m above the ground and 0.5 m away from the vertical edge connecting the two walls.

What is the shortest distance that the lizard would have to crawl in order to catch the fly?

In any right-angled triangle, there is a relationship between the lengths of the three sides of the triangle. In this chapter, we will learn about this relationship and how it can be applied to solve problems, such as the one above.

# 9.1 Pythagoras' Theorem

Fig. 9.1 shows a right-angled triangle $ABC$ where angle $C = 90°$.

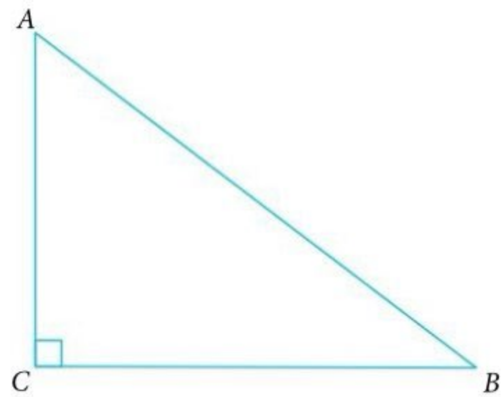



Fig. 9.1

In Book 1, we have learnt that the longest side of a triangle is opposite the largest angle. In a right-angled triangle, the right angle is the largest angle. Why?

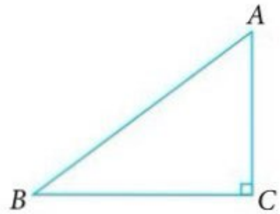
Thus in $\triangle ABC$, the side $AB$ is the longest side.

$AB$ is called the **hypotenuse** of $\triangle ABC$.

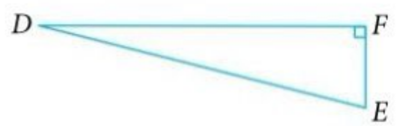
**Attention**

The term 'hypotenuse' does *not* apply to the longest side of an oblique triangle (a triangle with no right angle).

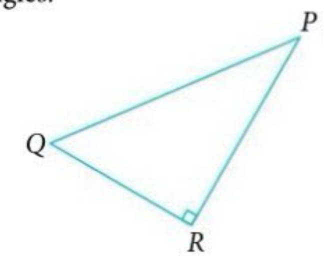
**Practise Now 1A** Identify the hypotenuse of each of the following right-angled triangles.




**(a)**




**(b)**




**(c)**

## Investigation Pythagoras' Theorem

**Part 1:** Determine the relationship between the lengths of the three sides of a right-angled triangle.
For this part of the Investigation, a ruler is required.

Fig. 9.2 shows three right-angled triangles where the lengths of the sides are *integer* values. Which side of each triangle is the hypotenuse?

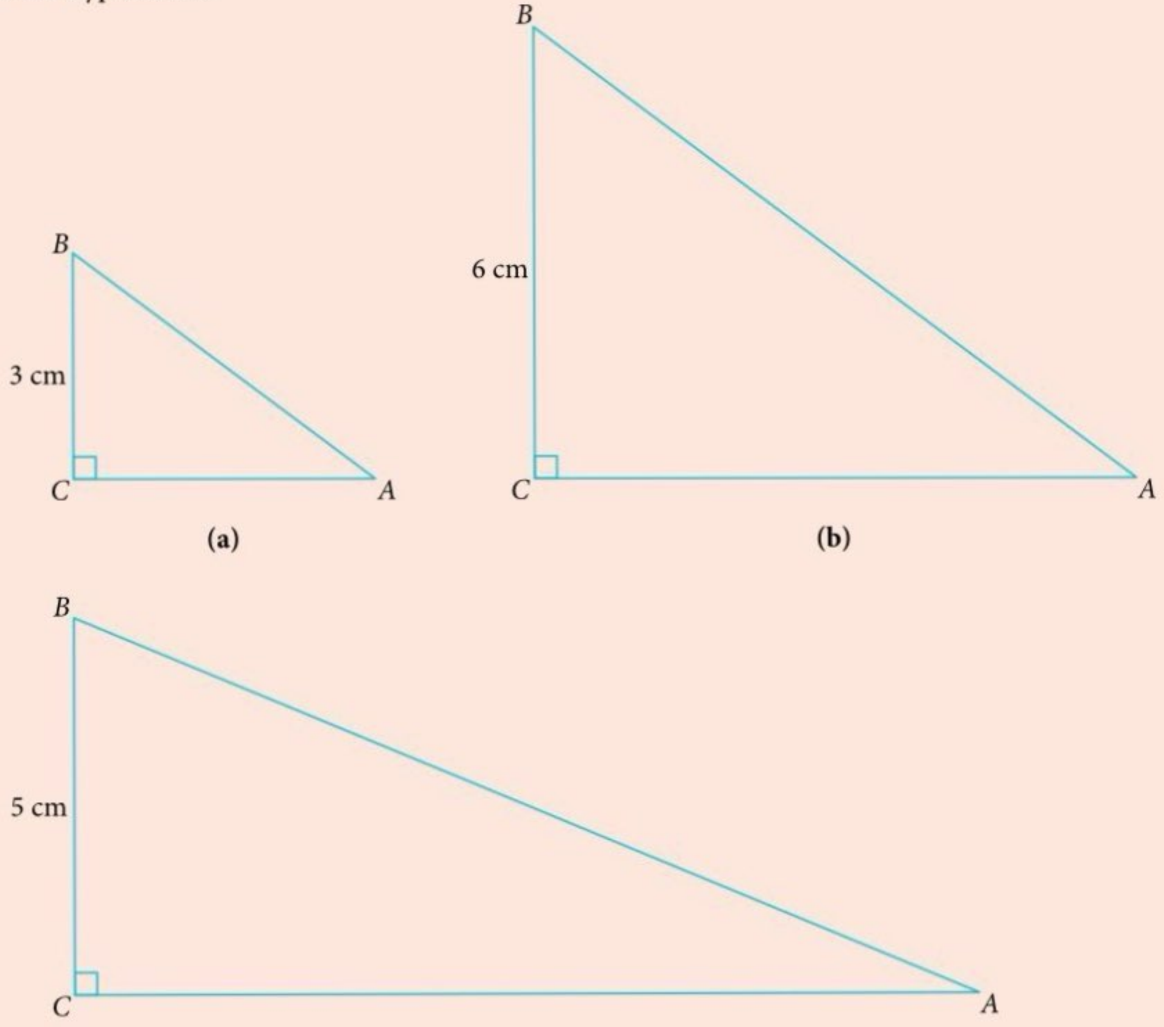



Fig. 9.2

1. For each triangle shown in Fig. 9.2, measure and write down the length of $AC$ and of $AB$ in Table 9.1.

| | $BC$ | $AC$ | $AB$ | $BC^2$ | $AC^2$ | $AB^2$ | $BC^2 + AC^2$ |
|---|---|---|---|---|---|---|---|
| **(a)** | 3 cm | | | 9 cm$^2$ | | | |
| **(b)** | 6 cm | | | 36 cm$^2$ | | | |
| **(c)** | 5 cm | | | 25 cm$^2$ | | | |

Table 9.1

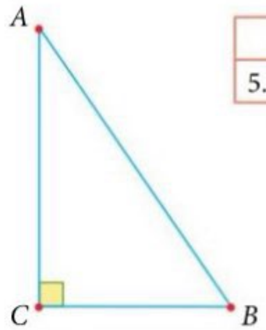
2. Complete Table 9.1. What do you notice about the value of $AB^2$ and that of $BC^2 + AC^2$ in Table 9.1?

**Part 2:** Use a geometry software template to determine the relationship between the lengths of the three sides of a right-angled triangle.

For this part of the Investigation, go to www.sl-education.com/tmsoupp2/pg274 or scan the QR code on the right and open the geometry software template 'Pythagoras' Theorem'.

**Pythagoras' Theorem**

What is the relationship between the value of $AB^2$ and that of $BC^2 + AC^2$?




| $BC$ | $AC$ | $AB$ | $BC^2$ | $AC^2$ | $AB^2$ | $BC^2 + AC^2$ |
|---|---|---|---|---|---|---|
| 5.17 cm | 7.55 cm | 9.15 cm | 26.72 cm$^2$ | 56.99 cm$^2$ | 83.71 cm$^2$ | 83.71 cm$^2$ |

Fig. 9.3

1. Fig. 9.3 is a screenshot of the template. Which side is the hypotenuse of the right-angled triangle $ABC$?
2. Click and move a point $A$, $B$ or $C$ to get five other right-angled triangles. Complete Table 9.2.

| | $BC$ | $AC$ | $AB$ | $BC^2$ | $AC^2$ | $AB^2$ | $BC^2 + AC^2$ |
|---|---|---|---|---|---|---|---|
| (a) | | | | | | | |
| (b) | | | | | | | |
| (c) | | | | | | | |
| (d) | | | | | | | |

Table 9.2

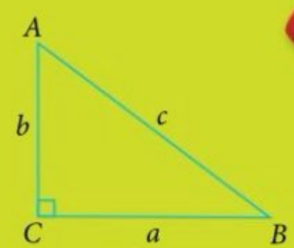
3. What do you notice about the value of $AB^2$ and that of $BC^2 + AC^2$ in Table 9.2?

From the above Investigation, we observe that *the square of the length of the hypotenuse of a right-angled triangle is always equal to the sum of the squares of the lengths of the other two sides*. This result, known as **Pythagoras' Theorem**, describes the relationship between the three sides of a right-angled triangle.

In a right-angled triangle $ABC$,

$AB^2 = BC^2 + AC^2$,
where $AB$ is the length of the hypotenuse of the triangle;

i.e. $c^2 = a^2 + b^2$.



## Thinking Time

Fig. 9.4 shows three square plots of land, $P$, $Q$ and $R$, connected by a triangular path.

Which plot(s) of land is/are the largest?

**(a)** $P$ **(b)** $Q$ **(c)** $R$ **(d)** $P + Q$

Explain your answer.

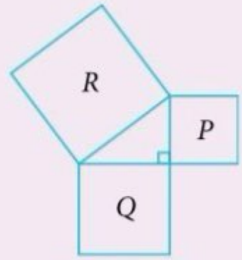
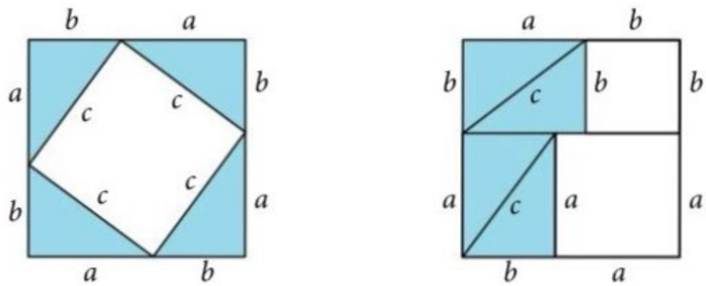



Fig. 9.4

There are more than 300 proofs of Pythagoras' Theorem. We shall now take a look at one of these proofs.

> **Information**
> In mathematics, a theorem is a statement that has been proven to be true.

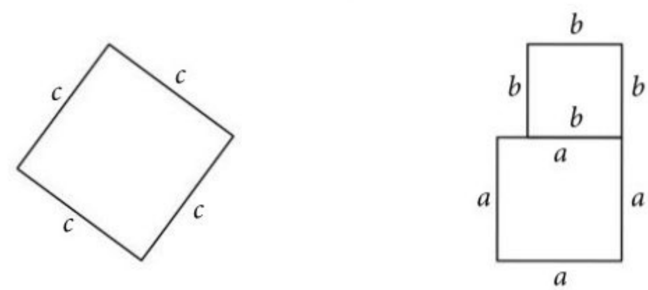
Consider the figures shown in Fig. 9.5. The eight right-angled triangles in blue are of the same size.




**(a)** **(b)**

Fig. 9.5

Fig. 9.5(**a**) and Fig. 9.5(**b**) are squares of sides $(a + b)$ units. Thus, they are of the same size. After removing four blue right-angled triangles from each figure, we obtain the figures shown in Fig. 9.6.




**(a)** **(b)**

Fig. 9.6

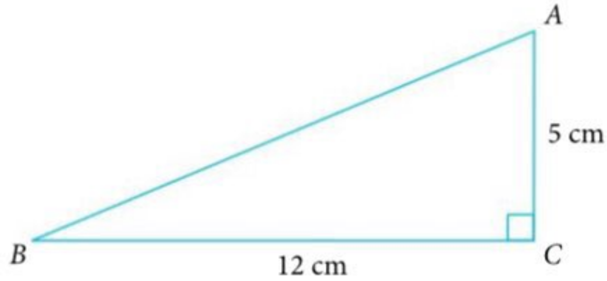
Since the total area of the four right-angled triangles removed from each figure in Fig. 9.5 is the same, the area of the figure in Fig. 9.6(**a**) is equal to the area of the figure in Fig. 9.6(**b**).

Therefore, $c^2 = a^2 + b^2$.

Worked Example 1

## Finding the length of the hypotenuse of a right-angled triangle

In $\triangle ABC$, $AC = 5$ cm, $BC = 12$ cm and $\angle ACB = 90°$. Calculate the length of $AB$.




### Solution

In $\triangle ABC$, $\angle ACB = 90°$.

Using Pythagoras' Theorem,

$$\begin{aligned} AB^2 &= BC^2 + AC^2 \quad \text{AB is the hypotenuse} \\ &= 12^2 + 5^2 \\ &= 144 + 25 \\ &= 169 \end{aligned}$$

$$\begin{aligned} \therefore AB &= \sqrt{169} \quad (\text{since } AB > 0) \\ &= 13 \text{ cm} \end{aligned}$$

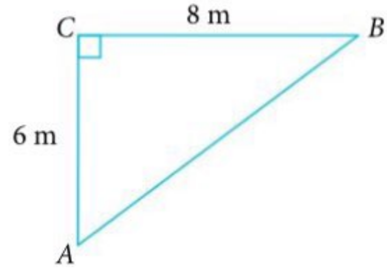
**Attention**

Pythagoras' Theorem can only be applied in a right-angled triangle.

**Practise Now 1B**

Similar and Further Questions
Exercise 9A
Questions 1(a)–(d), 2, 3

1. In $\triangle ABC$, $AC = 6$ m, $BC = 8$ m and $\angle ACB = 90°$.
Find the length of $AB$.

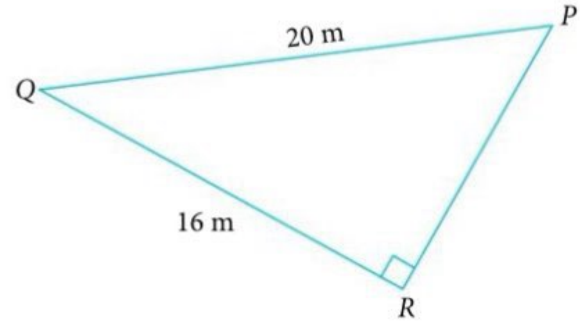



2. In triangle $ABC$, $AC = 24$ cm, $BC = 7$ cm and angle $ACB = 90°$.
Find the length of $AB$.

**Worked Example 2**

### Finding the length of the third side of a right-angled triangle

In triangle $PQR$, $PQ = 20$ m, $QR = 16$ m and angle $PRQ = 90°$. Calculate the length of $PR$.




**Solution**

In $\triangle PQR$, $\angle PRQ = 90°$.

Using Pythagoras' Theorem,

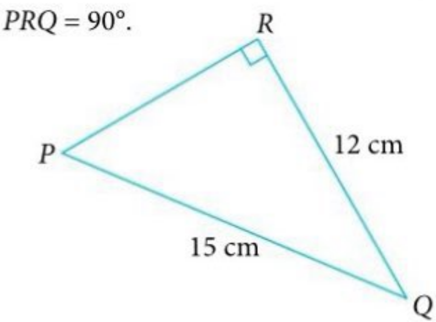
$$\begin{aligned} PQ^2 &= QR^2 + PR^2 \quad \text{PQ is the hypotenuse} \\ 20^2 &= 16^2 + PR^2 \\ PR^2 &= 20^2 - 16^2 \\ &= 400 - 256 \\ &= 144 \end{aligned}$$

$$\therefore \; PR = \sqrt{144} \text{ (since } PR > 0) = 12 \text{ m}$$

**Practise Now 2**

Similar and Further Questions
**Exercise 9A**
Questions 4(a)-(d), 5, 6, 8

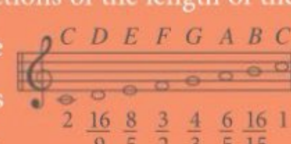
1. In triangle $PQR$, $PQ = 15$ cm, $QR = 12$ cm and angle $PRQ = 90°$. Find the length of $PR$.




2. In triangle $PQR$, $PQ = 35$ m, $PR = 28$ m and angle $PRQ = 90°$. Find the length of $QR$.

Pythagoras was a Greek philosopher and mathematician.
In addition to Pythagoras' Theorem, he also discovered the following:

- The musical note produced by a vibrating string of a certain length is exactly one octave lower than the note produced by a string of the same material and half that length.
- Other notes in the musical scale can be produced using certain fractions of the length of the string. For example, a string $\frac{4}{3}$ the length of a C-string produces the note G (one octave lower). The figure shows the various musical notes that can be produced for each given fraction of the length of a C-string.



Worked Example 3

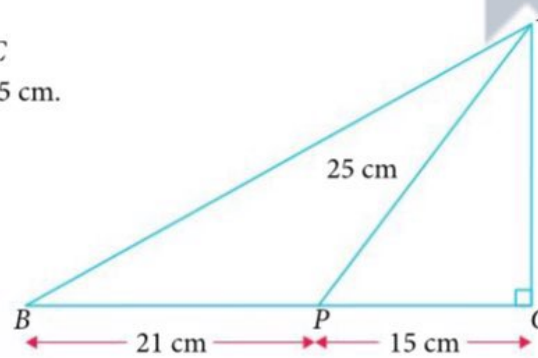
## Finding the lengths of unknown sides

In triangle $ABC$, angle $ACB = 90°$. $P$ lies on $BC$ such that $BP = 21$ cm, $PC = 15$ cm and $AP = 25$ cm.

**(a)** Calculate the length of

**(i)** $AC$, **(ii)** $AB$.

**(b)** Point $X$ lies on $AC$ such that $PX = 18$ cm. Does $X$ lie closer to $A$ or $C$? Explain your answer.




### Solution

**(a) (i)** In $\triangle APC$, $\angle ACB = 90°$.

Using Pythagoras' Theorem,

$$
\begin{aligned}
AP^2 &= PC^2 + AC^2 \qquad \text{AP is the hypotenuse}\\
25^2 &= 15^2 + AC^2\\
AC^2 &= 25^2 - 15^2\\
&= 625 - 225\\
&= 400\\
\therefore AC &= \sqrt{400} \ (\text{since } AC > 0)\\
&= 20 \text{ cm}
\end{aligned}
$$

**(ii)** In $\triangle ABC$, $\angle ACB = 90°$.

Using Pythagoras' Theorem,

$$
\begin{aligned}
AB^2 &= BC^2 + AC^2 \qquad \text{AB is the hypotenuse}\\
&= (21 + 15)^2 + 20^2\\
&= 36^2 + 20^2\\
&= 1296 + 400\\
&= 1696\\
\therefore AB &= \sqrt{1696} \ (\text{since } AB > 0)\\
&= 41.2 \text{ cm (to 3 s.f.)}
\end{aligned}
$$

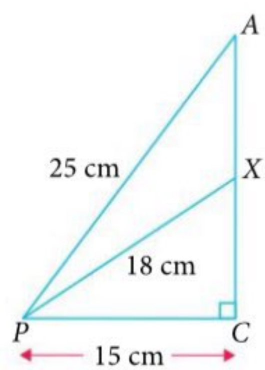
**Attention**

Since $\triangle ABP$ is not a right-angled triangle, Pythagoras' Theorem cannot be applied in $\triangle ABP$.

**(b)** In $\triangle CPX$, $\angle PCX = 90°$.

Using Pythagoras' Theorem,

$$
\begin{aligned}
PX^2 &= PC^2 + CX^2 \qquad \text{PX is the hypotenuse}\\
18^2 &= 15^2 + CX^2\\
CX^2 &= 18^2 - 15^2\\
&= 99\\
CX &= \sqrt{99} \ (\text{since } CX > 0)\\
&= 9.9499 \text{ cm (to 5 s.f.)}\\
AX &= AC - CX\\
&= 20 - \sqrt{99}\\
&= 10.050 \text{ cm (to 5 s.f.)}
\end{aligned}
$$




Since length of $CX <$ length of $AX$, $X$ lies closer to $C$.

**Practise Now 3**

Similar and Further Questions

**Exercise 9A**

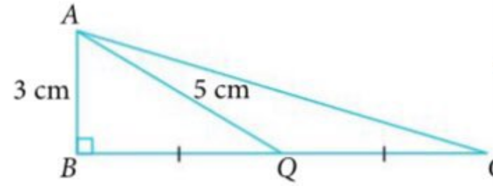
Questions 7, 9(a), (b), 10(a)–(e), 11(a)–(d), 12–14

**1.** In triangle $ABC$, $AB = 3$ cm and angle $ABC = 90°$. $Q$ lies on $BC$ such that $BQ = QC$ and $AQ = 5$ cm.

**(a)** Find the length of

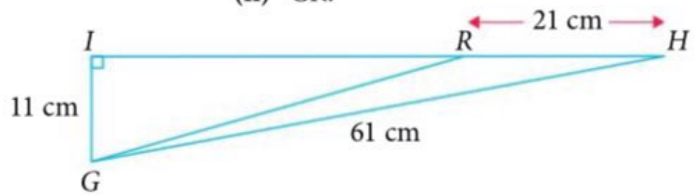
**(i)** $BQ$, **(ii)** $AC$.

**(b)** Point $X$ lies on $BC$ such that $AX = 7$ cm. Does $X$ lie closer to $Q$ or $C$? Explain your answer.



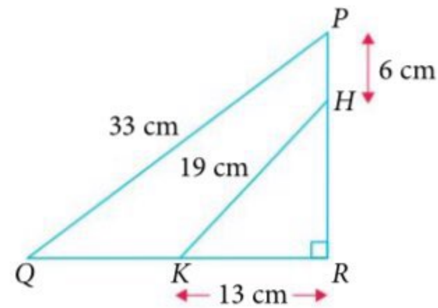

**2.** In $\triangle GHI$, $GH = 61$ cm and $GI = 11$ cm and $\angle GIH = 90°$. $R$ lies on $IH$ such that $RH = 21$ cm. Find the length of

**(i)** $HI$, **(ii)** $GR$.




**3.** In $\triangle PQR$, $PQ = 33$ cm and $\angle PRQ = 90°$. $H$ lies on $PR$ such that $PH = 6$ cm and $K$ lies on $QR$ such that $KR = 13$ cm. Find the length of

**(i)** $HR$, **(ii)** $QK$.




## Reflection

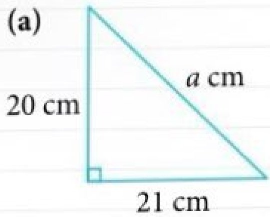
1. What type of triangle can I apply Pythagoras' Theorem on?
2. When applying Pythagoras' Theorem, where should I write the square of the length of the hypotenuse?

Basic | Intermediate | Advanced

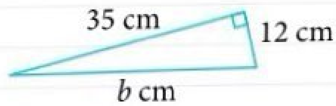
# Exercise 9A

**1.** Find the value of the unknown in each of the following right-angled triangles.

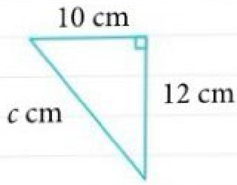
**(a)**



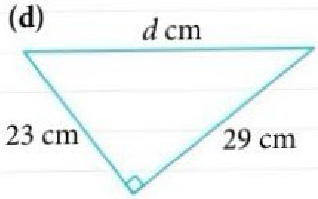

**(b)**




**(c)**




**(d)**

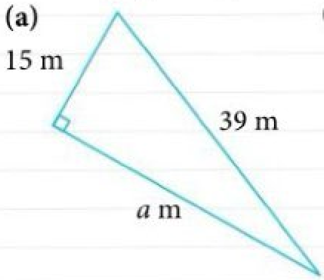



**2.** In $\triangle ABC$, $AB = 8$ cm, $BC = 15$ cm and $\angle ABC = 90°$. Find the length of $AC$.

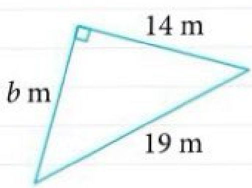
**3.** A triangle $DEF$ is right-angled at $E$, with $DE = 6.7$ m and $EF = 5.5$ m. Find the length of $DF$.

**4.** Find the value of the unknown in each of the following right-angled triangles.

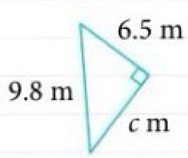
**(a)**



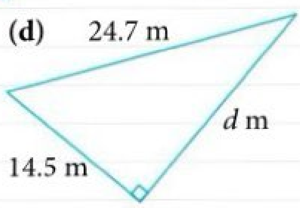

**(b)**




**(c)**




**(d)**

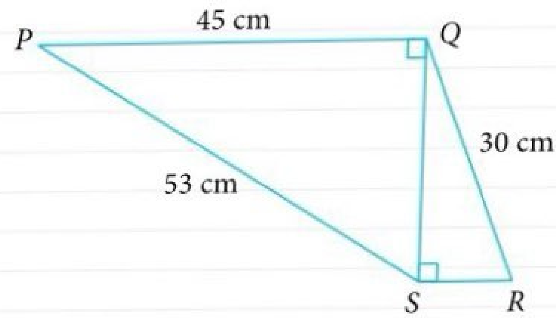



**5.** In triangle $GHI$, $GH = 33$ cm, $GI = 65$ cm and angle $GHI = 90°$. Find the length of $HI$.

**6.** A triangle $MNO$ is right-angled at $N$, with $NO = 11$ m and $MO = 14.2$ m. Find the length of $MN$.

**7.** In the figure, $\angle PQS = \angle QSR = 90°$. Given that $PQ = 45$ cm, $QR = 30$ cm and $PS = 53$ cm, find the length of

**(i)** $QS$, **(ii)** $SR$.

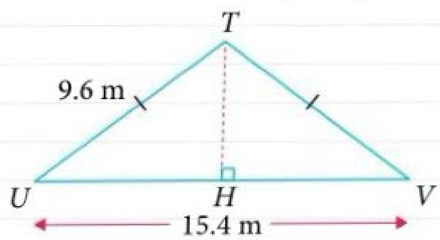



**8.** OPEN The length of the hypotenuse of a right-angled triangle is $\sqrt{34}$ cm. Find a possible set of lengths for the other two sides of the triangle.

**9.** The figure shows an isosceles triangle $TUV$ where $TU = TV = 9.6$ m and $UV = 15.4$ m.

**(a)** Find the height, $TH$, of the triangle.

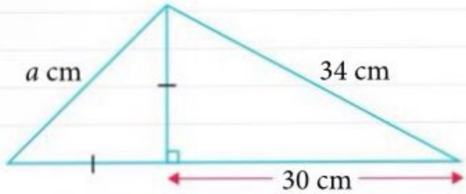
**(b)** Point $P$ lies on $TH$ such that $UP = 8$ m. Does $P$ lie closer to $H$ or $T$? Explain your answer.



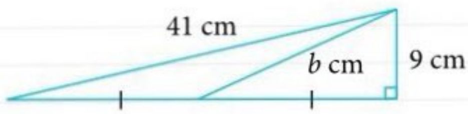
## Exercise 9A

**10.** Find the value of the unknown in each of the following figures.

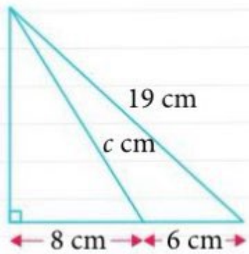
**(a)**



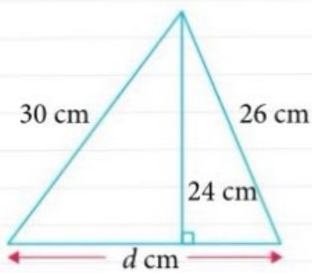

**(b)**



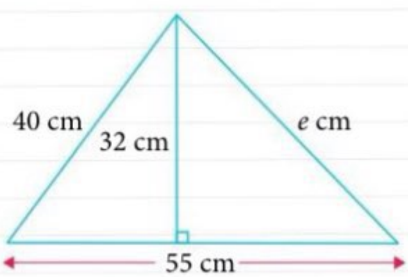

**(c)**




**(d)**

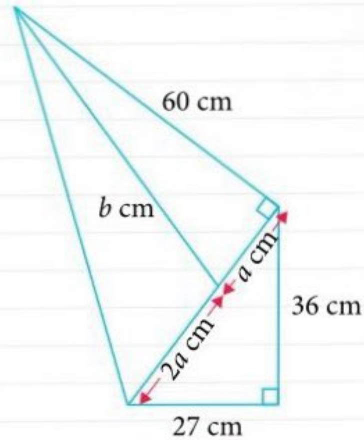



**(e)**

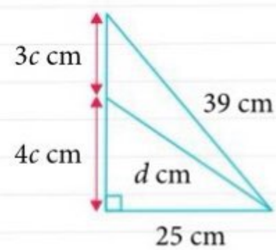



**11.** Find the value(s) of the unknown(s) in each of the following figures.

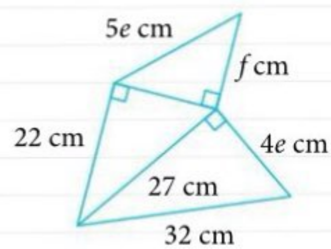
**(a)**



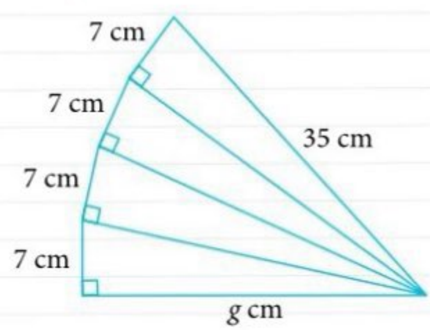

**(b)**




**(c)**

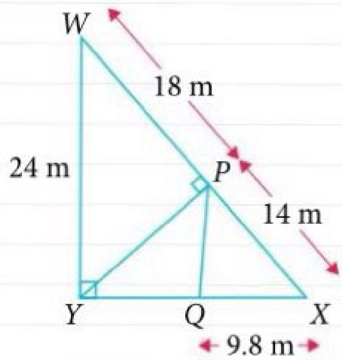



**(d)**

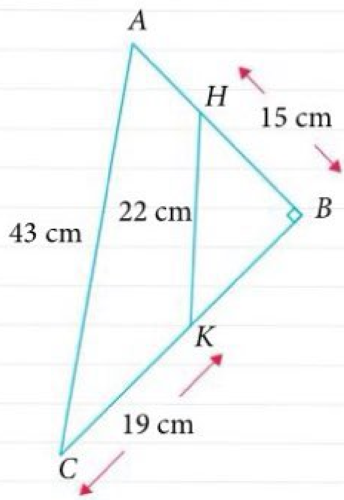


## Exercise 9A

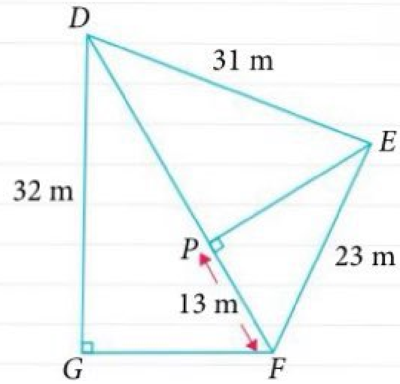
12. In $\triangle WXY$, $WY = 24$ m and $\angle WYX = 90°$. $P$ lies on $WX$ such that $YP$ is perpendicular to $WX$, $WP = 18$ m and $PX = 14$ m. $Q$ lies on $YX$ such that $QX = 9.8$ m. Find
    (i) the length of $YQ$, (ii) the area of $\triangle XPY$.




13. In triangle $ABC$, $AC = 43$ cm and angle $ABC = 90°$. $H$ lies on $AB$ such that $HB = 15$ cm and $K$ lies on $CB$ such that $CK = 19$ cm. Given that $HK = 22$ cm, find the length of $AH$.




14. In the figure, $\angle DGF = \angle EPF = 90°$ and $DPF$ is a straight line. Given that $DE = 31$ m, $EF = 23$ m, $DG = 32$ m and $PF = 13$ m, find the area of the figure.



# 9.2 Applications of Pythagoras' Theorem in real-world contexts

Geometrical shapes such as right-angled triangles are often used to **model** situations in real life. Pythagoras' Theorem is useful in fields such as civil engineering, architecture and navigation. In this section, we will learn how Pythagoras' Theorem can be applied in real-world contexts.

Big Idea

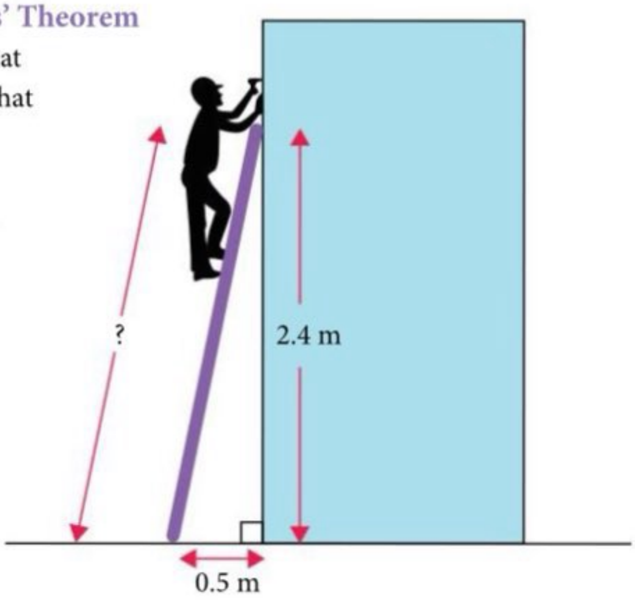
**Models**

Using geometrical diagrams to model real-world objects and phenomena can help us apply mathematical concepts and skills to solve related problems. For example, in this chapter, we can model situations involving right-angled triangles and apply the Pythagoras' Theorem to solve for the unknown sides. However, models are simplifications and idealisations of the real objects or phenomena and thus they have their limitations.

Worked Example 4

### Real-life application of Pythagoras' Theorem

A construction worker is on a ladder that is placed against a vertical wall. Given that the top of the ladder is 2.4 m above the ground and the foot of the ladder is placed 0.5 m from the wall for stability, calculate the length of the ladder.




**Solution**

Let the length of the ladder be $x$ m.
Using Pythagoras' Theorem,

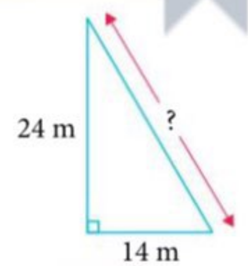
$x^2 = 2.4^2 + 0.5^2$    $x$ is the hypotenuse
$\quad = 5.76 + 0.25$
$\quad = 6.01$
$x = \sqrt{6.01}$ (since $x > 0$)
$\quad = 2.45$ (to 3 s.f.)

$\therefore$ the ladder is 2.45 m long.

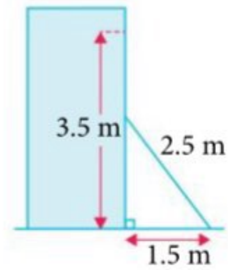
**Practise Now 4**

Similar and Further Questions
**Exercise 9B**
Questions 1–5, 8

1. A vertical pole of height 24 m is supported by a taut steel cable attached from the top of the pole to a point on level ground, 14 m from the foot of the pole. Find the length of the cable.



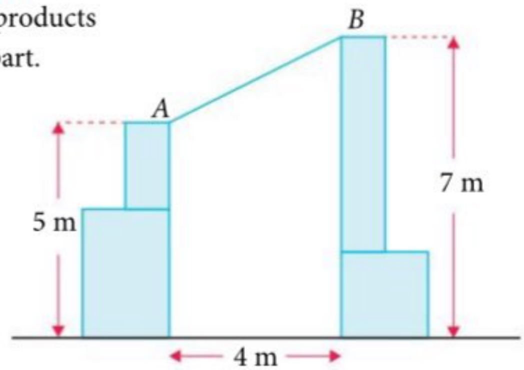

2. A ladder of length 2.5 m is placed against a vertical wall with its foot 1.5 m away from the base of the wall. Sara wants to hang a frame on the wall, 3.5 m above the ground. If Sara's height is 1.6 m, explain if she will be able to do so.




**Worked Example 5**

### Real-life application of Pythagoras' Theorem

In a factory, a sliding belt, $AB$, used to transport products is stretched between two vertical columns 4 m apart. The heights of the columns are 5 m and 7 m. Calculate the length of the belt.




**Solution**

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

**Stage 1: Understand the problem**

*What information is given?*
*What are we supposed to find?*

**Stage 2: Think of a plan**

*What have we learnt in this chapter that can help us solve this problem?*
*Pythagoras' Theorem can only be applied to a right-angled triangle. But there is no right-angled triangle in the diagram.*
*All we can see is a trapezium. Can we dissect the trapezium to obtain a right-angled triangle?*

**Stage 3: Carry out the plan**

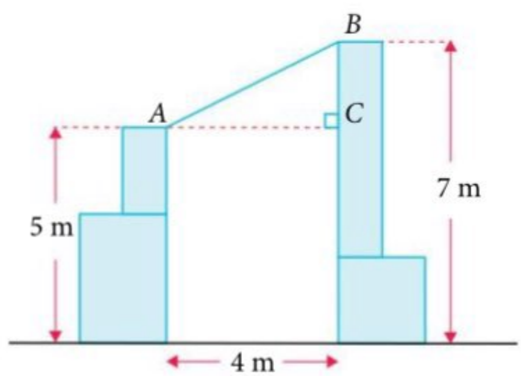
In $\triangle ABC$, $\angle C = 90°$,

$BC = 7 - 5$
$= 2$ m

Using Pythagoras' Theorem,

$AB^2 = BC^2 + AC^2$     $AB$ is the hypotenuse
$= 2^2 + 4^2$
$= 4 + 16$
$= 20$
$AB = \sqrt{20}$ (since $AB > 0$)
$= 4.47$ m (to 3 s.f.)

$\therefore$ the length of the belt is 4.47 m.



**Stage 4: Look back**

*What did we do to solve this problem?*

*If we see a trapezium in another question, what can we do to find out whether we can apply Pythagoras' Theorem to solve the problem?*

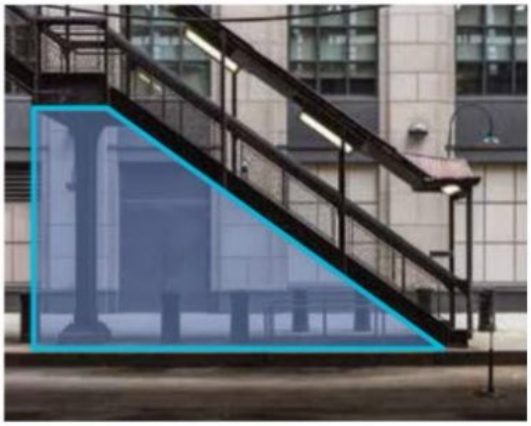
**Practise Now 5**

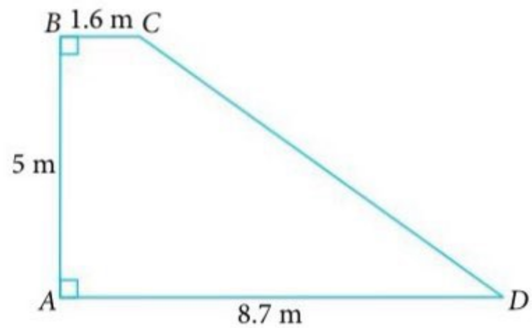
Similar and Further Questions

**Exercise 9B**

Questions 6, 9, 10, 13–15

The picture shows a trapezium-shaped section of an overhead bridge. Given that the height o the vertical scaffold $AB$ is 5 m and the lengths of $BC$ and $AD$ are 1.6 m and 8.7 m respectively find the length of $CD$.






## Class Discussion

**Modelling real-world phenomena**

Ken is standing 10 m away from a tree. The distance of his eyes from his feet is 1.8 m and the distance from his eyes to the top of the tree is 14 m.

**(a)** Use a geometrical shape to model this problem and find the height of the tree. What geometrical shape did you use? Explain your choice.

**(b)** In most cases, **models** are approximations or simplifications of the real-world objects or phenomena. Therefore, models come with assumptions and have limitations. What are some assumptions of your model?

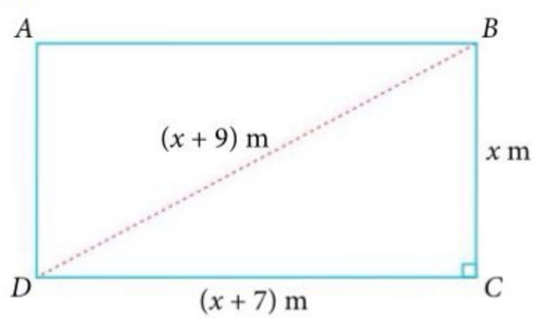
**Worked Example 6**

### Real-life application of Pythagoras' Theorem

An indoor court in the shape of a rectangle $ABCD$ has sides $(x + 7)$ m and $x$ m, and the length of the diagonal $BD$ is $(x + 9)$ m.

**(i)** Calculate the width of the court.

**(ii)** If Waseem runs at an average speed of 4 m/s around the court, how much time will he take to complete one round?



**Solution**

**(i)** In $\triangle BCD$, $\angle C = 90°$.

Using Pythagoras' Theorem,

$$BD^2 = CD^2 + BC^2 \qquad \text{BD is the hypotenuse}$$
$$(x + 9)^2 = (x + 7)^2 + x^2$$
$$x^2 + 18x + 81 = x^2 + 14x + 49 + x^2$$
$$x^2 - 4x - 32 = 0$$
$$(x + 4)(x - 8) = 0$$
$$x = -4 \text{ (N.A. since } x > 0) \text{ or } x = 8$$

$\therefore$ the width of the court is 8 m.

**Big Idea**

**Equivalence**

The algebraic equations in each line are equivalent because they have the same solution set of $x = -4$ or $x = 8$.

**(ii)** Length of the court $= 8 + 7$

$= 15$ m

Perimeter of the court $= 2(15 + 8)$

$= 46$ m

Time taken for Waseem to run one round $= \dfrac{46}{4}$

$= 11.5$ s

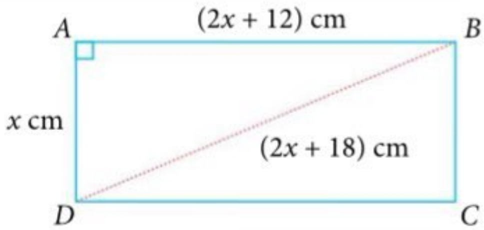
**Practise Now 6**

Similar and Further Questions

**Exercise 9B**

Questions 7, 11, 12

A dining table in the shape of a rectangle $ABCD$ has sides of length $(2x + 12)$ cm and $x$ cm, and the length of the diagonal $BD$ is $(2x + 18)$ cm. Determine if a tablecloth with an area of 2000 cm² can cover the entire surface of the table.




**Worked Example 7**

## Real-life application of Pythagoras' Theorem

Boat $A$ travels due East from Port $P$ for 3 hours to reach Port $Q$ and then travels due South for another 2 hours to reach Port $R$. Boat $B$ travels due North from Port $P$ for 2 hours to reach Port $S$ and then travels due East for 3 hours to reach Port $T$. If the average speeds of Boat $A$ and Boat $B$ are 12 km/h and 18 km/h respectively, calculate the shortest distance between

**(i)** Port $Q$ and Port $S$, **(ii)** Port $R$ and Port $T$.

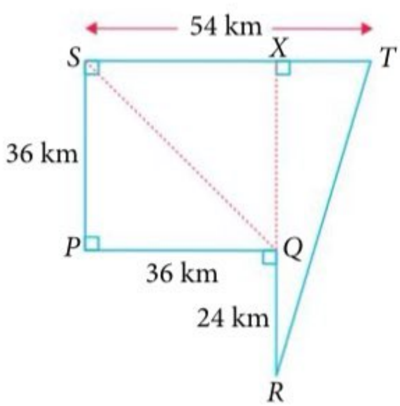
**Solution**

**(i)** $PQ = 12 \times 3 = 36$ km

$QR = 12 \times 2 = 24$ km

$PS = 18 \times 2 = 36$ km

$ST = 18 \times 3 = 54$ km




**Problem-solving Tip**

Sketch a figure to better visualise the problem.

In $\triangle PQS$,

$\angle P = 90°$.

Using Pythagoras' Theorem,

$QS^2 = PQ^2 + PS^2$ *QS is the hypotenuse*

$= 36^2 + 36^2$

$= 1296 + 1296$

$= 2592$

$QS = \sqrt{2592}$ (since $QS > 0$)

$= 50.9$ km (to 3 s.f.)

$\therefore$ the shortest distance between Port $Q$ and Port $S$ is 50.9 km.

**(ii)** Draw a perpendicular line from $Q$ to $ST$ cutting $ST$ at $X$.

In $\triangle RTX$, $\angle X = 90°$.

$TX = 54 - 36$

$= 18$ km

$RX = 24 + 36$

$= 60$ km

Using Pythagoras' Theorem,

$RT^2 = TX^2 + RX^2$ *RT is the hypotenuse*

$= 18^2 + 60^2$

$= 324 + 3600$

$= 3924$

$RT = \sqrt{3924}$ (since $RT > 0$)

$= 62.6$ km (to 3 s.f.)

$\therefore$ the shortest distance between Port $R$ and Port $T$ is 62.6 km.

**Practise Now 7**

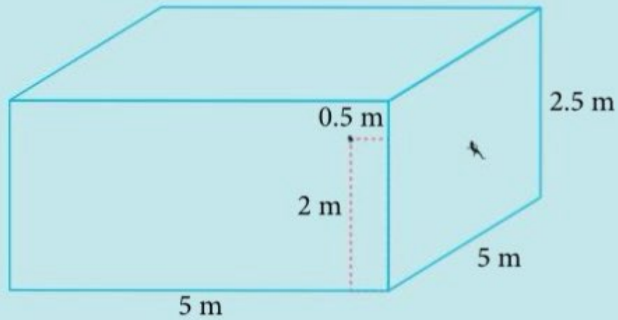
Similar and Further Questions
**Exercise 9B**
Question 16

A ship travels due South from Port $A$ for 1.2 hours to reach Port $B$ and then travels due East for another 1.7 hours to reach Jetty $C$. It then travels due South to Buoy $D$ which is 18 km away from Jetty $C$. From Buoy $D$, it travels 38 km due West to reach Island $E$. If the average speed of the ship is 10 km/h, find the shortest distance between

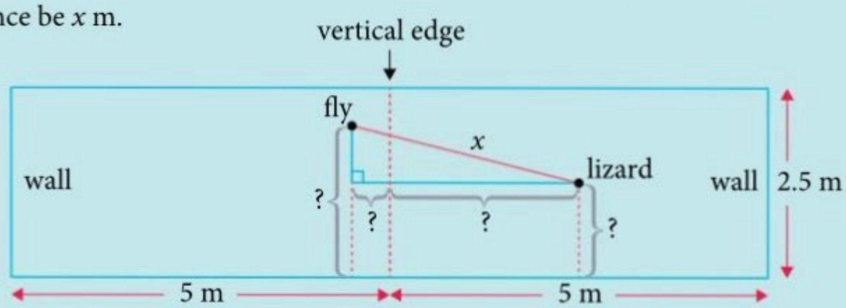
**(i)** Port $A$ and Jetty $C$, **(ii)** Port $A$ and Island $E$.

## Introductory Problem Revisited

How can we use Pythagoras' Theorem to solve the **Introductory Problem**? We can use a cuboid as a model for the living room.




For the shortest distance, the lizard would have to follow the path of a straight line to reach the fly. Let the shortest distance be $x$ m.




Using Pythagoras' Theorem, how can we find $x$?

## Reflection

1. What should I look out for before applying Pythagoras' Theorem?
2. What were some errors made when I attempted the questions in this section? How can I minimise these errors?

## Exercise 9B

**1.** Some cables are supported by a vertical post of height 47 m. The cables are attached from the top of the post to a point on level ground, 18 m from the foot of the post. Find the length of each cable.

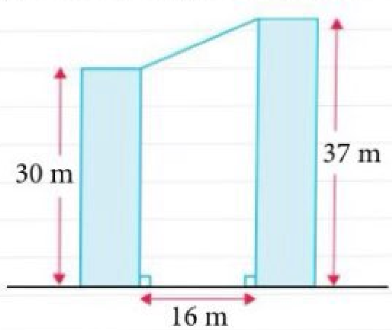
**2.** Each side of a square field is 50-m long. A barricade is to be placed along the diagonal of the field. Find the length of the barricade.

**3.** A rectangular swimming pool has a length of 50 m and a breadth of 30 m. During a swimming proficiency test, Bernard has to swim from one corner of the pool to the opposite corner. Find the distance Bernard has to swim.

**4.** A ladder of length 5 m is placed against a vertical wall with its foot 1.8 m away from the base of the wall. How far up the wall does the ladder reach?

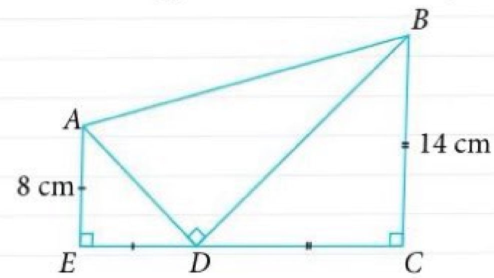
**5.** The height of a 30-inch television screen is 18 inches. Given that television screens are measured across the diagonal, i.e. the length of the diagonal is 30 inches, how wide is the screen?

**6.** Two vertical buildings of heights 30 m and 37 m are 16 m apart. A cable is pulled taut and attached from the top of one building to the top of the other building. Find the length of the cable.




**7.** A folded napkin has a triangular cross section of sides $x$ cm, $(x + 1)$ cm and $(x + 2)$ cm. If one of the angles of the triangle is 90°, find the value of $x$.

**8.** David folded a sheet of paper into an envelope, as shown in the figure. In the figure, $\angle AED = \angle ADB = \angle BCD = 90°$. To seal the envelope, he has to apply glue along $AD$ and $DB$. Given that $AE = ED = 8$ cm and $DC = BC = 14$ cm, find the total length of the sides along which the glue has to be applied to seal the envelope.

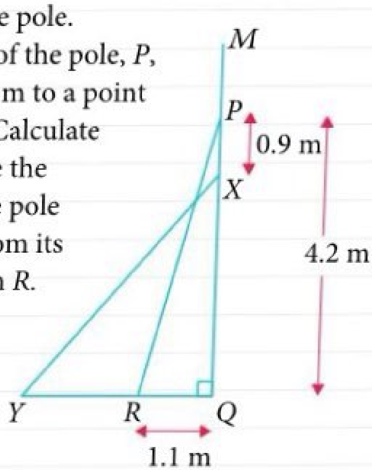



**9.** A table coaster in the shape of a rhombus has diagonals of lengths 10 cm and 24 cm. Find the length of each side of the coaster.

**10.** A straight pole $PR$ is leaning against a vertical wall $MQ$.

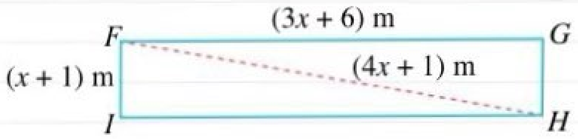
**(i)** Given that $PQ = 4.2$ m and $RQ = 1.1$ m, find the length of the pole.

**(ii)** The upper end of the pole, $P$, slides down 0.9 m to a point on the wall, $X$. Calculate $YR$, the distance the lower end of the pole has slid away from its original position $R$.



## Exercise 9B

**11.** A campsite in the shape of a rectangle $FGHI$ has sides $(3x + 6)$ m and $(x + 1)$ m. The length of the diagonal $FH$ is $(4x + 1)$ m. Circular stools with a diameter of 40 cm are to be placed along the entire length of $FG$. Find the number of stools required.




**12.** A garden in the shape of a rectangle has sides $x$ m and 12 m, and a diagonal of length $5\sqrt{x}$ m.

(i) Find a possible area of the garden.

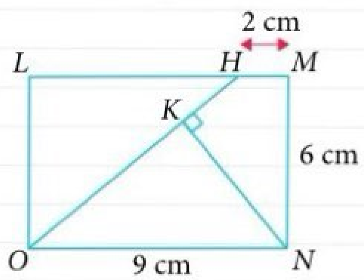
A landscaper charges \$110 per m² for landscaping services.

(ii) Using the area found in part (i), find the cost of landscaping half of the garden.

**13.** The top surface of a pouch is in the shape of a rectangle $LMNO$ with sides 9 cm and 6 cm.

(i) A zip is to be sewn along $OH$ such that $H$ is a point on $LM$ and $HM = 2$ cm. Find the length of the zip.

(ii) A second zip is to be sewn along $NK$ such that $NK$ is the perpendicular from $N$ to $OH$. Calculate the length of the second zip.

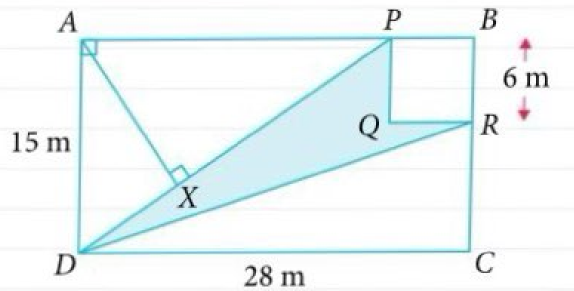



**14.** In the figure, $ABCD$ is a rectangle of sides 28 m and 15 m. $PBRQ$ is a square of sides 6 m.

(i) Find the area of the shaded region $DPQR$.

(ii) Calculate the length of $DP$.

(iii) Given that $X$ is a point on $DP$ such that $AX$ is perpendicular to $DP$, find the length of $AX$.




**15.** A designer is tasked to design two tables for children such that the tabletops are of different shapes as shown, and that the perimeter of each tabletop is 132 cm.


Square tabletop


Round tabletop

(a) Find

(i) the length of each side of the table with the square tabletop,

(ii) the radius of the table with the round tabletop.

(b) Hence, calculate the area of each tabletop. (Take $\pi$ to be $\frac{22}{7}$.)

The designer decides to design another table with a tabletop in the shape of an equilateral triangle as shown, such that the perimeter of the tabletop is also 132 cm.

Exercise 9B

(c) Find
  (i) the length of each side of the table,
  (ii) the area of the tabletop.

(d) Which shape of the tabletop should the designer choose if he wants it to have the most tabletop space? Justify your answer.

16. A courier travels due North at an average speed of 40 km/h for 6 minutes to collect a parcel, before travelling 10 km due East to deliver it. He then travels due South at an average speed of 30 km/h for 12 minutes to collect another parcel. Find the shortest distance between the courier and his starting point.

## 9.3 Converse of Pythagoras' Theorem

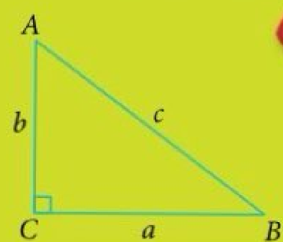
Pythagoras' Theorem states that if an angle in a triangle is a right angle, then the square of the length of the side opposite the right angle, which is the hypotenuse, is equal to the sum of the squares of the lengths of the other two sides.

The **converse of Pythagoras' Theorem** is also true. That is, if the square of the length of the *longest side* is equal to the sum of the squares of the lengths of the other two sides, then the angle opposite the *longest side* is a right angle.

In a triangle $ABC$, if $AB^2 = BC^2 + AC^2$,

i.e. $c^2 = a^2 + b^2$,

then the triangle is a right-angled triangle, where angle $C = 90°$.




**Big Idea**

**Equivalence**

When we consider Pythagoras' Theorem and its converse, the following two mathematical statements are equivalent.

Statement 1: A triangle is a right-angled triangle.

Statement 2: The square of the longest side of the triangle is equal to the sum of the squares of the two shorter sides.

When two mathematical statements are equivalent, it means that they imply each other. That is, if Statement 1 is true, then Statement 2 must also be true and vice versa.

Do you recall equivalent statements in other chapters?

**Worked Example 8**

**Determining if triangles are right-angled triangles given the lengths of three sides**

Determine if each of the following triangles is a right-angled triangle. For each right-angled triangle, state the right angle.

(a) $\triangle ABC$, given that $AB = 39$ cm, $BC = 15$ cm and $AC = 36$ cm

(b) $\triangle PQR$, given that $PQ = 28$ m, $QR = 20$ m and $PR = 19$ m

### Solution

**(a)** $AB$ is the longest side of $\triangle ABC$.

$$AB^2 = 39^2$$
$$= 1521$$
$$BC^2 + AC^2 = 15^2 + 36^2$$
$$= 225 + 1296$$
$$= 1521$$

Since $AB^2 = BC^2 + AC^2$, then by the converse of Pythagoras' Theorem, $\triangle ABC$ is a right-angled triangle where $\angle C = 90°$.

**(b)** $PQ$ is the longest side of $\triangle PQR$.

$$PQ^2 = 28^2$$
$$= 784$$
$$QR^2 + PR^2 = 20^2 + 19^2$$
$$= 400 + 361$$
$$= 761$$

Since $PQ^2 \neq QR^2 + PR^2$, $\triangle PQR$ is not a right-angled triangle.

**Problem-solving Tip**

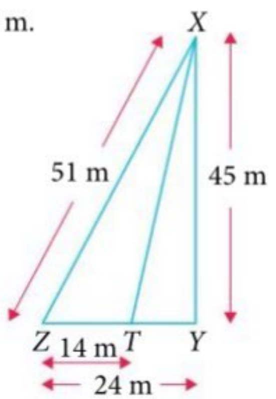
To "determine" means you need to show appropriate mathematical working to explain if a triangle is right-angled or not.

**Practise Now 8**

Similar and Further Questions
**Exercise 9C**
Questions 1(a)–(d), 2–5

1. Determine if each of the following triangles is a right-angled triangle. For each right-angled triangle, state the right angle.
   (a) $\triangle ABC$, given that $AB = 12$ cm, $BC = 10$ cm and $AC = 8$ cm
   (b) $\triangle PQR$, given that $PQ = 34$ m, $QR = 16$ m and $PR = 30$ m

2. $XYZ$ is a plot of land such that $XY = 45$ m, $YZ = 24$ m and $XZ = 51$ m.
   (i) Show that angle $XYZ = 90°$.
   (ii) A tree $T$ is located on $ZY$ such that $ZT = 14$ m. Find $TX$, the distance of the tree from $X$.




## Reflection

1. How can I determine if a triangle has a right angle if the angles are not given?
2. What have I learnt in this section or chapter that I am still unclear of?

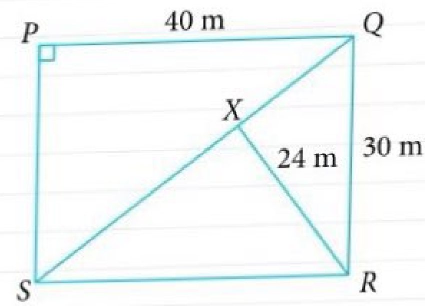
## Exercise 9C

1. Determine if each of the following triangles is a right-angled triangle. For each right-angled triangle, state the right angle.
   **(a)** $\triangle ABC$, given that $AB = 16$ cm, $BC = 63$ cm and $AC = 65$ cm
   **(b)** $\triangle DEF$, given that $DE = 24$ cm, $EF = 27$ cm and $DF = 21$ cm
   **(c)** $\triangle GHI$, given that $GH = 7.8$ m, $HI = 7.1$ m and $GI = 2.4$ m
   **(d)** $\triangle MNO$, given that $MN = \frac{5}{13}$ m, $NO = \frac{3}{13}$ m and $MO = \frac{4}{13}$ m

2. In triangle $PQR$, $PQ = 19$ cm, $QR = 24$ cm and $PR = 30$ cm. Show that triangle $PQR$ is not a right-angled triangle.

3. In triangle $STU$, $ST = \frac{7}{12}$ cm, $TU = \frac{5}{6}$ cm and $SU = \frac{1}{3}$ cm. Is triangle $STU$ a right-angled triangle? Explain your answer.

4. A rectangular grass patch, $PQRS$, has sides 40 m and 30 m. A straight path which cuts through the grass patch joins $S$ and $Q$. Lamppost $X$ is located on $SQ$ such that $SX : XQ = 16 : 9$ and $RX = 24$ m. Imran walks along the path and stops at a point which is nearest to $R$. Show that he stops at $X$.




5. The lengths of the sides $a$, $b$ and $c$, of a triangle are given by $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$, where $m$ and $n$ are positive integers and $m > n$. Show that the triangle is a right-angled triangle.

## Looking Back

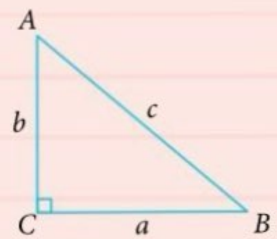
Geometrical shapes such as trapeziums and right-angled triangles can be used as **models** to represent real-world objects and phenomena. Coupled with Pythagoras' Theorem, knowledge of the properties of these shapes helps us to solve real-world problems in civil engineering, architecture and navigation. However, in most cases, models are approximations or simplification of the real-world objects and phenomena that come with assumptions and have limitations. For example, in the Class Discussion on page 285, some assumptions made would be: the tree is vertical and the ground is level.

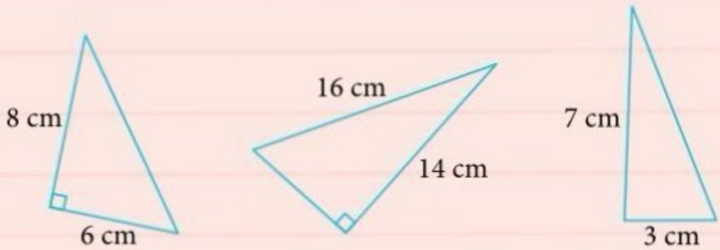
We have also seen an instance of **equivalence** in this chapter. Using Pythagoras' Theorem and its converse, we see that the statement 'Triangle is a right-angled triangle' is equivalent to the statement 'The square of the longest side of the triangle is equal to the sum of the squares of the two shorter sides'.

## Summary

1. In a right-angled triangle $ABC$ where angle $C$ is the right angle, the side opposite angle $C$ is the longest side and is called the **hypotenuse**.
   - Why is the side opposite angle $C$ the longest side?
   - Do you call the longest side of an oblique triangle 'hypotenuse' as well?

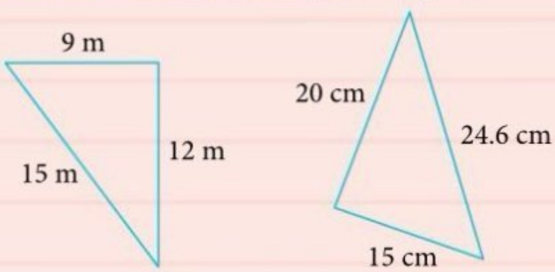
2. **Pythagoras' Theorem**
   In a right-angled triangle $ABC$, $AB^2 = BC^2 + AC^2$,
   where $AB$ is the length of the hypotenuse of the triangle; i.e. $c^2 = a^2 + b^2$.
   - Find the unknown length of each of the following triangles *if possible*.




(a) (b) (c)

3. **Converse of Pythagoras' Theorem**
   In a triangle $ABC$, if $AB^2 = BC^2 + AC^2$, i.e. $c^2 = a^2 + b^2$, then the triangle is a right-angled triangle, where angle $C = 90°$.
   - Are the following right-angled triangles?




(a) (b)

CHAPTER 10

# Trigonometric Ratios

Trigonometry, derived from the Greek words for triangle (*trigōnon*) and measure (*metron*), is a branch of mathematics that studies relationships between the sides and angles of triangles. Trigonometry enables us to measure unknown heights and distances indirectly. As we have seen in Chapter 9, triangles are often used to model real-world problems and have been studied throughout history, by ancient Egyptians for instance.

Today, trigonometry is still widely applied in various fields. For example, surveyors use an instrument called a theodolite to measure angles in horizontal planes and vertical planes during the construction of buildings and infrastructure, land surveying, and rocket launching.

In this chapter, we will begin our study of trigonometry by exploring the idea of trigonometric ratios and how it can be applied to solve real-world problems.

## Learning Outcomes

What will we learn in this chapter?

- What trigonometric ratios of acute angles are
- How to find the unknown sides and angles in right-angled triangles
- Why trigonometry has useful applications in real life

Introductory Problem


With a height of 8611 metres above sea level, K2 is the highest mountain in Pakistan. In contrast, the highest hill in Singapore is the Bukit Timah Hill at 163.63 metres. How do you think the heights of tall objects like the Bukit Timah Hill are determined? What were the assumptions made in the calculation?

In the previous chapter, we have learnt about the relationship between the sides of a right-angled triangle. In this chapter, we will focus on the relationships between the sides and angles of a right-angled triangle.

# 10.1 Trigonometric ratios

## A. Introduction to trigonometric ratios

### Investigation Trigonometric ratios

Go to www.sl-education.com/tmsoupp2/pg296 or scan the QR code on the right and open the geometry software template 'Trigonometric ratios'.

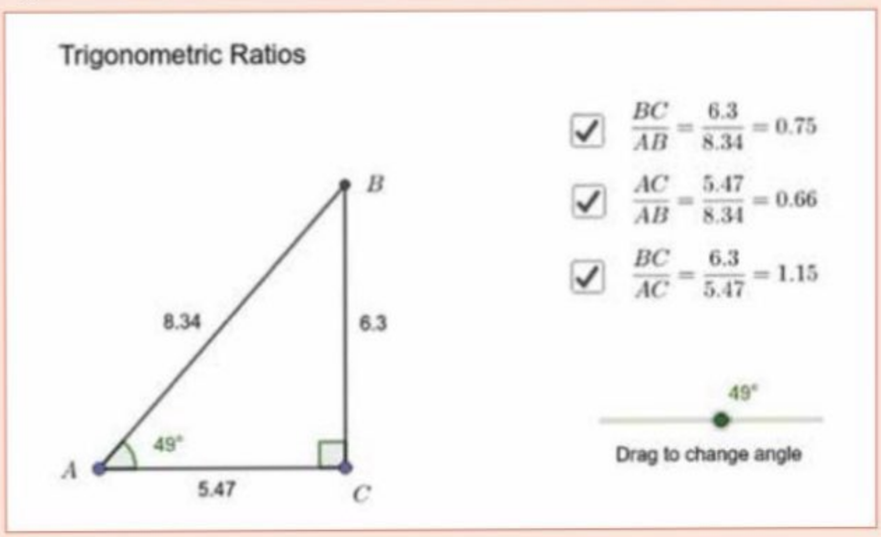



Fig. 10.1

1. The screen shows a right-angled triangle $ABC$. Adjust angle $A$ to 30°.
2. Drag point $C$ to vary the lengths of the sides of the triangle with angle $A$ fixed at 30° and angle $C$ at 90°. What do you call the family of right-angled triangles obtained as you change the lengths of their sides?
3. Copy and complete Table 10.1.
   **(a)** Record the lengths of $AB$, $BC$ and $AC$.
   **(b)** Find the ratios $\frac{BC}{AB}$, $\frac{AC}{AB}$ and $\frac{BC}{AC}$, giving each of your answers correct to 2 significant figures.

The first row has been completed as an example for you.

| Triangle | Angle $A$ | Length of $AB$ | Length of $BC$ | Length of $AC$ | $\frac{BC}{AB}$ | $\frac{AC}{AB}$ | $\frac{BC}{AC}$ |
|---|---|---|---|---|---|---|---|
| 1 | 30° | 4.33 | 2.16 | 3.75 | 0.50 | 0.87 | 0.58 |
| 2 | 30° | | | | | | |
| 3 | 30° | | | | | | |

Table 10.1

4. (a) Look at the three ratios in Table 10.1. What do you observe about the values?

   (b) What idea can you use to explain that the ratio $\frac{BC}{AB}$ will always be equal for any right-angled triangle $ABC$ with angle $A$ fixed at 30°?

   Hint: Refer to Question 2.

5. Change the size of angle $A$.

   Drag point $C$ to vary the lengths of the sides of the triangle. Copy and complete Table 10.2.

   (a) Record the lengths of $AB$, $BC$ and $AC$.

   (b) Find the ratios $\frac{BC}{AB}$, $\frac{AC}{AB}$ and $\frac{BC}{AC}$, giving each of your answers correct to 2 significant figures.

| Angle $A$ | Length of $AB$ | Length of $BC$ | Length of $AC$ | $\frac{BC}{AB}$ | $\frac{AC}{AB}$ | $\frac{BC}{AC}$ |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | | | | |
| | | | | | | |

Table 10.2

6. Fig. 10.2 shows a right-angled triangle $ABC$ where $\angle C = 90°$.

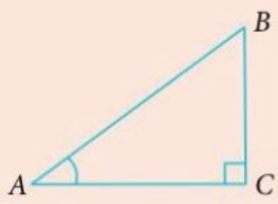



Fig. 10.2

   (a) We have learnt in Chapter 9 that the longest side of a right-angled triangle is called the **hypotenuse**. Label the hypotenuse in Fig. 10.2.

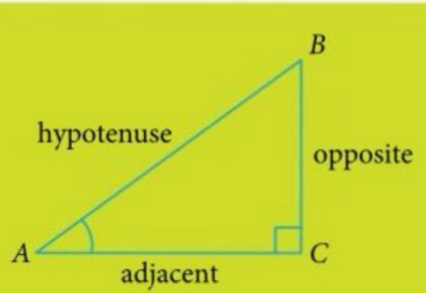
   (b) With reference to angle $A$ in Fig. 10.2, $BC$, the side opposite angle $A$ is also called the **opposite** side. Label the opposite side in Fig. 10.2.

   (c) With reference to angle $A$ in Fig. 10.2, there are two sides adjacent to angle $A$. One of them is the hypotenuse. The other side is called the **adjacent** side. Label the adjacent side in Fig. 10.2.

7. In terms of the hypotenuse (hyp), the opposite side (opp) and the adjacent side (adj), the ratio $\frac{BC}{AB}$ can be rewritten as $\frac{\text{opp}}{\text{hyp}}$. Write down an expression for $\frac{AC}{AB}$ and for $\frac{BC}{AC}$ respectively.

8. With reference to angle $B$ in Fig. 10.2, which side is the opposite side and which side is the adjacent side?

From the Investigation on pages 296 and 297, we observe that for a family of *similar* right-angled triangles where angle $A$ is fixed, the three ratios in Tables 10.1 and 10.2 are always *equal*.




In a right-angled triangle $ABC$, if $\angle C = 90°$,

then $\dfrac{BC}{AB} = \dfrac{\text{opp}}{\text{hyp}}$ is called the **sine** of $\angle A$, or $\sin A = \dfrac{\text{opp}}{\text{hyp}}$,

$\dfrac{AC}{AB} = \dfrac{\text{adj}}{\text{hyp}}$ is called the **cosine** of $\angle A$, or $\cos A = \dfrac{\text{adj}}{\text{hyp}}$,

$\dfrac{BC}{AC} = \dfrac{\text{opp}}{\text{adj}}$ is called the **tangent** of $\angle A$, or $\tan A = \dfrac{\text{opp}}{\text{adj}}$.

These three ratios of one length to another are known as **trigonometric ratios**. Since they are ratios, they are numbers *without any units*.

Additionally, the definitions of trigonometric ratios *only apply to acute angles in a right-angled triangle*, where $0° < A < 90°$.

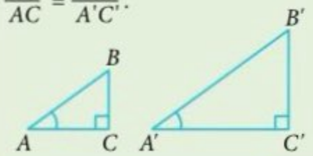
**Big Idea**

**Proportionality**

The corresponding sides of similar triangles are proportional.

E.g. if $\triangle ABC$ is enlarged to $\triangle A'B'C'$, we have:

$\dfrac{BC}{AB} = \dfrac{B'C'}{A'B'}$; $\dfrac{AC}{AB} = \dfrac{A'C'}{A'B'}$; $\dfrac{BC}{AC} = \dfrac{B'C'}{A'C'}$.




**Big Idea**

**Notations**

We use the notations $\sin A$, $\cos A$ and $\tan A$ to represent the sine, cosine and tangent of angle $A$ in a concise and precise manner.

**Problem-solving Tip**

The mnemonic '**TOA CAH SOH**' can be used to remember the three trigonometric ratios, where

- **TOA** represents $\tan = \dfrac{\text{opp}}{\text{adj}}$,
- **CAH** represents $\cos = \dfrac{\text{adj}}{\text{hyp}}$,
- **SOH** represents $\sin = \dfrac{\text{opp}}{\text{hyp}}$.

## Thinking Time

Consider $\triangle XYZ$ where $\angle X = 50°$ and $\angle Y = 90°$. Are the trigonometric ratios of $\angle X$ the same as those of $\angle A$ in the Investigation on pages 296 and 297, i.e. is $\sin 50° = \sin 30°$? Is $\cos 50° = \cos 30°$? Is $\tan 50° = \tan 30°$? In other words, do trigonometric ratios depend on the value of the angle? Explain your answer.

**Practise Now 1A**

Similar and Further Questions

**Exercise 10A**

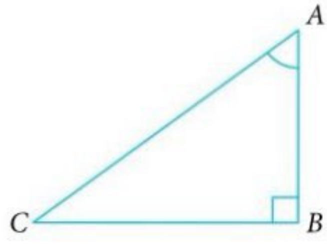
Questions 1(a), (b), 2(a), (b), 3, 5(a), (b)

**1.** In $\triangle ABC$, $\angle B = 90°$. Name

**(i)** the hypotenuse,

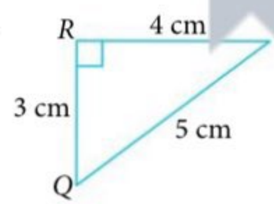
**(ii)** the side opposite $\angle A$,

**(iii)** the side adjacent to $\angle A$.



2. In $\triangle PQR$, $PR = 4$ cm, $RQ = 3$ cm, $PQ = 5$ cm and $\angle R = 90°$.
State the value of

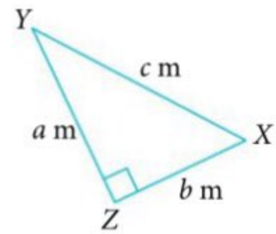
| | |
|---|---|
| **(i)** $\sin P$, | **(ii)** $\cos P$, |
| **(iii)** $\tan P$, | **(iv)** $\sin Q$, |
| **(v)** $\cos Q$, | **(vi)** $\tan Q$. |




3. In $\triangle XYZ$, $XY = c$ m, $YZ = a$ m, $XZ = b$ m and $\angle Z = 90°$.
**(a)** Write down an expression for

| | |
|---|---|
| **(i)** $\sin X$, | **(ii)** $\cos X$, |
| **(iii)** $\tan X$, | **(iv)** $\sin Y$, |
| **(v)** $\cos Y$, | **(vi)** $\tan Y$, |

in terms of $a$, $b$ and/or $c$.




OPEN **(b)** If $\tan X = \tan Y$, write down a possible set of values of $a$, $b$ and $c$.

## Reflection

What do I already know about similar triangles that could guide my understanding of trigonometric ratios?

## B. Finding trigonometric ratios given angles

Fig. 10.3 shows a right-angled triangle where $\angle A = 25°$ and $\angle C = 90°$.

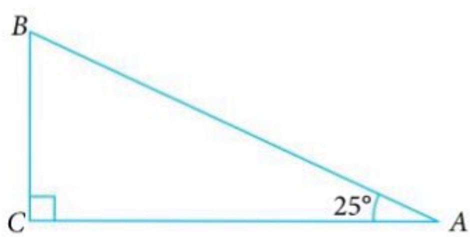



Fig. 10.3

**Internet Resources**

Trigonometric ratios are used to describe many natural phenomena. They are used in the study of acoustics, X-rays, light, etc.
Do you know of other uses of trigonometric ratios? Search the Internet to find out more.

If we measure the length of $AB$ and of $BC$ in Fig. 10.3, and find the ratio $\frac{BC}{AB}$, the value that we obtain is approximately 0.423.

We know that the ratio $\frac{BC}{AB}$ is equal to the trigonometric ratio $\sin A$. Therefore, $\sin 25° = \frac{BC}{AB} = 0.423$.

Let us vary the size of angle $A$. Fig. 10.4 shows a right-angled triangle where $\angle A = 26°$.

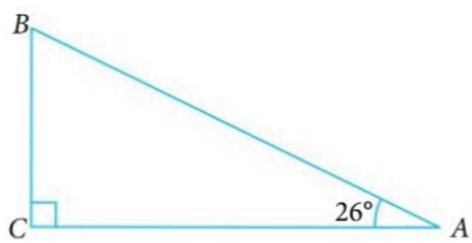



Fig. 10.4

If we measure the length of $AB$ and of $BC$ in Fig. 10.4 to find the ratio $\frac{BC}{AB}$, we obtain a value of approximately 0.438. This means $\sin 26° = 0.438$.

Like above, trigonometric ratios were computed manually and presented in tables in the past. Table 10.3 shows the trigonometric ratios of the angles 25°, 26° and 27°.

| Angle | Sine | Cosine | Tangent |
|---|---|---|---|
| 25° | 0.423 | 0.906 | 0.466 |
| 26° | 0.438 | 0.899 | 0.488 |
| 27° | 0.454 | 0.891 | 0.510 |

Table 10.3

It is tedious to have to draw different right-angled triangles to measure the lengths of the sides and find the trigonometric ratios. This method is also prone to large errors. The use of calculators can help us to compute the trigonometric ratios of any angle.

**Worked Example 1**

### Using calculator to evaluate trigonometric ratios

Use a calculator to evaluate each of the following.

**(a)** $\sin 32°$ **(b)** $\cos 15.3°$ **(c)** $\tan 25.96°$

**(d)** $2 \sin 37° + 5 \tan 56°$ **(e)** $\frac{3}{\cos 48.1°}$ **(f)** $\frac{\cos 57°}{\sin 46.5° + \tan 26.4°}$

**Solution**

**(a)** Sequence of calculator keys:

sin ( 3 2 ) =

$\therefore \sin 32° = 0.530$ (to 3 s.f.)

**(b)** Sequence of calculator keys:

cos ( 1 5 . 3 ) =

$\therefore \cos 15.3° = 0.965$ (to 3 s.f.)

**(c)** Sequence of calculator keys:

tan ( 2 5 . 9 6 ) =

$\therefore \tan 25.96° = 0.487$ (to 3 s.f.)

**(d)** Sequence of calculator keys:

2 sin ( 3 7 ) + 5 tan ( 5 6 ) =

$\therefore 2 \sin 37° + 5 \tan 56° = 8.62$ (to 3 s.f.)

**(e)** Sequence of calculator keys:

3 ÷ cos ( 4 8 . 1 ) =

$\therefore \frac{3}{\cos 48.1°} = 4.49$ (to 3 s.f.)

**Problem-solving Tip**

As we are computing the values of angles in degrees, remember to set the mode of your calculator to 'DEG'.

**(f)** Sequence of calculator keys:

cos ( 5 7 ) ÷ ( sin ( 4 6 . 5 ) +
tan ( 2 6 . 4 ) ) =

$$\therefore \frac{\cos 57°}{\sin 46.5° + \tan 26.4°} = 0.446 \text{ (to 3 s.f.)}$$

**Practise Now 1B**

Similar and Further Questions
**Exercise 10A**
Questions 4(a)–(f), 6(a)–(h)

Use a calculator to evaluate each of the following.

**(a)** cos 24° **(b)** tan 74.6° **(c)** sin 72.15°

**(d)** 3 sin 48° + 2 cos 39° **(e)** $\frac{5}{\tan 18.3°}$ **(f)** $\frac{\tan 48.3° - \sin 28.7°}{\cos 15° + \cos 35°}$

## Thinking Time

Do you agree with the following statements? Give reasons for your answer.

**(a)** The cosine of an acute angle is always less than 1.

**(b)** The sine of an acute angle can never be 0.

**(c)** The tangent of an acute angle is sometimes equal to 1.

Advanced | Intermediate | Basic

## Exercise 10A

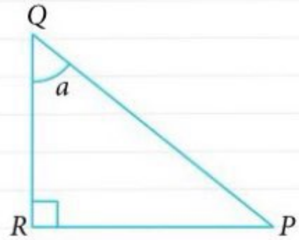
**1.** For each of the following right-angled triangles, name

**(i)** the hypotenuse,

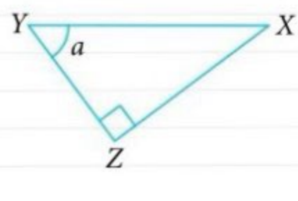
**(ii)** the side opposite ∠$a$,

**(iii)** the side adjacent to ∠$a$.

**(a)**




**(b)**



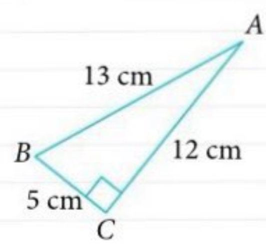

**2.** For each of the following right-angled triangles, state the value of

**(i)** sin $A$, **(ii)** cos $A$,

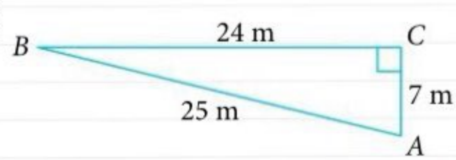
**(iii)** tan $A$, **(iv)** sin $B$,

**(v)** cos $B$, **(vi)** tan $B$.

**(a)**




**(b)**



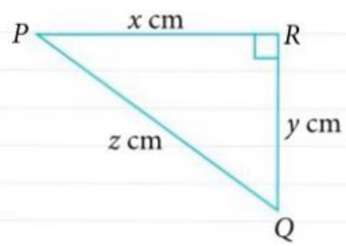
Basic Intermediate Advanced

## Exercise 10A

**3.** In $\triangle PQR$, $PR = x$ cm, $QR = y$ cm, $PQ = z$ cm and $\angle R = 90°$. Write down an expression for

**(i)** $\sin P$, **(ii)** $\cos P$,
**(iii)** $\tan P$, **(iv)** $\sin Q$,
**(v)** $\cos Q$, **(vi)** $\tan Q$,

in terms of $x$, $y$ and/or $z$.




**4.** Use a calculator to evaluate each of the following.

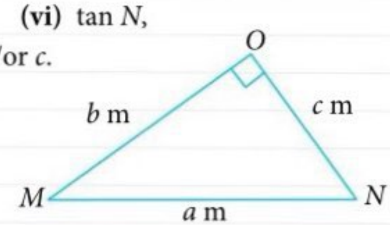
**(a)** $\tan 47°$
**(b)** $\sin 75.3°$
**(c)** $\cos 30.19°$
**(d)** $\sin 35° + \cos 49°$
**(e)** $2 \cos 42.3° + 3 \sin 16.8°$
**(f)** $\sin 71.6° \times \tan 16.7°$

**5.** In $\triangle MNO$, $MN = a$ m, $OM = b$ m, $ON = c$ m and $\angle O = 90°$.

**(a)** Write down an expression for

**(i)** $\sin M$, **(ii)** $\cos M$,
**(iii)** $\tan M$, **(iv)** $\sin N$,
**(v)** $\cos N$, **(vi)** $\tan N$,

in terms of $a$, $b$ and/or $c$.




OPEN **(b)** If $\tan M = \tan N$, write down a possible set of values of $a$, $b$ and $c$.

**6.** Use a calculator to evaluate each of the following.

**(a)** $\dfrac{5\tan 61.4°}{2\cos 10.3°}$ **(b)** $\dfrac{4(\sin 22.5°)^2}{\cos 67.5°}$

**(c)** $\dfrac{\tan 15° + \cos 33°}{\sin 78.4°}$ **(d)** $\dfrac{\tan 47.9°}{\cos 84° - \sin 63°}$

**(e)** $\dfrac{\cos 67° + \sin 89°}{\tan 63.4° \times \cos 15.5°}$ **(f)** $\dfrac{\sin 24.6° \div \cos 62.1°}{\tan 21° + \cos 14°}$

**(g)** $\dfrac{\sin 57° - \cos 73°}{\tan 15.3° \times \sin 83.4°}$ **(h)** $\dfrac{\cos 24.7° \times \sin 35.1°}{\tan 57° - \cos 15°}$

# 10.2 Applications of trigonometric ratios to find unknown sides of right-angled triangles

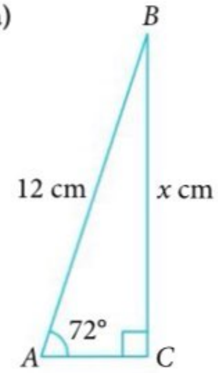
Worked Examples 2 and 3 illustrate how trigonometric ratios are used to find the lengths of the unknown sides of right-angled triangles.

Worked Example 2

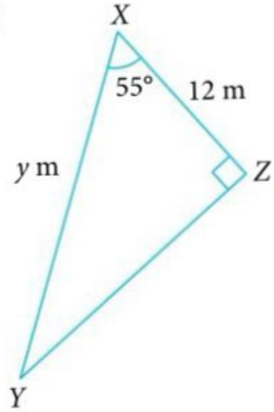
## Using sine, cosine and tangent ratios

Calculate the value of the unknown in each of the following right-angled triangles.

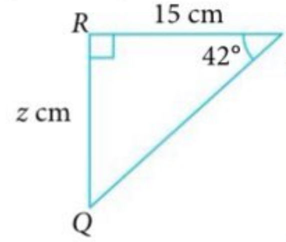
**(a)**




**(b)**




**(c)**




### Solution

**(a)** $\sin\angle BAC = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB}$

$\sin 72° = \frac{x}{12}$

$\therefore x = 12 \sin 72°$

$= 11.4$ (to 3 s.f.)

**(b)** $\cos\angle YXZ = \frac{\text{adj}}{\text{hyp}} = \frac{XZ}{XY}$

$\cos 55° = \frac{12}{y}$

$\therefore y = \frac{12}{\cos 55°}$

$= 20.9$ (to 3 s.f.)

**(c)** $\tan\angle QPR = \frac{\text{opp}}{\text{adj}} = \frac{QR}{PR}$

$\tan 42° = \frac{z}{15}$

$\therefore z = 15 \tan 42°$

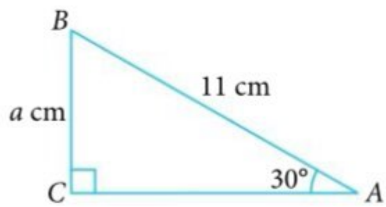
$= 13.5$ (to 3 s.f.)

**Practise Now 2**

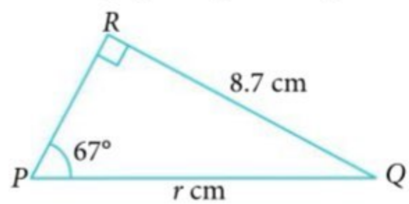
Similar and Further Questions
Exercise 10B
Questions 1(a), (b), 2(a), (b), 3(a), (b)

1. Find the value of the unknown in each of the following right-angled triangles.

**(a)**

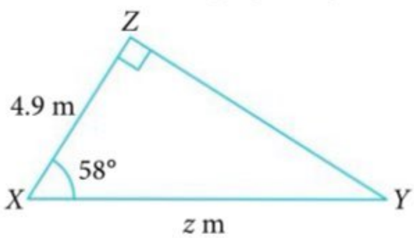



**(b)**

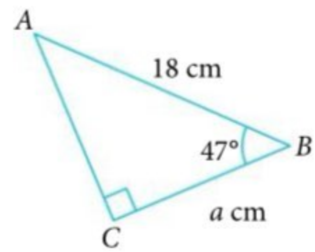



2. In each of the following right-angled triangles, find the value of the unknown.

**(a)**

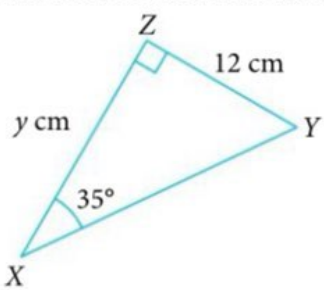



**(b)**

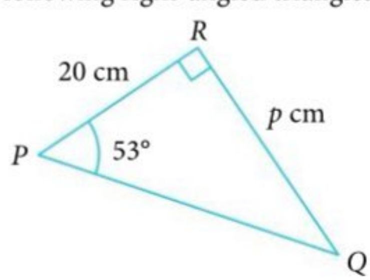



3. Find the value of the unknown in each of the following right-angled triangles.

**(a)**




**(b)**



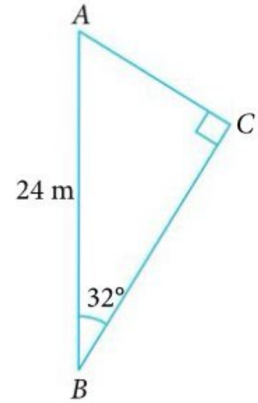
Worked Example 3

## Finding the lengths of unknown sides of a right-angled triangle

In $\triangle ABC$, $\angle C = 90°$. Given that $\angle B = 32°$ and $AB = 24$ m, calculate the length of

**(i)** $AC$,

**(ii)** $BC$.




**Solution**

**(i)** $\sin \angle ABC = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{AC}{AB}$

$$\sin 32° = \frac{AC}{24}$$

$$\therefore AC = 24 \sin 32°$$

$$= 12.7 \text{ m (to 3 s.f.)}$$

**(ii) Method 1:**

$$\cos \angle ABC = \frac{\text{adj}}{\text{hyp}} = \frac{BC}{AB}$$

$$\cos 32° = \frac{BC}{24}$$

$$\therefore BC = 24 \cos 32°$$

$$= 20.4 \text{ m (to 3 s.f.)}$$

**Method 2:**

In $\triangle ABC$, $\angle C = 90°$.

Using Pythagoras' Theorem,

$$AB^2 = AC^2 + BC^2$$

$$24^2 = 12.718^2 + BC^2$$ use $AC = 12.718$ (to 5 s.f.) or $AC = 24 \sin 32°$

$$BC^2 = 24^2 - 12.718^2$$

$$= 414.25 \text{ (to 5 s.f.)}$$

$$\therefore BC = \sqrt{414.25} \quad (\text{since } BC > 0)$$

$$= 20.4 \text{ m (to 3 s.f.)}$$

**Recall**

Intermediate working should be correct to 4 or 5 significant figures for better accuracy.

**Reflection**

Which method do you prefer? Why?

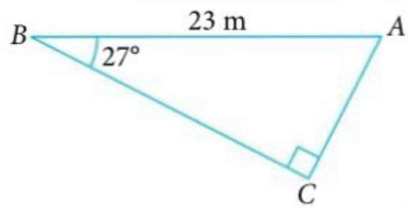
**Practise Now 3**

Similar and Further Questions
**Exercise 10B**
Questions 4(a)–(d), 5–8

**1.** In $\triangle ABC$, $\angle C = 90°$. Given that $\angle B = 27°$ and $AB = 23$ m, find the length of

**(i)** $BC$,

**(ii)** $AC$.




**2.** The figure shows a quadrilateral $PQRS$ where $\angle PQR = \angle PRS = 90°$. Given that $PR = 10.2$ m, $\angle QPR = 46°$ and $\angle PSR = 48°$, find

**(i)** the length of $PQ$,

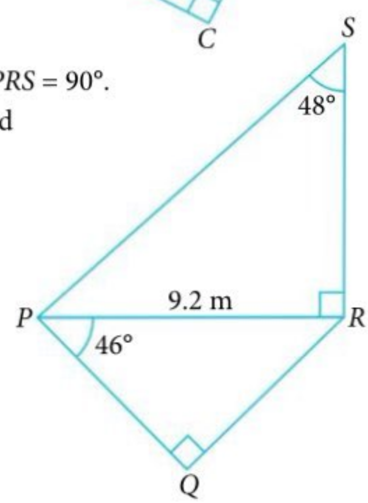
**(ii)** the length of $QR$,

**(iii)** the length of $PS$,

**(iv)** the length of $RS$,

**(v)** the perimeter of the quadrilateral $PQRS$,

**(vi)** the area of the quadrilateral $PQRS$.



## Reflection

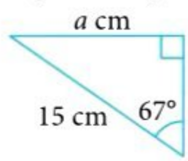
When applying trigonometric ratios to find the unknown sides of a right-angled triangle, how do I know which trigonometric ratio to use? What guides me in my decision?

Advanced | Intermediate | Basic

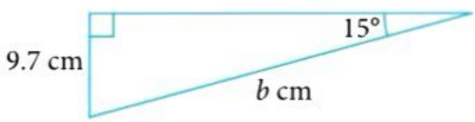
## Exercise 10B

**1.** Find the value of the unknown in each of the following right-angled triangles.

**(a)**

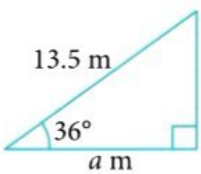



**(b)**

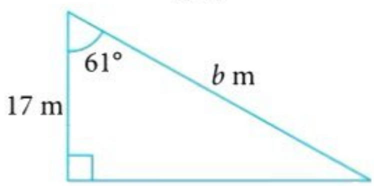



**2.** Find the value of the unknown in each of the following right-angled triangles.

**(a)**

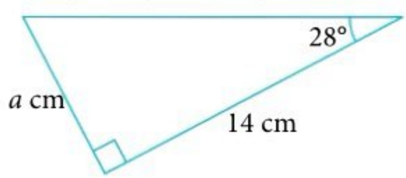



**(b)**

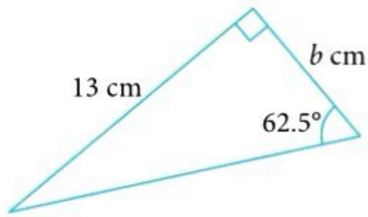



**3.** Find the value of the unknown in each of the following right-angled triangles.

**(a)**

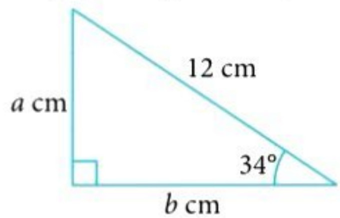



**(b)**

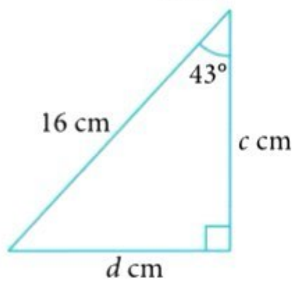



**4.** Find the values of the unknowns in each of the following right-angled triangles.

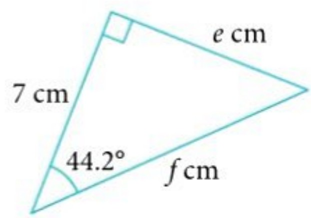
**(a)**




**(b)**

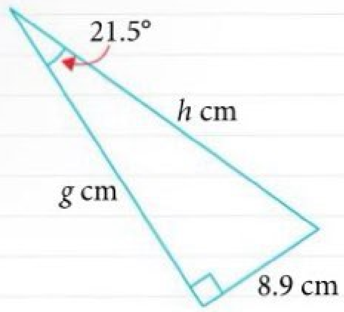



**(c)**



## Exercise 10B

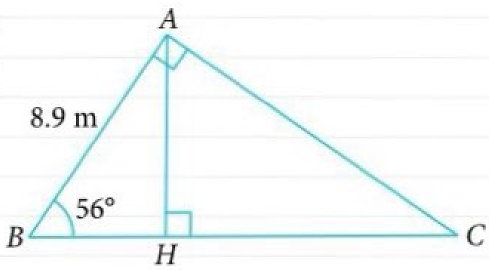
(d)




**5.** In $\triangle ABC$, $AB = 8.9$ m, $\angle BAC = 90°$ and $\angle ABC = 56°$. $H$ lies on $BC$ such that $AH$ is perpendicular to $BC$. Find the length of

**(i)** $AH$,

**(ii)** $HC$.



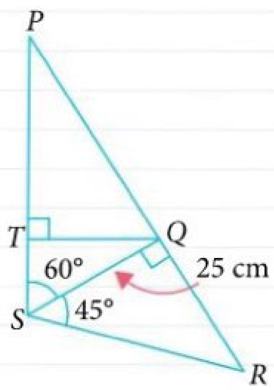

**6.** In the figure, $QS = 25$ cm, $\angle QSR = 45°$ and $\angle QST = 60°$.

Find the length of

**(i)** $TQ$,

**(ii)** $PT$,

**(iii)** $PR$.

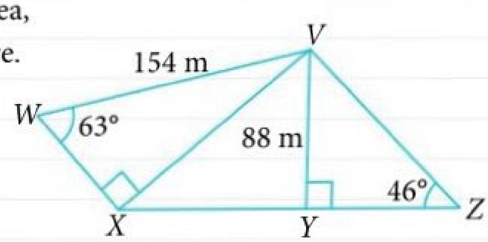



**7.** A figure $VWXYZ$ is made up of three right-angled triangles. Given that $VW = 154$ m, $VY = 88$ m, $\angle VXW = \angle VYZ = 90°$, $\angle VWX = 63°$, $\angle VZY = 46°$, find

**(i)** the perimeter,

**(ii)** the area,

of the figure.




**8.** If $y$ is inversely proportional to $(\tan x)^2$ and $y = 2$ when $x = 30°$, find the value of $y$ when this value of $x$ is doubled.

# 10.3 Applications of trigonometric ratios to find unknown angles in right-angled triangles

## A. Finding angles given trigonometric ratios

If we are given that $\sin A = 0.45$, how do we find the acute angle $A$?

Since the value of the trigonometric ratio $\sin A$ is 0.45, we can write $\sin A = \frac{45}{100}, \frac{9}{20}$ or $\frac{4.5}{10}$, etc.

Consider the trigonometric ratio $\sin A = \frac{4.5}{10}$. Recall that since $\sin A = \frac{\text{opp}}{\text{hyp}}$, then in a right-angled triangle, the length of the side opposite $\angle A$ is 4.5 cm and the length of the hypotenuse is 10 cm. Fig. 10.5 shows a right-angled triangle $ABC$ where $\angle C = 90°$, $BC = 4.5$ cm and $AB = 10$ cm.

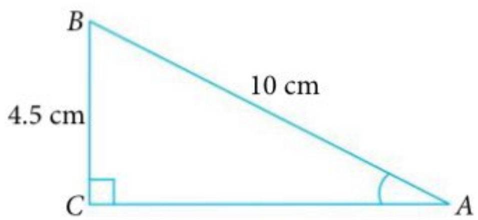



Fig. 10.5

Using a protractor, we measure $\angle A$ to be 27°.

## Using a calculator to find angles given trigonometric ratios

We can also use a calculator to find $A$ where $\sin A = 0.45$.

Sequence of calculator keys:

$\sin^{-1}$

SHIFT | sin | ( | 0 | . | 4 | 5 | ) | =

$\therefore A = \sin^{-1}(0.45)$
$= 26.7°$ (to 1 d.p.)

**Attention**

Unless otherwise stated, we leave angles in degrees correct to one decimal place.

**Practise Now 4A**

Similar and Further Questions
**Exercise 10C**
Questions 1(a)–(c)

Use a calculator to find each of the following angles, given its trigonometric ratio.

**(a)** $\sin A = 0.78$

**(b)** $\cos B = 0.35$

**(c)** $\tan C = 1.23$

## B. Applications of trigonometric ratios to find unknown angles in right-angled triangles

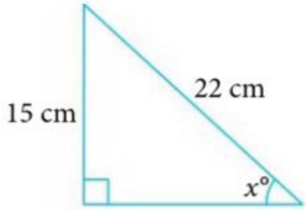
In Section 10.2, we have learnt how to use trigonometric ratios to find the lengths of the unknown sides of right-angled triangles. Let us now use trigonometric ratios to find the unknown angles in right-angled triangles.

Worked Example 4

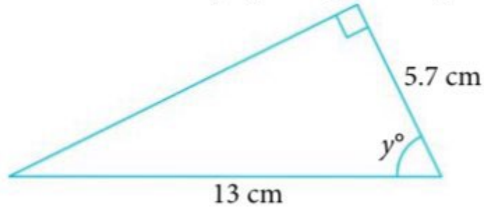
**Finding unknown angles in right-angled triangles**

Calculate the value of the unknown in each of the following right-angled triangles.

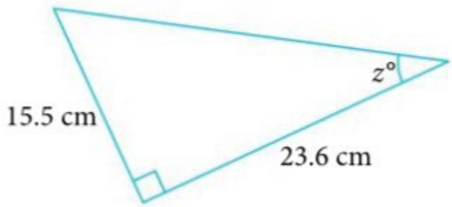
**(a)**




**(b)**




**(c)**




**Solution**

**(a)** $\sin x° = \dfrac{15}{22}$

$x° = \sin^{-1}\left(\dfrac{15}{22}\right)$

$= 43.0°$ (to 1 d.p.)

$\therefore x = 43.0$

**(b)** $\cos y° = \dfrac{5.7}{13}$

$y° = \cos^{-1}\left(\dfrac{5.7}{13}\right)$

$= 64.0°$ (to 1 d.p.)

$\therefore y = 64.0$

**(c)** $\tan z° = \dfrac{15.5}{23.6}$

$z° = \tan^{-1}\left(\dfrac{15.5}{23.6}\right)$

$= 33.3°$ (to 1 d.p.)

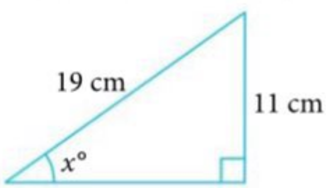
$\therefore z = 33.3$

**Practise Now 4B**

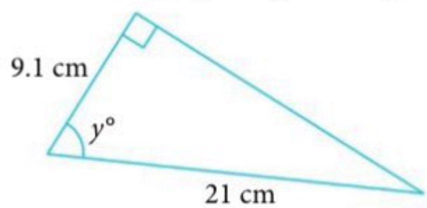
Similar and Further Questions
Exercise 10C
Questions 2(a)–(i)

Find the value of the unknown in each of the following right-angled triangles.

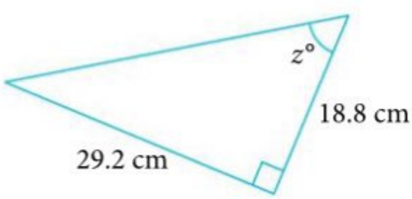
**(a)**




**(b)**




**(c)**



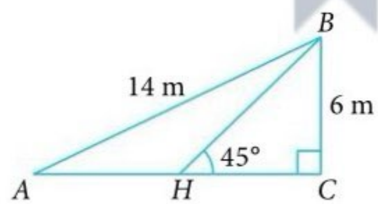
**Worked Example 5**

### Finding unknowns using trigonometric ratios

In $\triangle ABC$, $AB = 14$ m, $BC = 6$ m and $\angle ACB = 90°$. $H$ lies on $AC$ such that $\angle BHC = 45°$. Calculate

**(i)** $\angle BAC$,

**(ii)** the length of $AH$.




**Solution**

**(i)** In $\triangle ABC$,

$$\sin \angle BAC = \frac{6}{14}$$

$$\therefore \angle BAC = \sin^{-1}\left(\frac{6}{14}\right)$$

$$= 25.4° \text{ (to 1 d.p.)}$$

**(ii)** In $\triangle BHC$,

$$\tan 45° = \frac{6}{HC}$$

$$HC = \frac{6}{\tan 45°}$$

$$= 6 \text{ m}$$

In $\triangle ABC$, $\angle ACB = 90°$.

Using Pythagoras' Theorem,

$$AB^2 = AC^2 + BC^2$$

$$14^2 = AC^2 + 6^2$$

$$AC^2 = 14^2 - 6^2$$

$$= 196 - 36$$

$$= 160$$

$$AC = \sqrt{160} \text{ (since } AC > 0)$$

$$\therefore AH = AC - HC$$

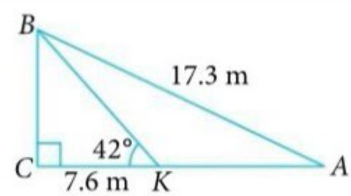
$$= \sqrt{160} - 6$$

$$= 6.65 \text{ m (to 3 s.f.)}$$

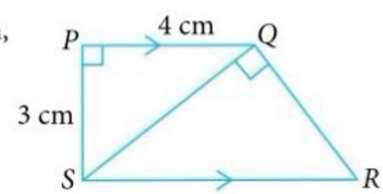
**Practise Now 5**

Similar and Further Questions
**Exercise 10C**
Questions 3–8

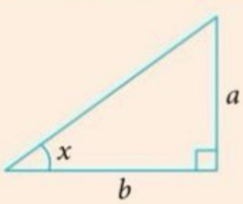
1. In $\triangle ABC$, $AB = 17.3$ m and $\angle ACB = 90°$. $K$ lies on $CA$ such that $CK = 7.6$ m and $\angle BKC = 42°$. Find
   **(i)** $\angle BAC$,
   **(ii)** the length of $KA$.




2. The figure shows a trapezium $PQRS$ in which $PQ$ is parallel to $SR$. Given that $PQ = 4$ cm and $PS = 3$ cm, find
   **(i)** $\angle PQS$,
   **(ii)** the length of $QR$.




## Journal Writing




It is given that $\tan x = \frac{a}{b}$, where $a < b$.

Determine if $\angle x$ is more or less than 45°. Explain your reasoning.

## Exercise 10C

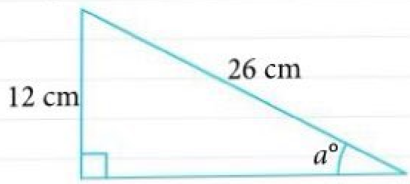
**1.** Use a calculator to find each of the following angles, given its trigonometric ratio.

**(a)** $\sin A = 0.527$ **(b)** $\cos B = 0.725$

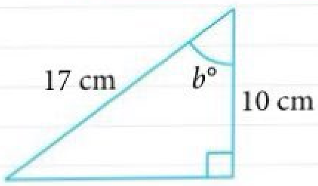
**(c)** $\tan C = 2.56$

**2.** Find the value of the unknown in each of the following right-angled triangles.

**(a)**

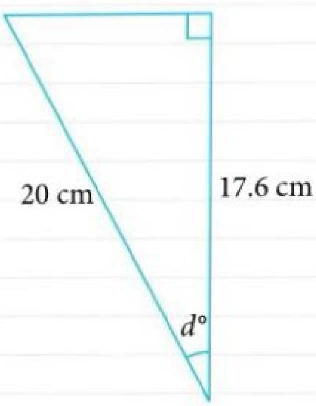



**(b)**

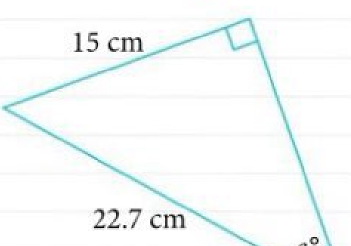



**(c)**

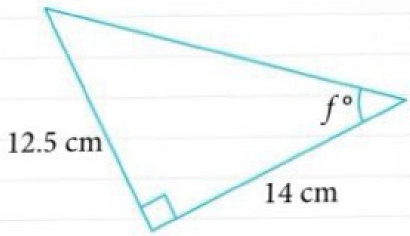

**(d)**



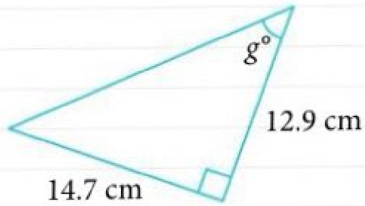

**(e)**



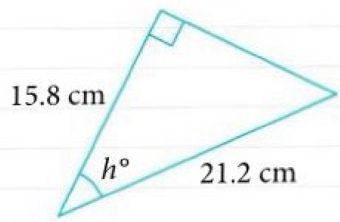

**(f)**



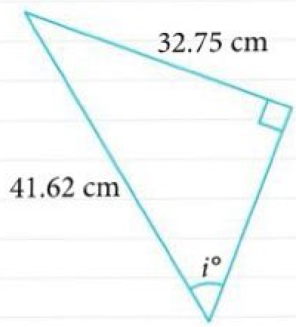

**(g)**




**(h)**



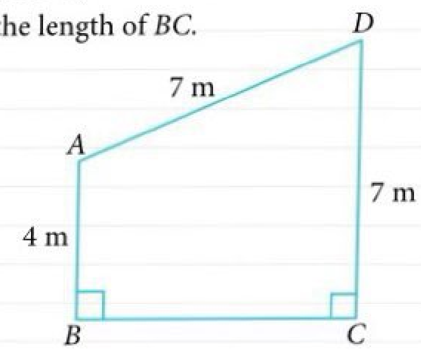

**(i)**




**3.** The figure shows a quadrilateral $ABCD$ where $\angle ABC = \angle BCD = 90°$. Given that $AB = 4$ m and $DA = DC = 7$ m, find

**(i)** $\angle ADC$,

**(ii)** the length of $BC$.

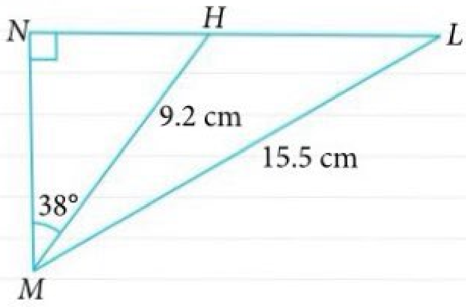


## Exercise 10C

**4.** In $\triangle LMN$, $LM = 15.5$ cm and $\angle LNM = 90°$. $H$ lies on $NL$ such that $HM = 9.2$ cm and $\angle HMN = 38°$. Find

**(i)** $\angle MLN$,

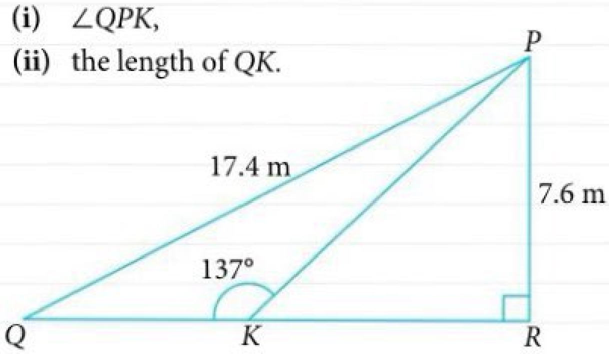
**(ii)** the length of $HL$.



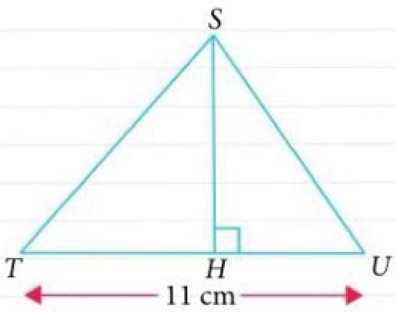

**5.** In $\triangle PQR$, $PQ = 17.4$ m and $PR = 7.6$ m. $K$ lies on $QR$ such that $\angle PKQ = 137°$. Find

**(i)** $\angle QPK$,

**(ii)** the length of $QK$.



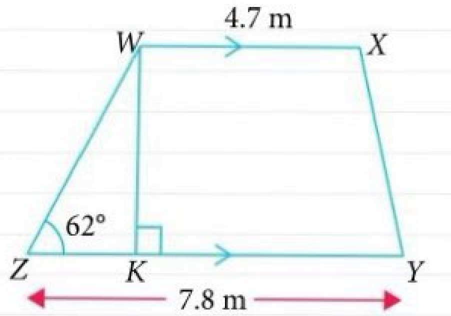

**6.** The figure shows a triangle $STU$ where $TU = 11$ cm. $H$ lies on $TU$ such that the length of $TH$ is 120% of the length of $HU$ and angle $SHU = 90°$. Given that the area of triangle $STH$ is 21 cm², find angle $TSU$.



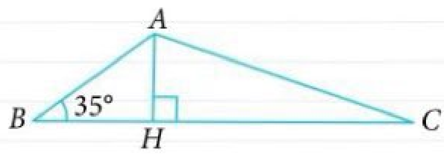

**7.** The figure shows a trapezium $WXYZ$ in which $WX$ is parallel to $ZY$. It is given that $WX = 4.7$ m, $ZY = 7.8$ m and $\angle WZY = 62°$. $K$ lies on $ZY$ such that the ratio of the length of $WK$ to the length of $ZY$ is $6 : 13$. Find

**(i)** $\angle XYZ$,

**(ii)** the perimeter of the trapezium $WXYZ$.




**8.** The figure shows a triangle $ABC$ where angle $ABC = 35°$. $H$ lies on $BC$ such that the length of $HC$ is twice that of $BH$. Find angle $ACB$.



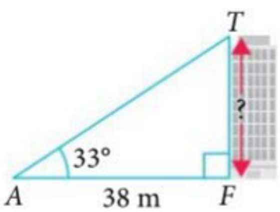
# 10.4 Applications of trigonometric ratios in real-world contexts

Trigonometry is commonly used to find the heights of buildings and mountains, the distance of the shore from a point in the sea and the distance between celestial bodies, etc. In this section, we will learn how to apply trigonometric ratios to solve problems in real-world contexts.

**Worked Example 6**

### Finding height of building

A point $A$ on level ground is 38 m away from the foot $F$ of a building $TF$. Given that $AT$ makes an angle of 33° with the horizontal, calculate the height of the building.




**Solution**

$$\tan 33° = \frac{TF}{38}$$

$$TF = 38 \tan 33°$$

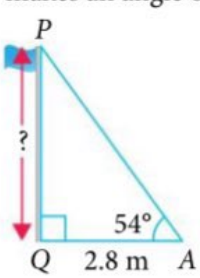
$$= 24.7 \text{ m (to 3 s.f.)}$$

$\therefore$ the height of the building is 24.7 m.

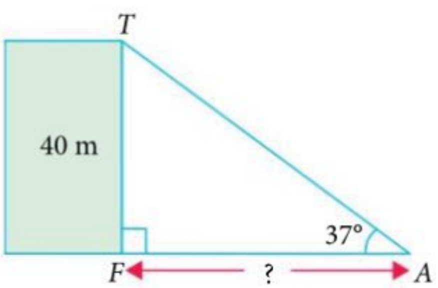
**Practise Now 6**

Similar and Further Questions
**Exercise 10D**
Questions 1–3, 7, 13

1. A point $A$ on level ground is 2.8 m away from the foot $Q$ of a flagpole $PQ$. Given that $AP$ makes an angle of 54° with the horizontal, find the height of the flagpole.



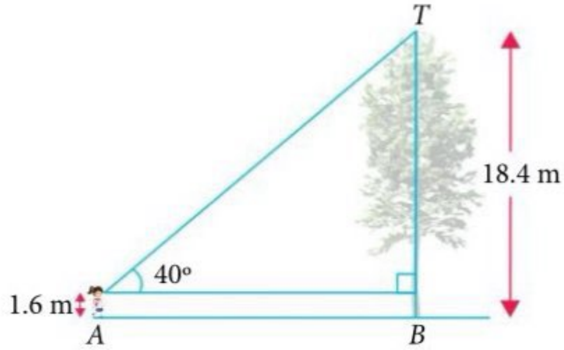

2. The height of a tower $TF$ is 40 m. Given that $A$ is a point on level ground such that $AT$ makes an angle of 37° with the horizontal, find the distance $FA$.



3. Li Ting uses a clinometer to measure the height of a tree *TB*. From where she stands, the angle of elevation of the tree is 40° and the height of the tree is calculated to be 18.4 m. Given that the clinometer is 1.6 m above the ground, find the distance *AB* between Li Ting and the foot of the tree.

**Information**

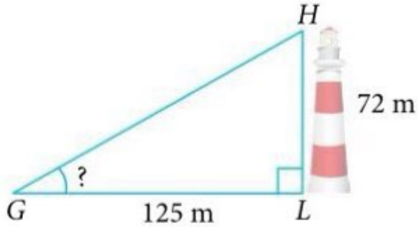
A clinometer is a tool that measures angles of inclination, usually the angle between the ground or the observer and a tall object, so as to obtain the height of the object.




**Worked Example 7**

### Finding angle from ground

A lighthouse *HL* is 72 m tall. Given that a point *G* on level ground is 125 m away from the foot *L* of the lighthouse, calculate $\angle HGL$.




**Solution**

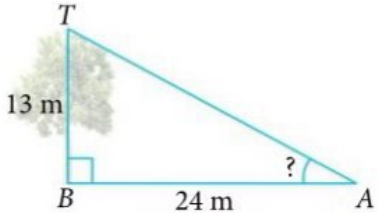
$\tan \angle HGL = \dfrac{72}{125}$

$\therefore \angle HGL = 29.9°$ (to 1 d.p.)

**Practise Now 7**

Similar and Further Questions
Exercise 10D
Question 4

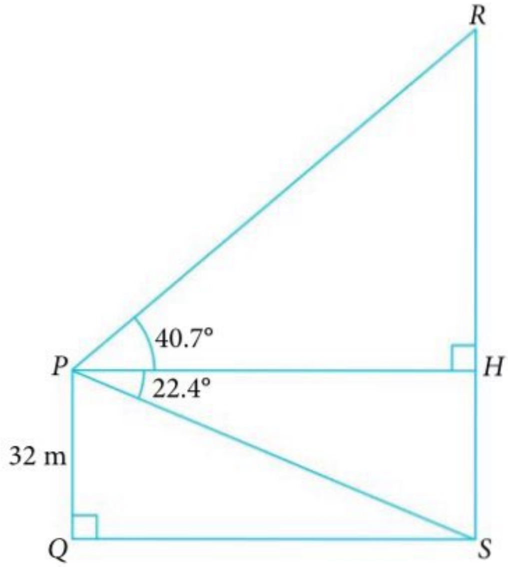
A tree *TB* is 13 m tall. Given that a point *A* on level ground is 24 m away from the foot *B* of the tree, find $\angle TAB$.



Worked Example 8

### Real-life application of trigonometric ratios

The height of a building $PQ$ is 32 m. $RHS$ is another building and $PH$ is a horizontal sky bridge linking the buildings. Given that angle $RPH = 40.7°$ and angle $HPS = 22.4°$, calculate the height of the building $RHS$.




**Solution**

$HS = PQ = 32$ m

In $\triangle PSH$,

$$\tan 22.4° = \frac{32}{PH}$$

$$PH = \frac{32}{\tan 22.4°}$$

$$= 77.638 \text{ m (to 5 s.f.)}$$

In $\triangle PRH$,

$$\tan 40.7° = \frac{RH}{PH}$$

$$RH = PH \tan 40.7°$$

$$= 77.638 \tan 40.7°$$

$$= 66.779 \text{ m (to 5 s.f.)}$$

$$RS = RH + HS$$

$$= 66.779 + 32$$

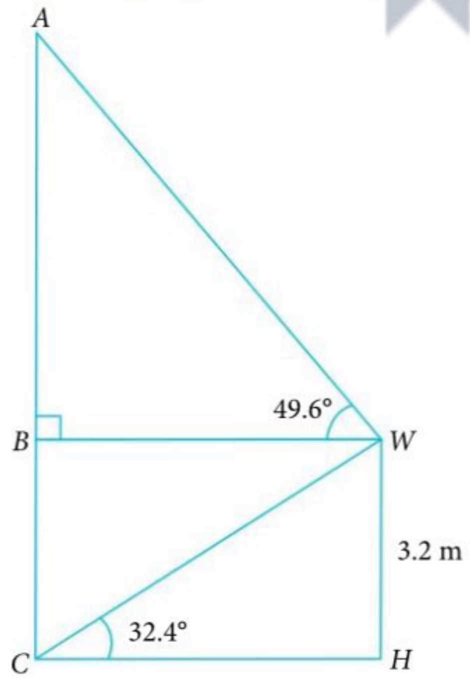
$$= 98.8 \text{ m (to 3 s.f.)}$$

$\therefore$ the height of the building $RHS$ is 98.8 m.

**Practise Now 8**

Similar and Further Questions
Exercise 10D
Question 8

The height of a warehouse $WH$ is 3.2 m. $ABC$ is a vertical mast in front of the warehouse. $BW$ is a horizontal cable attached to the mast from the top of the warehouse. Given that angle $AWB = 49.6°$ and angle $WCH = 32.4°$, find the height of the mast.



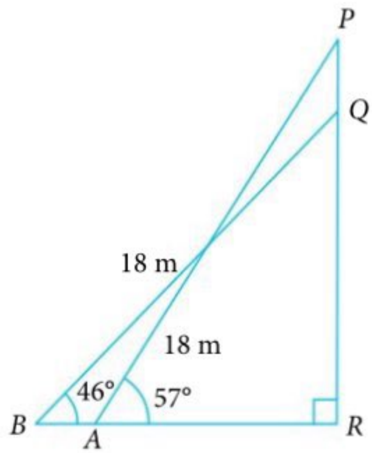

**Worked Example 9**

## Real-life application of trigonometric ratios

When a ladder of length 18 m leans against the top edge of a window of a building, it forms an angle of 57° with the ground. When the ladder leans against the lower edge of the same window, it forms an angle of 46° with the ground. Calculate the height of the window.

**Solution**

In the figure, $P$ and $Q$ represent the top and lower edge of the window respectively. $AP$ and $BQ$ represent the ladder when it is at two different positions.




In $\triangle APR$,

$$\sin 57° = \frac{PR}{18}$$

$$PR = 18 \sin 57°$$

$$= 15.096 \text{ m (to 5 s.f.)}$$

In $\triangle BQR$,

$$\sin 46° = \frac{QR}{18}$$

$$QR = 18 \sin 46°$$

$$= 12.948 \text{ m (to 5 s.f.)}$$

$$PQ = PR - QR$$

$$= 15.096 - 12.948$$

$$= 2.15 \text{ m (to 3 s.f.)}$$

$\therefore$ the height of the window is 2.15 m.

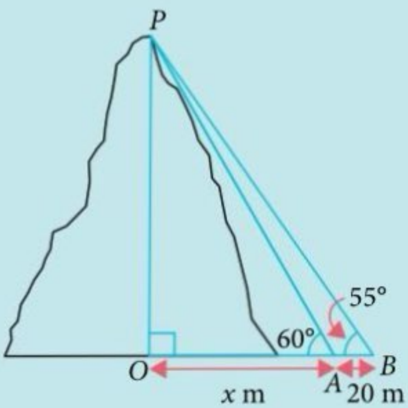
**Practise Now 9**

Similar and Further Questions
**Exercise 10D**
Questions 5, 6, 9–12

When a straight pole of length 14.5 m leans against the top edge of a signboard on a building, it forms an angle of 53° with the ground. When the pole leans against the lower edge of the same signboard, it forms an angle of 42° with the ground. Find the height of the signboard.

## Introductory Problem Revisited

We do not need to climb a hill to find its height. Instead, trigonometric ratios can be applied.




Let the height of the hill, denoted by $OP$, be $h$ metres. A surveyor stands at $A$, where $OA$ is perpendicular to $OP$. Using a theodolite, he measures angle $OAP$. Suppose angle $OAP$ is 60°. Then he stands at $B$, which is 20 metres away from $A$, and measures angle $OBP$. Suppose angle $OBP$ is 55°. Can you find the value of $h$? What are the assumptions made?

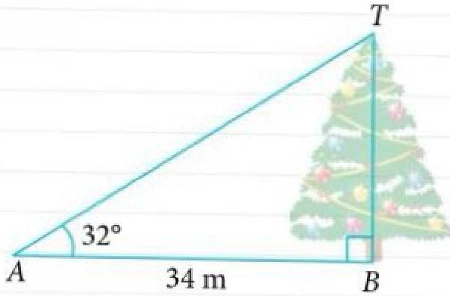
Discuss your solution with your classmates.

## Reflection

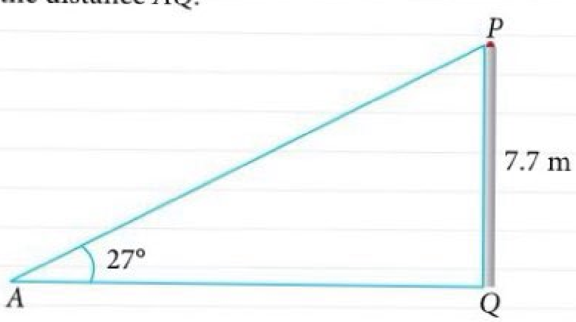
1. In right-angled triangles, how are the lengths of the three sides related? What about the lengths of the sides and the acute angles?
2. What have I learnt in this section or chapter that I am still unsure of?

## Exercise 10D

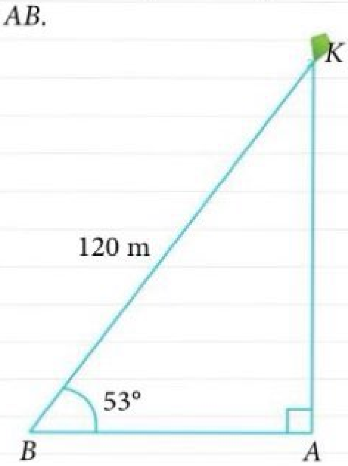
**1.** A point $A$ on level ground is 34 m away from the foot $B$ of a Christmas tree $TB$. Given that $AT$ makes an angle of 32° with the horizontal, find the height of the Christmas tree.



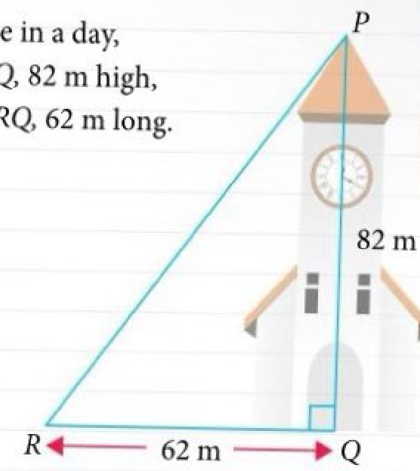

**2.** The height of a vertical post $PQ$ is 7.7 m. Given that $A$ is a point on level ground such that $AP$ makes an angle of 27° with the horizontal, find the distance $AQ$.




**3.** Joyce stands at $B$ and flies a kite. The kite is vertically above $A$. The string $BK$ of length 120 m, attached to the kite, makes an angle of 53° to the horizontal. Assuming the string is taut, find the distance $AB$.




**4.** At a certain time in a day, a clock tower $PQ$, 82 m high, casts a shadow $RQ$, 62 m long. Find $\angle PRQ$.

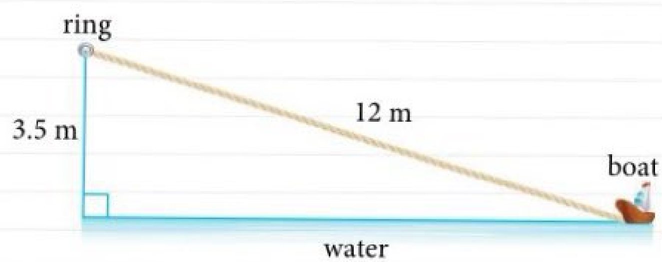



**5.** A ladder of length 5 m leans against a nail on a vertical wall. It forms an angle of 60° with the ground. Find

**(i)** the height of the nail above the ground,

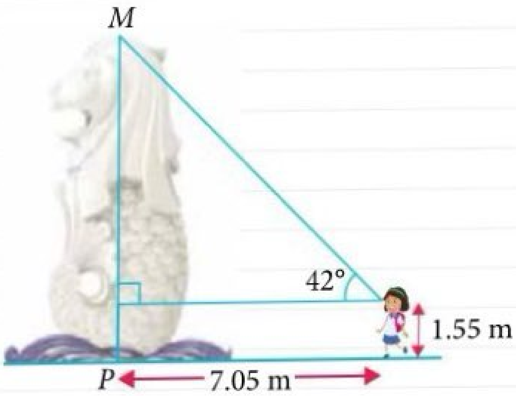
**(ii)** the distance of the foot of the ladder from the base of the wall.

**6.** A boat is tied to a rope of length 12 m which is attached to a ring that is 3.5 m above the water. Assuming that the rope is taut, find the angle it makes with the water.

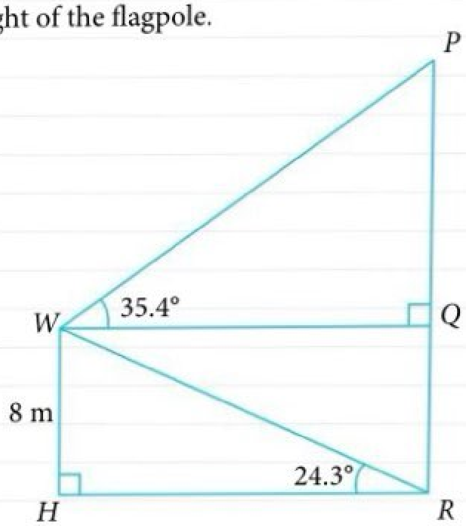


## Exercise 10D

**7.** Sara is standing 7.05 m away from the statue of the Merlion $MP$ at Merlion Park, Singapore. The height of her eyes from her feet is 1.55 m. Given that the angle of elevation of the top of the statue from her eyes is 42°, find the height of the statue.

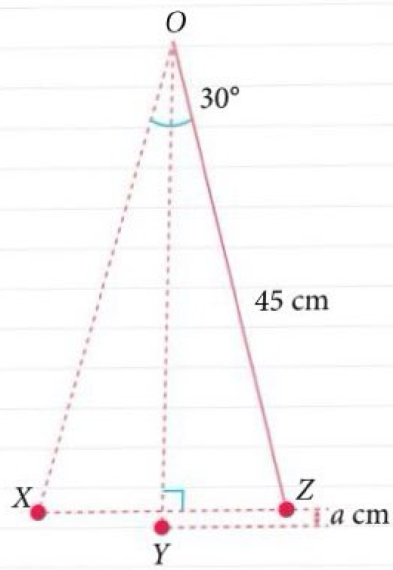



**8.** The height of a building $WH$ is 8 m. $PQR$ is a flagpole and $WQ$ is a horizontal cable connected to the top of the building and to the flagpole. Given that $\angle PWQ = 35.4°$ and $\angle WRH = 24.3°$, find the height of the flagpole.




**9.** A plank of length 4 m rests against a wall 1.8 m high such that 1.2 m of the plank lies beyond the wall. Find the angle the plank makes with the wall.

**10.** A pendulum of length 45 cm swings backwards and forwards from $X$ to $Z$ passing through $Y$, the lowest point of oscillation. Given that $\angle XOZ = 30°$, find the height, $a$ cm, in which the pendulum bob rises above $Y$.




**11.** When a ladder of length 2.5 m leans against the top edge of a window of a building, it forms an angle of 55° with the ground. When the ladder leans against the lower edge of the same window, it forms an angle of 38° with the ground. Find the height of the window, giving your answer in centimetres.

## Exercise 10D

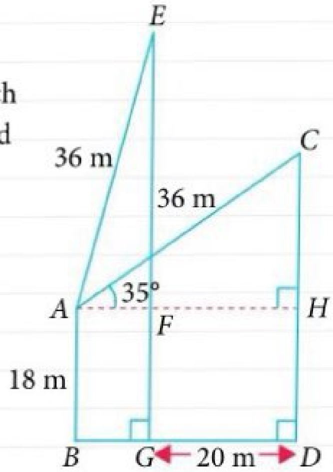
**12.** A crane stands on level ground. It is represented by a vertical tower $AB$ of height 18 m and a jib $AC$ of length 36 m. A vertical cable hangs from $C$ and is attached to a load at $D$. The jib is inclined at an angle of 35° to the horizontal line $AH$.

**(i)** Find $CD$.

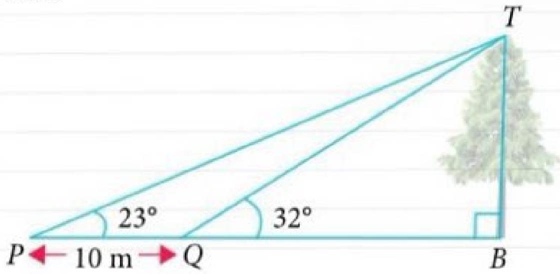
The load is lifted from $D$ and the jib is rotated in a vertical plane about $A$. When the jib is at the position $AE$, the load is lowered and placed on the level ground at the point $G$, which is vertically below $E$. The line $EG$ cuts the line $AH$ at $F$ and $GD = 20$ m. Find

**(ii)** $EF$,

**(iii)** the angle in which the jib has rotated in the vertical plane about $A$.



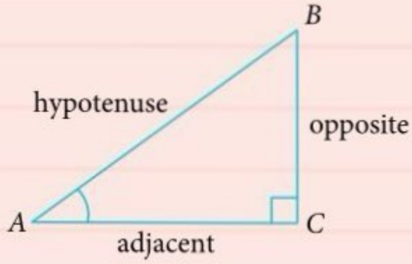

**13.** Two points $P$ and $Q$, 10 m apart on level ground, are due West of the foot $B$ of a tree $TB$. Given that $\angle TPB = 23°$ and $\angle TQB = 32°$, find the height of the tree.




## Looking Back

In this chapter, we begin our exploration of trigonometry by studying the relationships between the lengths of sides and the sizes of angles in triangles. In particular, we looked at a family of similar right-angled triangles and learnt that the ratio of any two sides in relation to a given angle in right-angled triangles is constant regardless of the size of the triangles. Trigonometric ratios quantify the idea of **proportionality** between the sides of similar right-angled triangles. This simple yet powerful notion of trigonometric ratios will open up new avenues of exploration as we continue to deepen our knowledge of trigonometry and apply these ideas to other kinds of triangles in Book 4.

Summary




In $\triangle ABC$, if $\angle C = 90°$,

then $\dfrac{BC}{AB} = \dfrac{\text{opp}}{\text{hyp}}$ is called the **sine** of $\angle A$, or $\sin A = \dfrac{\text{opp}}{\text{hyp}}$,

$\dfrac{AC}{AB} = \dfrac{\text{adj}}{\text{hyp}}$ is called the **cosine** of $\angle A$, or $\cos A = \dfrac{\text{adj}}{\text{hyp}}$,

$\dfrac{BC}{AC} = \dfrac{\text{opp}}{\text{adj}}$ is called the **tangent** of $\angle A$, or $\tan A = \dfrac{\text{opp}}{\text{adj}}$.

- State the ratios $\sin B$, $\cos B$ and $\tan B$ in terms of the lengths of the sides of $\triangle ABC$.

# CHAPTER 11

## Volume, Surface Area and Symmetry of Prisms and Cylinders


Water is a precious resource that we all must conserve. In 2019, the global water demand for all uses was 4600 trillion litres per year, or 12.6 trillion litres per day. This volume is enough to fill more than 5 million Olympic-sized pools each day! What is the volume of water a swimming pool can hold?

In this chapter, we will learn how to find the volume and surface area of various solids and their applications in the real world.

### Learning Outcomes

What will we learn in this chapter?

- How to convert between $cm^3$ and $m^3$
- What prisms and cylinders are
- How to find the volume and surface area of cubes, cuboids, prisms and cylinders
- Why volume and surface areas of prisms and cylinders have useful applications in real life
- What plane and rotational symmetries of 3D solids are

## Introductory Problem

Gold is one of the most precious metals found in the Earth's crust because it is rare, difficult to obtain and corrosion-resistant. Counterfeiters may claim that their products are made from pure gold and sell them at a premium price. Do you know how to test for the purity of gold in a piece of jewellery?

# 11.1 Conversion of units

Volume is a **measure** of the amount of space an object occupies, which enables us to quantify the capacity of the object. An object that occupies more space has a greater volume.

Volume is measured using units such as millilitres (ml), litres (*l*), cubic centimetres ($cm^3$) and cubic metres ($m^3$).

### Class Discussion — Measurements in daily lives

1. **(i)** In everyday life, we often encounter units of measurement of volume. Search the Internet to find out Pakistanis' average water consumption for daily activities, e.g. washing the dishes and taking a shower. What unit of measurement was used for this volume? Determine the activity which requires the greatest amount of water on average.

   **(ii)** Check your utilities bill and find out the amount of water used in your home last month. Suggest some ways to reduce the average water consumption in your household.

2. **(i)** Some recipes require ingredients to be measured using a teaspoon. What is the volume, in ml, of one teaspoon of liquid?

   **(ii)** Health practitioners recommend that we drink at least 8 glasses of water daily. Approximately how many litres does this correspond to?

We have learnt how to convert from one unit of area to another in Book 1. To convert from one unit of volume to another, we need to know that:

$$1 \text{ ml} = 1 \text{ cm}^3$$
$$1\ l = 1000 \text{ ml} = 1000 \text{ cm}^3$$

Worked Example 1

## Converting between different units

Express

**(a)** $1 \text{ m}^3$ in $\text{cm}^3$,

**(b)** $1 \text{ cm}^3$ in $\text{m}^3$,

**(c)** $5 \text{ m}^3$

**(i)** in $\text{cm}^3$, **(ii)** in millilitres,

**(d)** $80\,000 \text{ cm}^3$

**(i)** in $\text{m}^3$, **(ii)** in litres.

### Solution

**(a)**
$$\begin{aligned} 1 \text{ m} &= 100 \text{ cm} \\ (1 \text{ m})^3 &= (100 \text{ cm})^3 \\ &= 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm} \\ 1 \text{ m}^3 &= 1\,000\,000 \text{ cm}^3 \end{aligned}$$

**(b)**
$$\begin{aligned} 100 \text{ cm} &= 1 \text{ m} \\ 1 \text{ cm} &= \frac{1}{100} \text{ m} \\ (1 \text{ cm})^3 &= \left(\frac{1}{100} \text{ m}\right)^3 \\ &= \frac{1}{100} \text{ m} \times \frac{1}{100} \text{ m} \times \frac{1}{100} \text{ m} \\ &= \frac{1}{1\,000\,000} \text{ m}^3 \\ 1 \text{ cm}^3 &= 0.000\,001 \text{ m}^3 \end{aligned}$$

**(c) (i)**
$$\begin{aligned} 1 \text{ m}^3 &= 1\,000\,000 \text{ cm}^3 \\ 5 \text{ m}^3 &= 5 \times 1\,000\,000 \text{ cm}^3 \\ &= 5\,000\,000 \text{ cm}^3 \end{aligned}$$

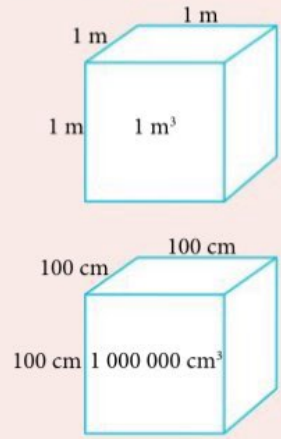
**(ii)**
$$\begin{aligned} 1 \text{ cm}^3 &= 1 \text{ ml} \\ 5\,000\,000 \text{ cm}^3 &= 5\,000\,000 \text{ ml} \\ \therefore 5 \text{ m}^3 &= 5\,000\,000 \text{ ml} \end{aligned}$$

**(d) (i)**
$$\begin{aligned} 1\,000\,000 \text{ cm}^3 &= 1 \text{ m}^3 \\ 80\,000 \text{ cm}^3 &= \frac{80\,000}{1\,000\,000} \text{ m}^3 \\ &= 0.08 \text{ m}^3 \end{aligned}$$

**(ii)**
$$\begin{aligned} 1 \text{ cm}^3 &= 1 \text{ ml} \\ 80\,000 \text{ cm}^3 &= 80\,000 \text{ ml} \\ &= \frac{80\,000}{1000}\, l \\ &= 80\, l \end{aligned}$$

**Attention**

The volume of a cube of sides 1 m is $1 \text{ m}^3$, which is equal to $1\,000\,000 \text{ cm}^3$.



**Practise Now 1**

Similar and Further Questions
Exercise 11A
Questions 1(a), (b), 2(a), (b)

Express
**(a)** $10 \text{ m}^3$
  **(i)** in $\text{cm}^3$,  **(ii)** in millilitres,
**(b)** $165\,000 \text{ cm}^3$
  **(i)** in $\text{m}^3$,  **(ii)** in litres.

# 11.2 Three-dimensional solids

We have learnt in Chapter 12 of Book 1 (Perimeter and Area of Plane Figures) that a flat surface, like the floor or the surface of a whiteboard, has two dimensions (2D). A solid, on the other hand, has three dimensions (3D) – length, breadth and height (or thickness or depth).

## A. Visualising 3D solids

### Investigation Visualising 3D solids

**Part 1: Angles of 3D solid**

1. Look at a school desk or table with a rectangular top (ignore the rounded corners, if any). Fig. 11.1 shows a photo of a school desk viewed from the top.
Measure the angles of the two corners of the rectangular top, $\angle ABC$ and $\angle BCD$.
Do you get 90° for both angles?

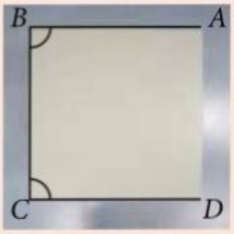
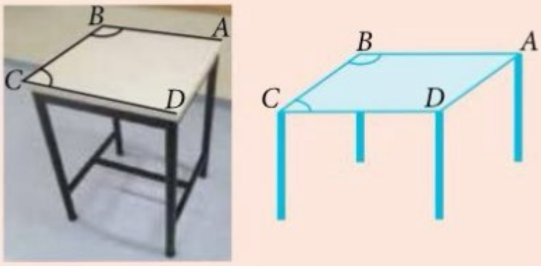



Fig. 11.1

2. Fig. 11.2(**a**) shows a photo of the same desk viewed from the side and Fig. 11.2(**b**) shows a sketch of the same desk.
Measure $\angle ABC$ and $\angle BCD$ again. Do you get 90°, smaller than 90°, or larger than 90°?




**(a)** **(b)**

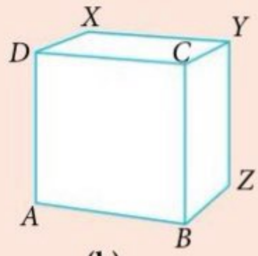
Fig. 11.2

We see that drawing a 3D solid or object on a flat surface *may make a right angle look smaller or larger than 90°*.

**Part 2: Lines and lengths of 3D solid**

Fig. 11.3(**a**) shows a photo of a cube and Fig. 11.3(**b**) shows a sketch of the same cube.

(**a**)




(**b**)

Fig. 11.3

1. From Fig. 11.3(**b**), identify
   (i) the line segments which are parallel to each other,
   (ii) the line segments which are perpendicular to each other,
   (iii) the shapes that are squares.
2. Joyce claimed that the plane *ADYZ* in Fig. 11.3(**b**) is a square. Do you agree? Explain your answer.

## B. Nets of 3D solids

### Investigation Cubes, cuboids, prisms and cylinders

For **Part 1** and **Part 2** of this Investigation, paper boxes in the shape of a cube and a cuboid are required.

**Part 1: Draw a cube and a cuboid**

1. Place the paper boxes on a table.
2. Observe the boxes by looking at each box from different angles, e.g. from the left, right, front, back.
3. Draw each box by copying its shape. A drawing of a cube and of a cuboid is given in Table 11.1.

**Part 2: Draw a net of a cube and of a cuboid**

1. Consider the box in the shape of a cube. Cut along the edges of the box such that all the surfaces can be laid flat in one piece.
2. Copy and complete Table 11.1 by drawing the nets of the solids that you have cut open.
3. Compare the nets you have drawn with those that your classmates drew. Are the nets identical? Can a solid have different nets?

Repeat Steps 1 to 3 for the box in the shape of a cuboid.

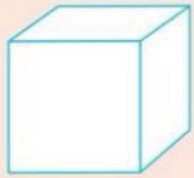
| Name | Figure | Net |
|---|---|---|
| **Cube** |  | |
| **Cuboid** |  | |

Table 11.1

**Part 3: Form a solid when the net is given**

1. Copy each of the following nets onto a sheet of paper and cut them out.
2. Try to fold them into the corresponding solids.
3. Copy and complete Table 11.2 by drawing the solids formed from the nets.

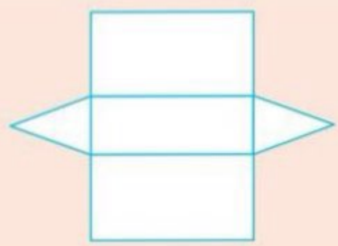
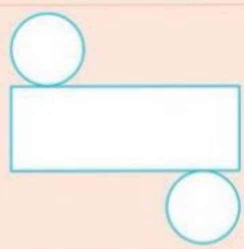
| Name | Figure | Net |
|---|---|---|
| **Triangular prism** | |  |
| **Cylinder** | |  |

Table 11.2

**Part 4: Draw a triangular prism and a cylinder**

1. Place the triangular prism and the cylinder from **Part 3** on a table.
2. Observe the solids by looking at each solid from different angles, e.g. from the left, right, front, back.
3. Draw each solid from a different angle as the one you have drawn in Table 11.2.

## Reflection

1. What do I already know about geometrical 2D shapes (or plane figures) that could guide my visualisation of 3D solids and the drawing of the nets of 3D solids?
2. A solid can have different nets. What do these different nets have in common? Why?

# 11.3 Volume and surface area of cubes and cuboids

In Chapter 12 of Book 1, we learnt about the perimeter and area of a 2D shape (or a plane figure). For a 3D object, we use the surface area as a measure of the boundary of the object, and the volume as a measure of the space enclosed within the surface of the object.

| | 2D shapes (plane figures) | 3D objects |
|---|---|---|
| **Measure of boundary** | Perimeter | Surface area |
| **Measure of space enclosed within the boundary** | Area | Volume |

Table 11.3

## A. Volume of cubes and cuboids (Recap)

We have learnt that the **volume** of an object is the amount of space it occupies.
The formula for the volume of a **cube** and of a **cuboid** is given in Table 11.4. A net of each of them is also provided.

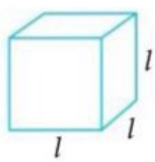
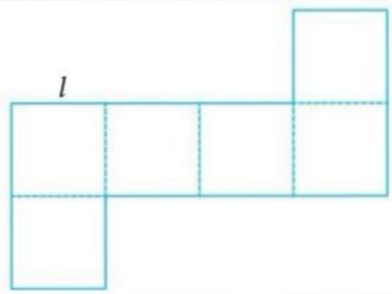
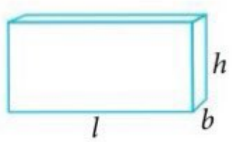
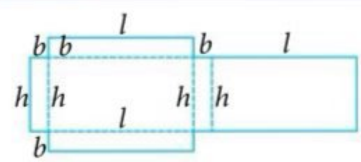
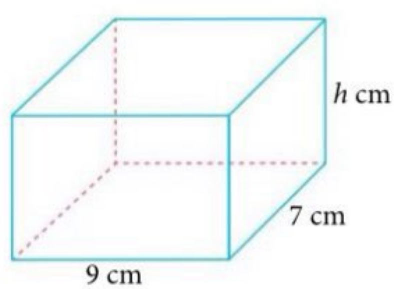
| Name | Figure | Volume | Net |
|---|---|---|---|
| **Cube** |   | $l^3$ |   |
| **Cuboid** |   | $lbh$ |   |

Table 11.4

**Worked Example 2**

### Volume of cube and cuboid

A cuboid, with dimensions 9 cm by 7 cm by $h$ cm, has a volume of 378 cm$^3$.




**(i)** Calculate the height, $h$, of the cuboid.

**(ii)** If the cuboid is melted to form a cube of length $l$ cm, find the value of $l$.

**(iii)** If the cuboid is melted to form cubes of length 3 cm, how many cubes can be obtained?

**(iv)** Li Ting said if the cuboid is cut to form cubes of length 3 cm, the maximum number of cubes that can be obtained will be the same as that found in part **(iii)**. Is Li Ting correct? Explain your answer.

**Solution**

**(i)** Volume of cuboid $= 9 \times 7 \times h$

$= 63h\ \text{cm}^3$

$63h = 378$

$h = \dfrac{378}{63}$

$= 6$

**(ii)** Volume of cube $= l \times l \times l$

$= 378\ \text{cm}^3$

$\therefore l = \sqrt[3]{378}$

$= 7.23$ (to 3 s.f.)

**(iii)** Volume of each cube $= 3 \times 3 \times 3$

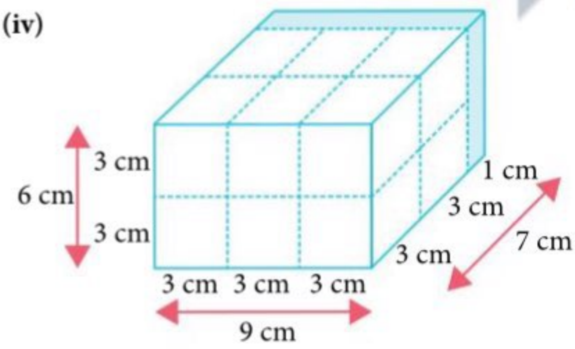
$= 27\ \text{cm}^3$

Number of cubes that can be obtained

$= \dfrac{378}{27}$

$= 14$

**(iv)**




Li Ting is not correct.

$9 \div 3 = 3$

$6 \div 3 = 2$

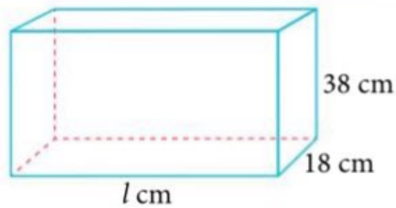
$7 \div 3 = 2$ R 1

The maximum number of cubes that can be obtained should be $3 \times 2 \times 2 = 12$.

**Practise Now 2**

Similar and Further Questions
**Exercise 11A**
Questions 3, 6(a), (b), 7

1. A cuboid, with dimensions $l$ cm by 18 cm by 38 cm, has a volume of 35 568 cm$^3$.




   **(i)** Find the length, $l$, of the cuboid.
   **(ii)** The cuboid is melted to form cubes of length 2 cm. How many cubes can be obtained?
   **(iii)** Ali says that if the cuboid is cut into cubes of length 2 cm, the maximum number of cubes obtained will be the same as that in part **(ii)**. Is Ali correct? Explain your answer.

2. An open rectangular tank is 55 cm long, 35 cm wide and 36 cm high. If it is initially half-filled with water, find the depth of water in the tank after 7700 cm$^3$ of water is added.

**Worked Example 3**

## Problem involving volume of cuboid

Calculate the volume of wood used in making an open rectangular box 2 cm thick, given that its internal dimensions are 54 cm by 46 cm by 18 cm.

**Solution**

External volume $= (54 + 2 + 2) \times (46 + 2 + 2) \times (18 + 2)$

$= 58 \times 50 \times 20$

$= 58\ 000\ \text{cm}^3$

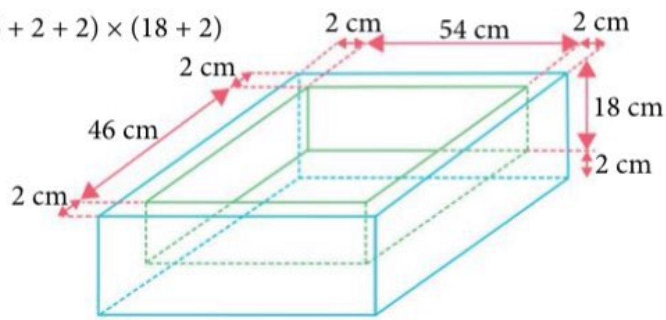
Internal volume $= 54 \times 46 \times 18$

$= 44\ 712\ \text{cm}^3$

$\therefore$ Volume of wood used

$= 58\ 000 - 44\ 712$

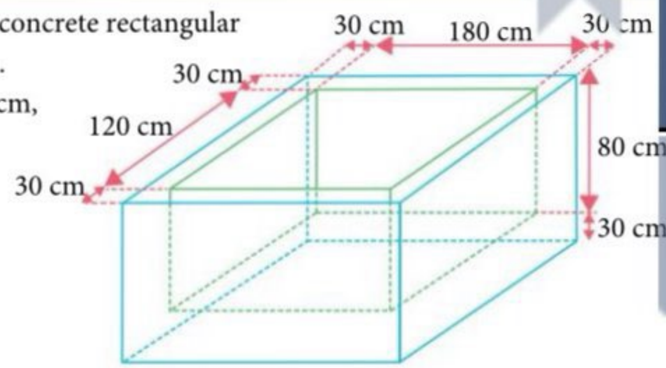
$= 13\ 288\ \text{cm}^3$



**Practise Now 3**

Similar and Further Questions
**Exercise 11A**
Questions 8–10, 15, 16

The internal dimensions of an open concrete rectangular tank are 180 cm by 120 cm by 80 cm. If the concrete has a thickness of 30 cm, find the volume of concrete used, leaving your answer in $m^3$.




**Introductory Problem Revisited**

The concept of volume can be used to test for pure gold. Density is a measure of mass per unit volume. The density of gold is 19.3 $g/cm^3$. This means that 1 $cm^3$ of gold weighs 19.3 g. With the knowledge of the density of gold, we can test if a piece of jewellery is made up of pure gold by carrying out a water displacement test.

First, the piece of jewellery is weighed. Then, the piece of jewellery is lowered into a graduated cylinder filled with water. The rise in water level is recorded and the increase in volume is calculated.

Based on the density of gold, 1 $cm^3$ of pure gold weighs 19.3 g. So for example, if a gold bangle weighs 29 g, we can calculate that: $29 \div 19.3 = 1.50$ (to 3 s.f.). This means that if the bangle is made of pure gold, the increase in volume should be 1.50 $cm^3$, as the amount of space it occupies should correspond to 1.50 $cm^3$.

## B. Surface area of cubes and cuboids

**Class Discussion** — **Surface area of cubes and cuboids**

1. Refer to the nets of a cube and a cuboid in Table 11.4 and fill in the blanks below.
   A cube has ______ surfaces. Each surface is in the shape of a ______.
   The area of each face is ______.
   ∴ The total surface area of a cube is ______.
   A cuboid has ______ surfaces. Each surface is in the shape of a ______.
   ∴ The total surface area of a cuboid is ______.
2. What is the relationship between the area of each face of the net and the total surface area of the object?
3. Verify your answers for Questions 1 and 2 with your classmate.

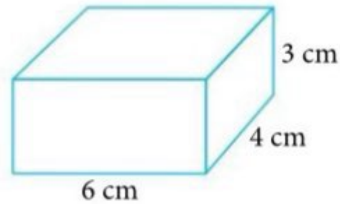
From the above Class Discussion, we can observe that **total surface area** is a **measure** of the total area occupied by the surface of the object, which is equal to the *area of all the faces* of the net. In particular, we have:

- Total surface area of a cube $= 6l^2$
- Total surface area of a cuboid $= 2(lb + lh + bh)$

**Worked Example 4**

### Surface area of cuboid

A cuboid is 6 cm long, 4 cm wide and 3 cm high. Calculate
(i) its volume, (ii) its total surface area.




**Solution**

(i) Volume of the cuboid $= 6 \times 4 \times 3$
$= 72 \text{ cm}^3$

(ii) Surface area of the cuboid $= 2(6 \times 4 + 6 \times 3 + 4 \times 3)$
$= 108 \text{ cm}^2$

**Practise Now 4**

Similar and Further Questions
**Exercise 11A**
Questions 4(a)–(f), 5(a)–(d), 11–14, 17, 18

1. A cuboid is 8 cm long, 5 cm wide and 10 cm high. Find
   (i) its volume, (ii) its total surface area.
2. An open rectangular tank of length 16 cm and breadth 9 cm contains water to a height of 8 cm. Find
   (i) the volume of water in the tank, giving your answer in litres,
   (ii) the surface area of the tank that is in contact with the water.
3. A metal cube has a volume of 27 cm³. It is to be painted on all its faces. Find the total area of the faces that will be coated with paint.

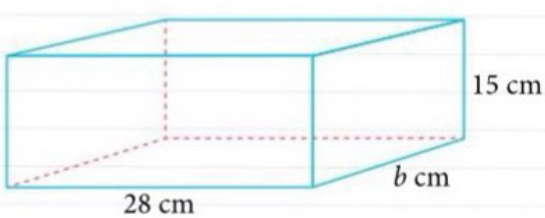
Basic Intermediate Advanced

## Exercise 11A

1. (a) Express each of the following in cm³.
   (i) 4 m³ (ii) 0.5 m³
   (b) Express each of the following in m³.
   (i) 250 000 cm³ (ii) 67 800 cm³

2. Express
   (a) 0.84 m³
   (i) in cm³, (ii) in millilitres,
   (b) 2560 cm³
   (i) in m³, (ii) in litres.

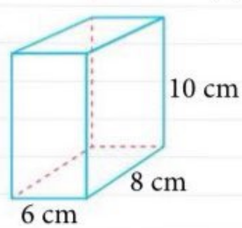
3. A cuboid, with dimensions 28 cm by $b$ cm by 15 cm, has a volume of 6720 cm³.



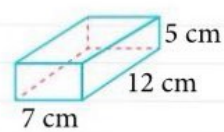

   (i) Find the breadth, $b$, of the cuboid.
   (ii) The cuboid is melted to form smaller cubes of length 4 cm. How many cubes can be obtained?

4. For each of the following cuboids, find
   (i) its volume, (ii) its total surface area.

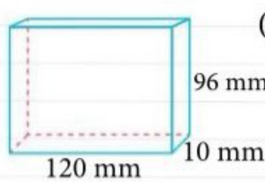
(a)



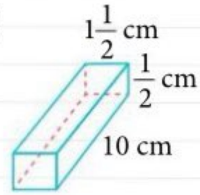

(b)




(c)

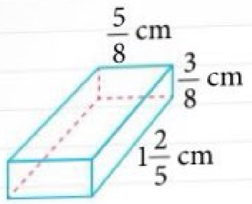



(d)

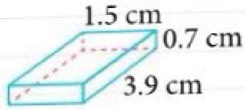


## Exercise 11A

(e)




(f)




**5.** Complete the table for each cuboid.

| | Length | Breadth | Height | Volume | Total surface area |
|---|---|---|---|---|---|
| (a) | 24 mm | 18 mm | 5 mm | | |
| (b) | 5 cm | 3 cm | | 120 cm$^3$ | |
| (c) | | 6 cm | 3.5 cm | 52.5 cm$^3$ | |
| (d) | 12 m | | 6 m | 576 m$^3$ | |

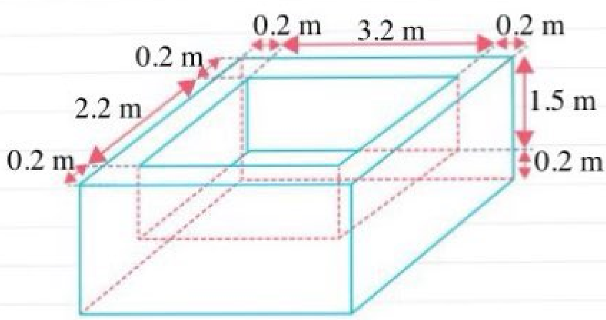
**6.** A rectangular block of cheese is 0.24 m long, 0.19 m wide and 0.15 m high.

**(a)** If the block of cheese is moulded to form a cube, find the length of each side of the cube.

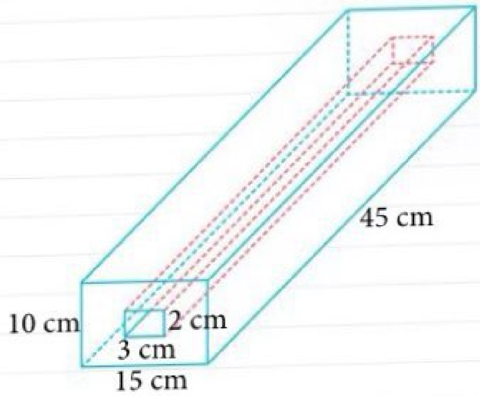
**(b)** Find the number of 2-cm cubes of cheese that can be cut from the rectangular block.

**7.** An open rectangular tank that is 4 m long, 2 m wide and 4.8 m high, is initially three-quarters filled with water. Find the depth of water in the tank after 4000 litres of water are added to it.

**8.** The internal dimensions of an open, wooden rectangular box are 3.2 m by 2.2 m by 1.5 m. If the wood has a thickness of 0.2 m, find the volume of wood used.




**9.** Find the volume of the hollow glass structure.




**10.** 2.85 million cubic metres of earth were required to fill the disused quarry.

**(i)** If each truck could carry a maximum load of 6.25 m$^3$ of earth per trip, how many trips would it take to fill the entire quarry?

**(ii)** The cost of transporting each truckload of earth was $55. How much did it cost to fill the quarry?

**(iii)** Given that the site of the quarry has a land area of approximately 3 hectares, find the cost to fill 1 m$^2$ of the land. (1 hectare = 10 000 m$^2$)

**11.** An open rectangular tank of length 0.2 m and breadth 0.15 m contains water to a height of 0.16 m. Find

**(i)** the volume of water in the tank, giving your answer in litres,

**(ii)** the surface area of the tank that is in contact with the water.

**12.** A fish tank measuring 80 cm by 40 cm contains water to a height of 35 cm. Find

**(i)** the volume of water in the tank, giving your answer in litres,

**(ii)** the surface area of the tank that is in contact with the water, giving your answer in m$^2$.

**13.** A metal cube has a volume of 1331 cm$^3$. It is to be painted on all its faces. Find the total area of the faces that will be coated with paint.

Basic Intermediate Advanced

# Exercise 11A

**14.** The total surface area of a cube is $433.5\text{ cm}^2$. Find its volume.

**15.** A trough, in the form of an open rectangular box, is 1.85 m long, 45 cm wide and 28 cm deep externally. If the trough is made of wood 2.5 cm thick, find the volume of wood used to make this trough, giving your answer in $\text{m}^3$.

**16.** The cross section of a drain is a rectangle 30 cm wide. If water 3.5 cm deep flows through the drain at a rate of 22 cm/s, how many litres of water will flow through in one minute?

**17.** A cuboid of length 12 cm and breadth 9 cm has a total surface area of $426\text{ cm}^2$.

**(i)** Find the height of the cuboid.

**(ii)** Hence, find its volume.

**(iii)** The cuboid is melted to form smaller cuboids with dimensions 5 cm by 3 cm by 2 cm. How many smaller cuboids can be obtained?

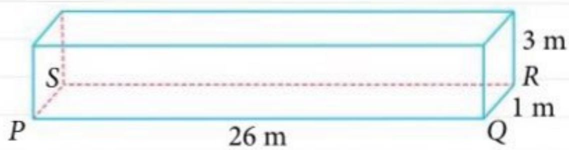
**(iv)** The cuboid is cut to form cubes of length 3 cm. Bernard said that since the volume of each cube is $27\text{ cm}^3$, the maximum number of cubes that can be obtained will be equal to $\dfrac{\text{volume of cuboid}}{27}$. Is Bernard correct? Explain your answer.

**18.** Three rooms, each in the shape of a cuboid, where $PQRS$ is the floor, are as shown.

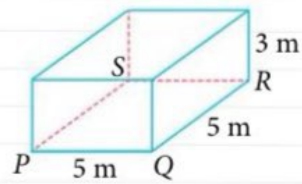
Room A: $PQ = 26$ m; $QR = 1$ m; height = 3 m

Room B: $PQ = QR = 5$ m; height = 3 m

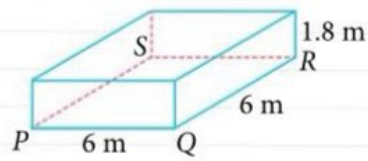
Room C: $PQ = QR = 6$ m; height = 1.8 m




**Room A**




**Room B**




**Room C**

**(i)** Find the floor area and the volume of each room.

**(ii)** Which room feels the most spacious? Does a larger floor area or a greater volume necessarily make a room feel more spacious? Explain your answer.

# 11.4 Volume and surface area of prisms

## A. Introduction to prisms

Fig. 11.4(**a**) shows a rectangular piece of cardboard. A large number of identical rectangular pieces of cardboard are stacked up to form a cuboid as shown in Fig. 11.4(**b**).

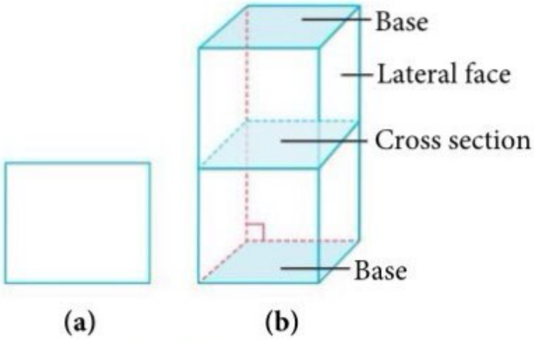



Fig. 11.4

The top and bottom pieces of cardboard are the **bases** of the cuboid. The bases are *parallel* to each other and are *identical* rectangles. A cuboid has a **uniform cross section**. Any horizontal cross section of the cuboid is parallel to the bases and is a rectangle that is identical to them. The other faces are the **lateral faces** of the cuboid. A cuboid is an example of a **prism**.

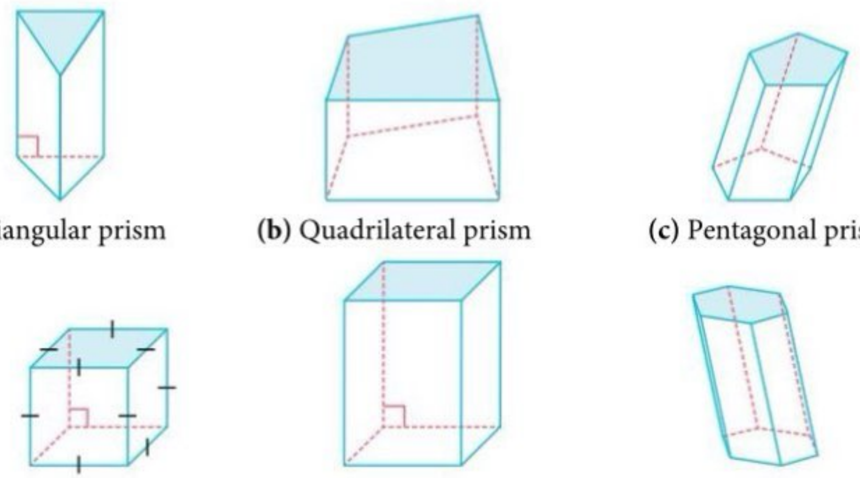
We can conclude that:

A prism has a uniform polygonal cross section.

**Information**

The figure shows a counterexample of a prism.
It does not have a uniform cross section.

Fig. 11.5 shows some examples of prisms with different bases. A base of each prism is shaded. A prism is named after its polygonal base. Can you name the last two prisms?



(**a**) Triangular prism (**b**) Quadrilateral prism (**c**) Pentagonal prism

(**d**) Cube or square prism (**e**) Cuboid or ______ prism (**f**) ______ prism

Fig. 11.5

**Information**

Here are some real-life examples of prisms:


Name the prisms. Can you think of other real-life examples of prisms?

## B. Right prisms

For the triangular prism in Fig. 11.5(**a**), all the *lateral faces* are *perpendicular* to the bases. Such prisms are called **right prisms**. Non-right prisms are called **oblique prisms**. The pentagonal prism in Fig. 11.5(**c**) is an example of an oblique prism. Can you identify another right prism and another oblique prism from Fig. 11.5?

### Thinking Time

1. (i) What is the shape of all the lateral faces of a right prism?
   (ii) What is the shape of all the lateral faces of an oblique prism?
2. Draw a square prism that is not a cube.

In this section, we will study only right prisms. Therefore, the term 'prism' refers to a right prism.

## C. Volume of prism

### Investigation — Volume of prism

Each prism shown in Fig. 11.6 below is made of unit cubes which are 1 unit long by 1 unit wide and 1 unit high. Can you explain why these are prisms?

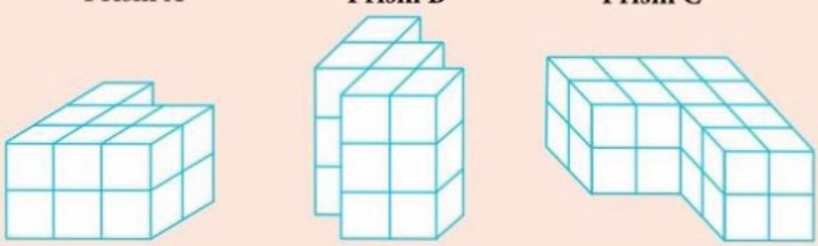



Fig. 11.6

1. Copy and complete Table 11.5.

| Prism | Area of cross-sectional base (base area) | Distance between cross-sectional bases (height) | Volume of prism (by counting unit cubes) |
|---|---|---|---|
| A | | | |
| B | | | |
| C | | | |

Table 11.5

2. What is the relationship between the volume of prism, area of cross-sectional base (base area) and distance between cross-sectional bases (height)?
3. Fig. 11.7 shows a triangular prism.
   Explain why the following formula is true:
   volume of prism = area of cross section × distance between cross-sectional bases.
   **Hint:** Joining two identical triangular prisms forms a cuboid.

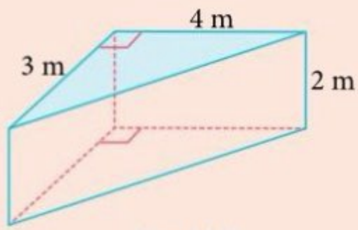



Fig. 11.7

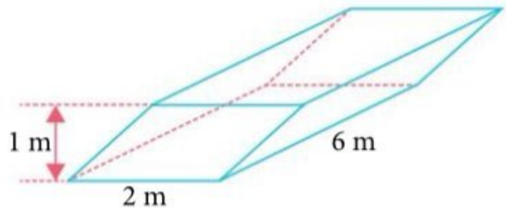
To conclude, we have:

**Volume of prism** = area of cross section × distance between cross-sectional bases
= base area × height

**Worked Example 5**

## Volume of prism

Calculate the volume of the prism, whose base is a parallelogram.




### Solution

Base area = area of parallelogram
$= 2 \times 1$
$= 2\ \text{m}^2$

Volume of the prism = base area × height
$= 2 \times 6$
$= 12\ \text{m}^3$

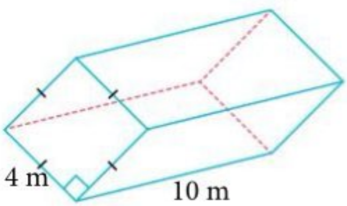
**Problem-solving Tip**

Do not confuse the base and the height of a parallelogram with the base area and the height of a prism.

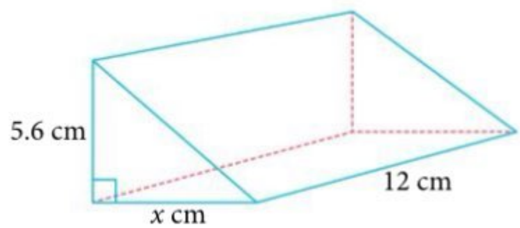
**Practise Now 5**

Similar and Further Questions
**Exercise 11B**
Questions 1(a)–(f), 2(a)–(d), 3

**1.** Find the volume of the prism.



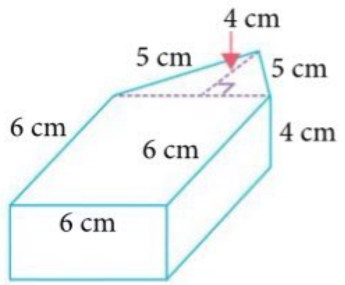

**2.** The volume of the prism is 151.2 cm³. Find the value of $x$.



## D. Surface area of prism

In Section 11.3, we have learnt that the total surface area of a cuboid is a **measure** of the total area occupied by the surface of the cuboid. This is equal to the area of all the faces of the net. We shall now extend this concept to find the total surface area of a prism.

Let us consider a pentagonal prism as shown in Fig. 11.8(**a**).
A net of the prism is shown in Fig. 11.8(**b**), where the red dotted lines indicate the folds.



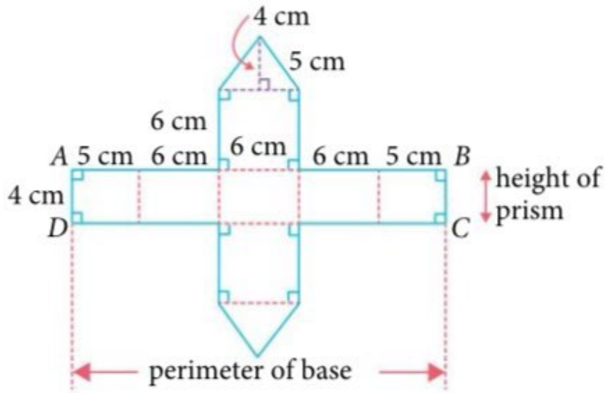




Fig. 11.8

From Fig. 11.8(**b**),

Total surface area of prism = area of rectangle $ABCD + 2 \times$ area of pentagonal base

$$= AB \times BC + 2 \times \left(6 \times 6 + \frac{1}{2} \times 6 \times 4\right)$$
$$= \text{perimeter of base} \times \text{height of prism} + 2 \times (36 + 12)$$
$$= (5 + 6 + 6 + 6 + 5) \times 4 + 2 \times 48$$
$$= 28 \times 4 + 96$$
$$= 112 + 96$$
$$= 208 \text{ cm}^2$$

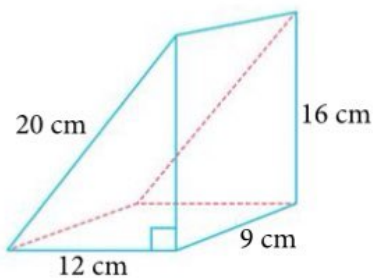
To conclude, we have:

**Total surface area of prism** = total area of the lateral faces + 2 × base area
= perimeter of the base × height + 2 × base area

**Worked Example 6**

### Surface area of prism

Calculate
(i) the volume, (ii) the total surface area,
of the prism.



**Solution**

(i) Volume of the prism $= \text{base area} \times \text{height}$

$$= \left(\frac{1}{2} \times 12 \times 16\right) \times 9 \quad \text{the base is a right-angled triangle}$$

$$= 96 \times 9$$

$$= 864\ \text{cm}^3$$

(ii) Total surface area of the prism $= \text{perimeter of base} \times \text{height} + 2 \times \text{base area}$

$$= (12 + 16 + 20) \times 9 + 2 \times 96$$

$$= 48 \times 9 + 192$$

$$= 432 + 192$$

$$= 624\ \text{cm}^2$$

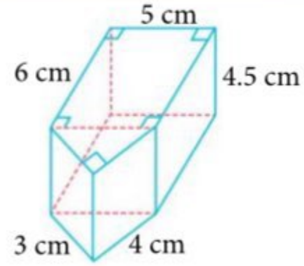
**Practise Now 6**

Similar and Further Questions
**Exercise 11B**
Questions 4(a), (b), 5

Calculate

**(i)** the volume, **(ii)** the total surface area,

of the prism.



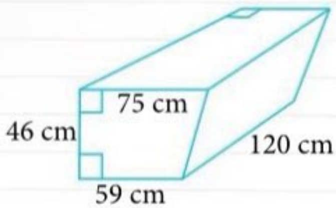

## Reflection

1. What do I already know about the volume and the surface area of cubes and cuboids that could guide my learning of the volume and the surface area of prisms?
2. What have I learnt in this section that I am still unclear of?

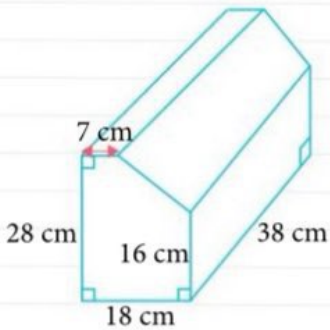
## Exercise 11B

1. By first identifying the base, find the volume of each of the following prisms.

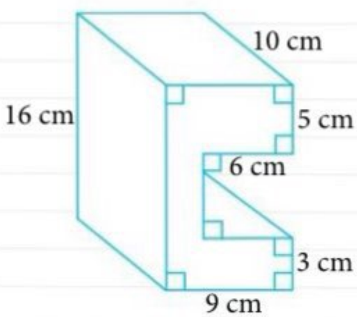
(a)



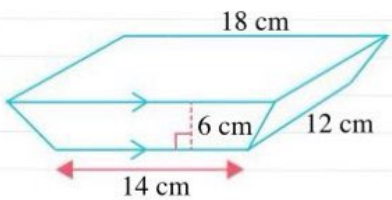

(b)



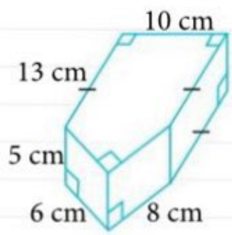

(c)



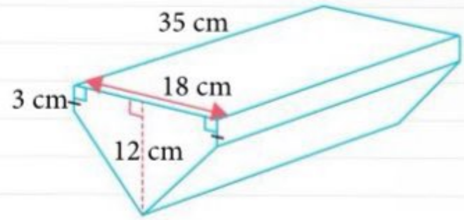

(d)



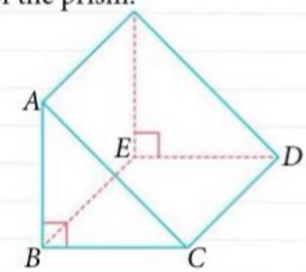

(e)




(f)



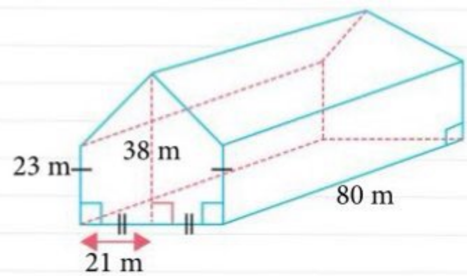

2. The figure shows a prism standing on a horizontal, rectangular base *BCDE*. $\triangle ABC$ is a vertical cross section of the prism.




Complete the table.

| | *AB* | *BC* | *CD* | Area of $\triangle ABC$ | Volume of prism |
|---|---|---|---|---|---|
| (a) | 3 cm | 4 cm | 7 cm | | |
| (b) | 9 cm | | 11 cm | 63 cm$^2$ | |
| (c) | | 15 cm | 300 cm | | 72 000 cm$^3$ |
| (d) | 24.6 cm | 7.8 cm | | | 38 376 cm$^3$ |

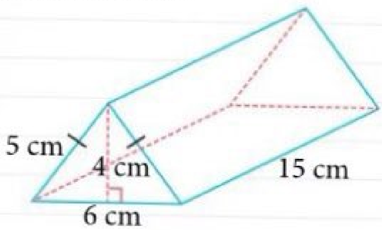
3. The figure shows an empty hall. Without taking into consideration the thickness of the walls and the roof, find the air space in the hall.



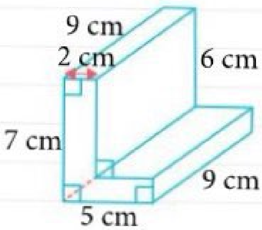
## Exercise 11B

**4.** For each of the following prisms, find
**(i)** its volume,
**(ii)** its total surface area.

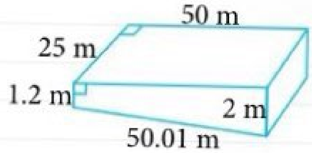
(a)




(b)




**5.** A swimming pool is 50 m long and 25 m wide. It is 1.2 m deep at the shallow end and 2 m deep at the other end.




Find
**(i)** the volume of water in the pool when it is full,
**(ii)** the area of the pool which is in contact with the water.

# 11.5 Volume and surface area of cylinders

## A. Cylinders

Fig. 11.9(**a**) shows a PKR 1 coin . A number of PKR 1 coins has been stacked vertically to form a circular tower in Fig. 11.9(**b**).

(a) (b)

Fig. 11.9

Every PKR 1 coin in the circular tower is *parallel* and *identical* to one another. What is the shape of the cross section?

This tower is in the shape of a **cylinder** (see Fig. 11.10). A cylinder has a *uniform* circular cross section. The cylinder shown in Fig. 11.10 is known as a right circular cylinder.
In this section, we will study only right circular cylinders. Therefore, the term 'cylinder' refers to a right circular cylinder.

Fig. 11.10

Thinking Time

Is a cylinder a prism? Explain your answer.

## B. Volume of cylinder

In Section 11.4, we have learnt:

**Volume of prism** = area of cross section × distance between cross-sectional bases
= base area × height

Let us learn how to find the volume of a cylinder by comparing a cylinder with a prism that has a regular polygonal base.

Investigation **Comparison between cylinder and prism**

1. Fig. 11.11 shows **(a)** a regular pentagon, **(b)** a regular hexagon, **(c)** a regular 12-gon, and **(d)** a regular 16-gon inside a circle respectively.

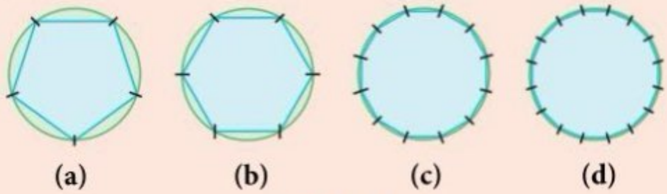



Fig. 11.11

If the number of sides of a *regular* polygon is increased indefinitely, what will the polygon start to look like?

2. Fig. 11.12**(a)**–**(c)** show a sequence of regular right prisms, i.e. right prisms with regular polygonal bases.
If the number of sides of the regular polygonal base of a prism is increased indefinitely, what will the prism start to look like?

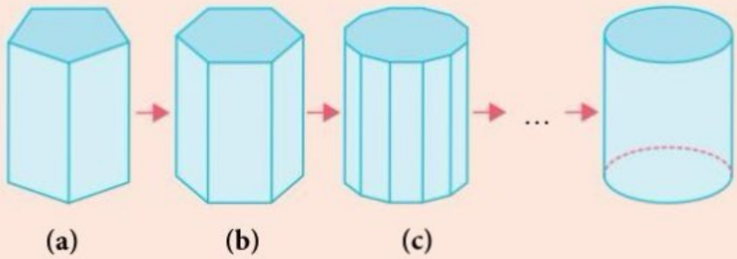



Fig. 11.12

In many ways, a cylinder is *like* a prism. However, a cylinder is *not* a prism because the base of a prism must be a polygon.

Since a cylinder is like a prism (see Fig. 11.12), by analogy, the formula for the volume of a cylinder should be the same as the formula for the volume of a prism. We have:

**Attention**

Although a circle can be visualised as a regular polygon with an infinite number of sides, a circle is *not* a polygon because a polygon must have a *finite* number of sides.

**Recall**

Area of a circle = $\pi r^2$

**Volume of cylinder**

= area of cross section × distance between cross-sectional bases

= base area × height

= $\pi r^2 h$,

where $r$ = base radius and $h$ = height of the cylinder.

**Worked Example 7**

### Volume of cylinder

The diameter of the base of a cylinder is 14 cm and its height is half of its base radius. Calculate the volume of the cylinder.

**Solution**

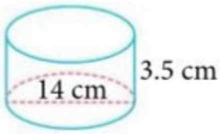
Base radius = 14 ÷ 2 = 7 cm

Height of the cylinder = $\frac{1}{2} \times 7$ = 3.5 cm

Volume of the cylinder = $\pi r^2 h$

$= \pi(7)^2(3.5)$

$= 539$ cm³ (to 3 s.f.)




**Practise Now 7**

Similar and Further Questions
Exercise 11C
Questions 1–3, 7, 8

1. The diameter of the base of a cylinder is 18 cm and its height is 2.5 times its base radius. Find the volume of the cylinder.
2. The volume of a cylindrical can of pineapple juice is 1000 cm³ and the diameter of its base is 12 cm. Find the height of the can of pineapple juice.

**Worked Example 8**

### Problem involving volume of pipe

A pipe of radius 2.8 cm discharges water at a rate of 3 m/s. Calculate the volume of water discharged per minute, giving your answer in litres.

**Solution**

Since water is discharged through the pipe at a rate of 3 m/s, i.e. 300 cm/s, in 1 second, the volume of water discharged is the volume of water that fills the pipe to a length of 300 cm as shown.


volume of water discharged in 1 second

**Just For Fun**

A glass is half-filled with water. Without measuring the volume of the water, how do you determine that the volume of the water is *exactly* half the volume of the glass?

In 1 second, volume of water discharged
$= \text{volume of pipe of length } 300 \text{ cm}$
$= \pi r^2 h$
$= \pi(2.8)^2(300)$
$= 2352\pi \text{ cm}^3$

In 1 minute, volume of water discharged $= 2352\pi \times 60$
$= 443\ 000 \text{ cm}^3$ (to 3 s.f.)
$= 443\ l$

**Practise Now 8**

Similar and Further Questions
**Exercise 11C**
Questions 9–11, 14

1. A pipe of radius 0.6 cm discharges petrol at a rate of 2.45 m/s. Find the volume of petrol discharged in 3 minutes, giving your answer in litres.

2. A pipe of diameter 0.036 m discharges water at a rate of 1.6 m/s into a cylindrical tank with a base radius of 3.4 m and a height of 1.4 m. Find the time required to fill the tank, giving your answer correct to the nearest minute.

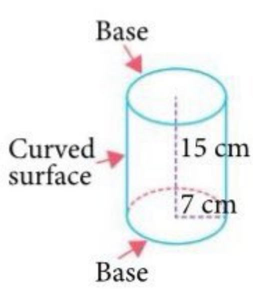
**Journal Writing**

Think about some building structures and various items that you have come across in your daily lives. Which of these are prisms and which of these are cylinders? Are you able to make sketches of them? Can you think of any reasons why they are shaped as prisms or cylinders?

## C. Surface area of cylinder

In Section 11.4, we have learnt that the total surface area of a prism is equal to the area of all the faces of the net. We shall now extend this concept to find the total surface area of a cylinder.

Fig. 11.13(**a**) shows a *closed* cylinder which has a base radius of 7 cm and a height of 15 cm. Recall from the Investigation in Section 11.2 that its corresponding net is as shown in Fig. 11.13(**b**).



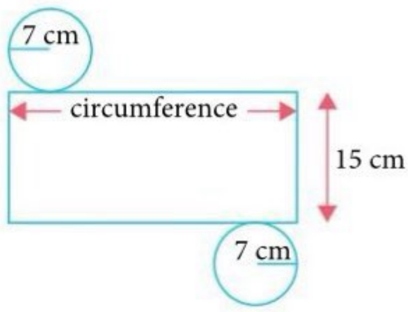




Fig. 11.13

The net of the cylinder consists of two ______ and one ______.
To find the total surface area of the cylinder, we need to know the area of the two circles and that of the rectangle.
The area of the two circles is $2\pi r^2 = 2 \times \pi \times 7^2 =$ ______.
How do we find the area of the rectangle? Notice that the length of the rectangle is the same as the circumference of the circular base, i.e. the length of the rectangle is $2\pi r = 2 \times \pi \times 7 =$ ______. Hence, the area of the rectangle is ______.
$\therefore$ the total surface area of the cylinder in Fig. 11.13(**a**) is ______.

## Thinking Time

Now consider a *closed* cylinder with a base radius of $r$ and a height of $h$.
Can you find a general formula for its total surface area? Explain your answer.

To conclude, we have:

**Total surface area of closed cylinder** $= 2 \times \text{base area} + \text{curved surface area}$
$= 2\pi r^2 + 2\pi rh$

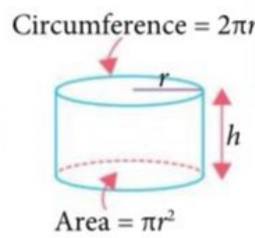



Fig. 11.14

## Class Discussion — Total surface area of other types of cylinders

We have just learnt how to find the total surface area of a closed cylinder.
Discuss how you can obtain the total outer surface area of
**(a)** an open cylinder,
**(b)** a pipe of negligible thickness,
by drawing the net of each of the two cylinders.
Hint: An open cylinder is open on one end while a pipe has two open ends.

## Worked Example 9

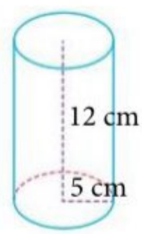
### Surface area of cylinder

A closed metal cylindrical container has a base radius of 5 cm and a height of 12 cm.
**(i)** Calculate the total surface area of the container.
The lid of the container is now removed. The exterior of the container, including the base, is painted green.
**(ii)** Express the area of the container that is painted as a percentage of the total surface area found in part **(i)**.

**Solution**

**(i)** Total surface area of the container

$$
\begin{aligned}
&= 2\pi r^2 + 2\pi rh \\
&= 2\pi(5)^2 + 2\pi(5)(12) \\
&= 50\pi + 120\pi \\
&= 170\pi \\
&= 534 \text{ cm}^2 \text{ (to 3 s.f.)}
\end{aligned}
$$




**(ii)** Area of the container that is painted $= \pi r^2 + 2\pi rh$

an open cylinder has only one base and a curved surface

$$
\begin{aligned}
&= \pi(5)^2 + 2\pi(5)(12) \\
&= 25\pi + 120\pi \\
&= 145\pi
\end{aligned}
$$

Required percentage $= \dfrac{145\pi}{170\pi} \times 100\%$
$= 85.3\%$ (to 3 s.f.)

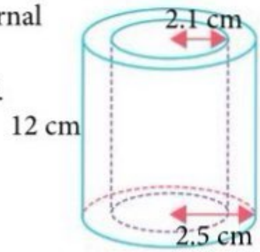
**Problem-solving Tip**
Leave intermediate working and answers in terms of $\pi$ for better accuracy.

Practise Now 9

Similar and Further Questions
Exercise 11C
Questions 4(a)–(c), 5(a)–(d), 6, 12, 13, 15, 16

1. A closed metal cylindrical can has a base radius of 3.5 cm and a height of 10 cm.
   **(i)** Find the total surface area of the can.
   The lid of the can is now removed. The exterior of the container, including the base, is painted purple.
   **(ii)** Find the ratio of the area of the can that is painted, to the total surface area found in part **(i)**.

2. The figure shows a section of a steel pipe of length 12 cm. The internal and external radii of the pipe are 2.1 cm and 2.5 cm respectively.
   **(i)** Show that the area of the cross section of the pipe is $1.84\pi$ cm$^2$.
   **(ii)** Find the internal curved surface area of the pipe.
   **(iii)** Hence, find the total surface area of the pipe.




## Reflection

1. What do I already know about the volume and the surface area of prisms that could guide my learning of the volume and the surface area of cylinders?
2. What have I learnt in this section that I am still unclear of?

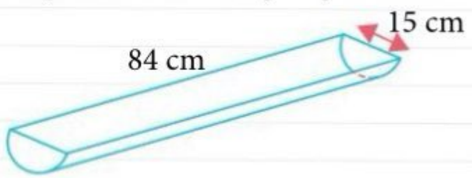
Basic | Intermediate | Advanced

## Exercise 11C

1. The diameter of the base of a cylinder is 0.4 m and its height is $\frac{3}{4}$ of its base radius. Find the volume of the cylinder, giving your answer in litres.

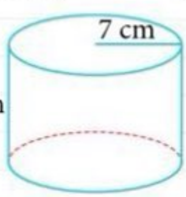
2. 150 litres of water are poured into a cylindrical drum of diameter 48 cm. Find the depth of water in the drum.

3. The figure shows a drinking trough in the shape of a half-cylinder. Find its capacity in litres.

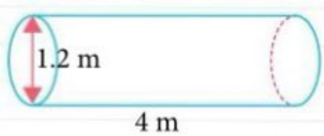



4. For each of the following closed cylinders, find
   **(i)** its volume, **(ii)** its total surface area.

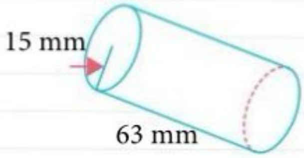
   **(a)**




   **(b)**




   **(c)**



## Exercise 11C

5. Complete the table for each closed cylinder.

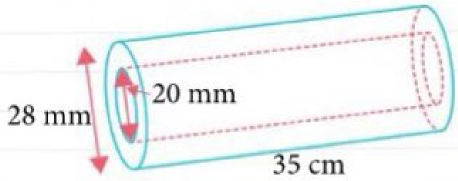
| | Diameter | Radius | Height | Volume | Total surface area |
|---|---|---|---|---|---|
| (a) | | | 14 cm | 704 $cm^3$ | |
| (b) | | | 20 cm | 12 320 $cm^3$ | |
| (c) | 4 cm | | | 528 $cm^3$ | |
| (d) | | 4 m | | 1056 $m^3$ | |

6. In a toy factory, 200 wooden closed cylinders of diameter 35 mm and height 7 cm have to be painted. What is the total surface area, in $cm^2$, that needs to be painted? (Take $\pi$ to be 3.142.)

7. A tank in the shape of a cylinder of diameter 2.4 m and height 6.4 m contains oil to the brim. Find the number of complete cylindrical containers of base radius 8.2 cm and height 28 cm which can be filled by the oil in the tank.

8. A copper cylindrical rod of diameter 14 cm and length 47 cm is melted and recast into a wire of diameter 8 mm. Find the length of the wire, giving your answer in metres.

9. The figure shows a metal pipe of length 35 cm. The internal and external diameters of the pipe are 20 mm and 28 mm respectively. Find the volume of metal used in making the pipe, giving your answer in cubic centimetres.




10. A pipe of diameter 2.4 cm discharges water at a rate of 2.8 m/s. Find the volume of water discharged in half an hour, giving your answer in litres.

11. A pipe of diameter 64 mm discharges water at a rate of 2.05 mm/s into an empty cylindrical tank of diameter 7.6 cm and height 2.3 m. Find the time required to fill the tank, giving your answer correct to the nearest minute.

12. An open rectangular tank of length 18 cm and breadth 16 cm contains water to a depth of 13 cm. The water is poured into a cylindrical container of diameter 17 cm. Find
    - (i) the volume of water in the tank,
    - (ii) the height of water in the cylindrical container,
    - (iii) the surface area of the cylindrical container that is in contact with the water.

13. A closed steel cylindrical container has a diameter of 186 mm and its height is $\frac{1}{3}$ of its base radius.
    - (i) Find the total surface area of the container, giving your answer in square centimetres.

    The lid of the container is now removed.
    The exterior of the container, including the base, is painted indigo.
    - (ii) Express the area of the container that is painted as a fraction of the total surface area found in part **(i)**.

14. 69.8 mm of rainfall was recorded over an area of 32 $km^2$ one day. If the rainwater falling onto the area was drained through two channels each with a cross-sectional area of 18 $m^2$ at a rate of 26.4 m/s, find the time, to the nearest minute, required for the channels to drain off the rain.

## Exercise 11C

15. An open rectangular tank of length 32 cm and breadth 28 cm contains water to a depth of 19 cm. 2580 circular metal discs of diameter 23 mm and height 4 mm are dropped into the tank. Find
    (i) the new height of water in the tank,
    (ii) the surface area of the tank that is in contact with the water after the discs have been added.

16. A pipe of length 15 cm has an internal radius of 3.8 cm. The thickness of the pipe is 0.8 cm. Find the total surface area of the pipe.

# 11.6 Volume and surface area of composite solids

In this section, we shall learn how to solve problems involving the volume and surface area of composite solids.

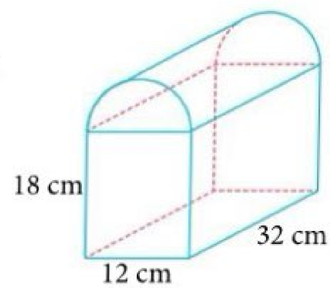
**Worked Example 10**

### Volume and surface area of composite solid

The figure shows a glass block made up of a rectangular prism of dimensions 32 cm by 12 cm by 18 cm and half a cylinder with a diameter of 12 cm. Calculate

(i) the volume, (ii) the total surface area,

of the glass block.




**Solution**

(i) **Method 1:**

Volume of the rectangular prism $= 32 \times 12 \times 18$

$= 6912 \text{ cm}^3$

Volume of the half-cylinder $= \frac{1}{2}\pi(6)^2(32)$

$= 576\pi \text{ cm}^3$

$\therefore$ Volume of the glass block $= 6912 + 576\pi$

$= 8720 \text{ cm}^3$ (to 3 s.f.)

**Method 2:**

Cross-sectional area of the glass block $= 12 \times 18 + \frac{1}{2}\pi(6)^2$

$= (216 + 18\pi) \text{ cm}^2$

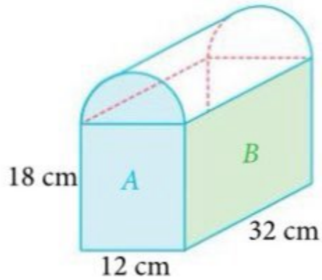
$\therefore$ Volume of the glass block $= (216 + 18\pi) \times 32$

$= 8720 \text{ cm}^3$ (to 3 s.f.)

**Reflection**

Which of the two methods do you prefer? Why?

**(ii)**




Total surface area of the glass block

$= 2 \times$ area of region $A + 2 \times$ area of region $B$ + area of base + curved surface area

$$= 2\times\left[12\times18+\frac{1}{2}\pi(6)^2\right]+2\times32\times18+32\times12+\frac{1}{2}\times2\pi(6)(32)$$

$= 432 + 36\pi + 1152 + 384 + 192\pi$

$= 1968 + 228\pi$

$= 2680 \text{ cm}^2$ (to 3 s.f.)

**Attention**

Do not find the sum of the total surface area of the rectangular prism and that of the half-cylinder to obtain the total surface area of the glass block. Why?

**Practise Now 10**

Similar and Further Questions

**Exercise 11D**

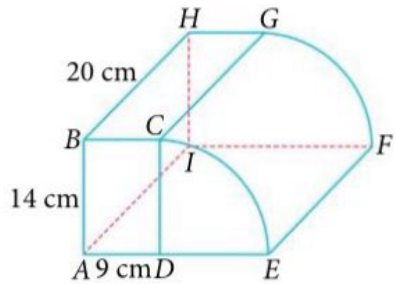
Questions 1-3, 4-7, 9

**1.** The figure shows a closed container of a uniform cross section, which consists of a rectangle $ADCB$ and a quadrant $DEC$ of a circle with centre $D$. Given that $AB = 14$ cm, $AD = 9$ cm and $AI = BH = CG = EF = 20$ cm, find

**(i)** the volume,

**(ii)** the total surface area,

of the container.

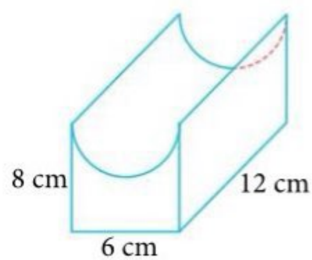



**2.** The figure shows a solid rectangular prism of dimensions 12 cm by 6 cm by 8 cm, with a half-cylinder of diameter 6 cm horizontally carved out of it. Find

**(i)** the volume,

**(ii)** the total surface area,

of the solid.




## Reflection

1. What is the difference between the volume and the surface area of a solid?
2. **(a)** What do I have to do to find the volume of a composite solid?
   **(b)** What do I have to do to find the surface area of a composite solid?
   **(c)** Are my approaches to parts **(a)** and **(b)** the same?
3. What have I learnt in this section or chapter that I am still unclear of?

# 11.7 Symmetry in right prisms and cylinders

## A. Plane symmetry

In Book 1, we have learnt that a plane figure that exhibits *line symmetry* can be divided by a line into two halves that are mirror images along the line. A rectangle is an example of a plane figure with line symmetry. It has two lines of *symmetry* as shown in Fig. 11.15.

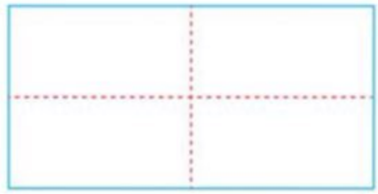


Fig. 11.15

In three dimensions, some solids can be divided by a *plane* into halves that are *symmetrical about the plane*. They exhibit **plane symmetry**. In Fig. 11.16, a cuboid with rectangular faces is cut symmetrically by two different planes. Each plane is called a **plane of symmetry** of the cuboid.

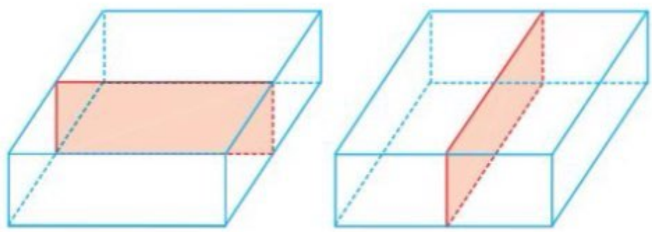



Fig. 11.16

Does the cuboid have other planes of symmetry? Sketch them in Fig. 11.16(**b**).

## B. Rotational symmetry

In Book 1, we learnt that a plane figure has *rotational symmetry* if it maps onto itself *more than once* during a full 360° rotation about its centre. For instance, the rectangle $ABCD$ maps onto itself twice in a full rotation about its centre $O$, as shown in Fig. 11.17. Thus, it has rotational symmetry with an *order of rotational symmetry of 2*.

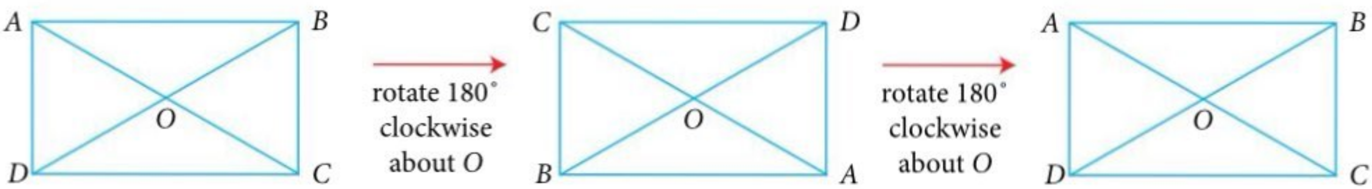



Fig. 11.17

In three dimensions, solids rotate about an *axis*. In Fig. 11.18, a cuboid with rectangular face $ABCD$ is rotated about the axis $OP$ passing through the centres of $ABCD$ and its parallel face.

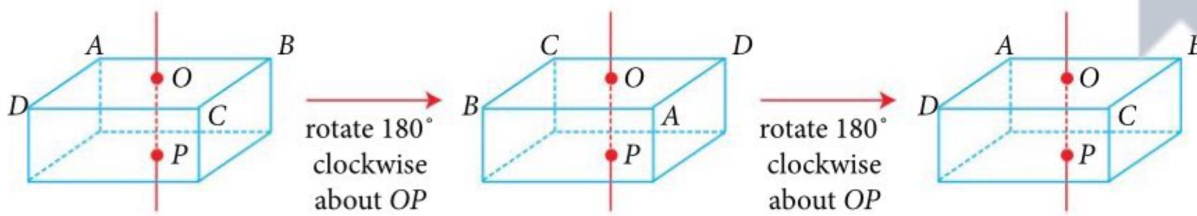



Fig. 11.18

Observe that the cuboid maps onto itself *twice* in a full rotation about $OP$. We say that the cuboid has a *rotational symmetry of order 2* about $OP$. $OP$ is the *axis of symmetry*.

Aside from $OP$, does the cuboid in Fig. 11.18 have other axes of symmetry? Draw and label these axes in Fig. 11.18.

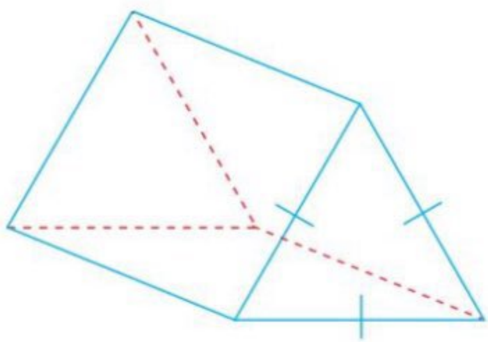
**Worked Example 11**

### Plane and rotational symmetries in equilateral triangular prism

The figure shows a prism with an equilateral triangular base.

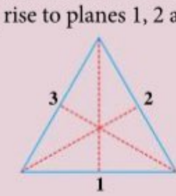
**(a)** Sketch the planes of symmetry of the prism.

**(b)** Sketch the axes of symmetry and the order of rotational symmetry of the prism about each axis.



**Problem-solving Tip**

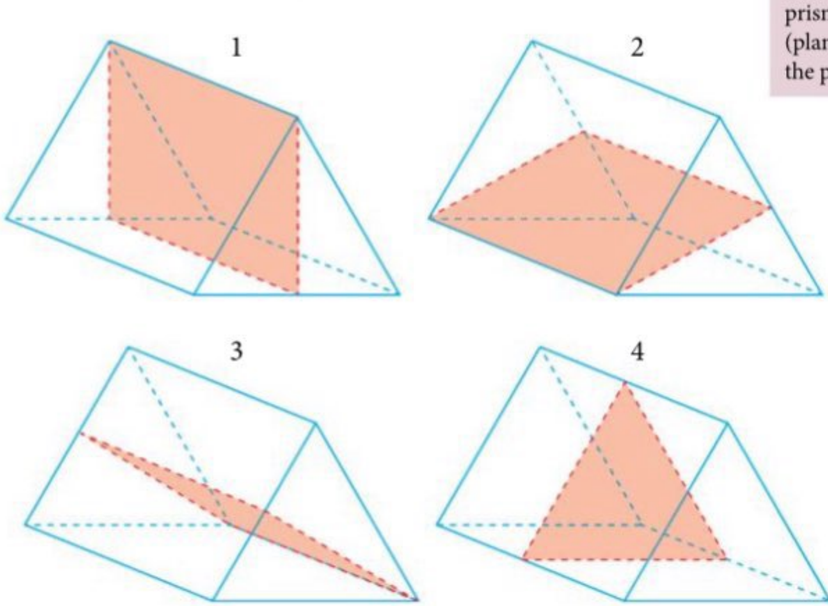
For a prism, every pair of corresponding lines of symmetry of the bases form the edges of a plane. In this worked example, there are 3 lines of symmetry (lines 1, 2 and 3) in the base, giving rise to planes 1, 2 and 3.




Since the solid is also a right prism, another plane of symmetry (plane 4) is the cross section of the prism.

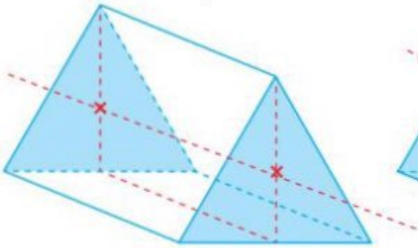
**Solution**

**(a)** The four planes of symmetry are:



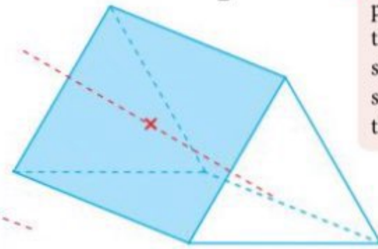
**(b)** The axes of symmetry are:

1



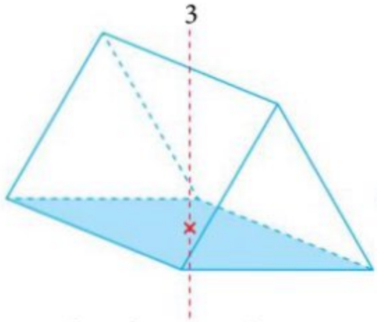
Order of rotational symmetry = 3

2



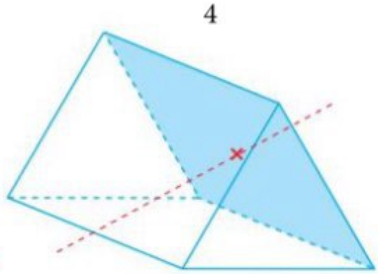
Order of rotational symmetry = 2

3



Order of rotational symmetry = 2

4



Order of rotational symmetry = 2

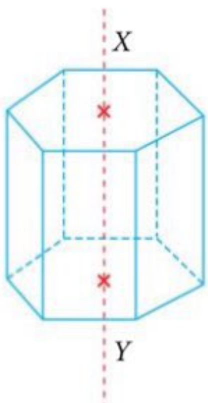
**Attention**

Each of the axes is perpendicular to, and passes through, the centre of the shaded planes. Thus, the axis of symmetry in 2 to 4 also passes through the edge of the prism.

**Practise Now 11**

Similar and Further Questions
**Exercise 11D**
Questions 8(a), (b), 10

The figure shows a right prism with a regular hexagonal base with an axis of symmetry $XY$.




**(a)** Find the number of planes of symmetry of the prism.
**(b)** State the order of rotational symmetry about the axis $XY$.

## Investigation — Plane and rotational symmetries of a cylinder

Fig. 11.19(**a**) and (**b**) show a plane of symmetry and an axis of symmetry of a cylinder respectively.

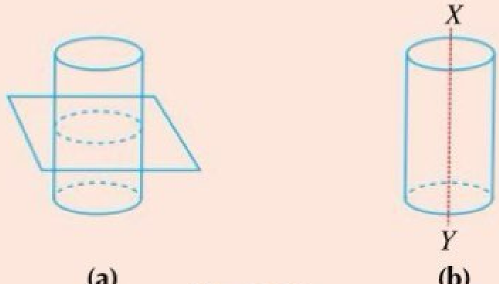



Fig. 11.19

The base of a cylinder is a circle.

1. (**a**) How many lines of symmetry does a circle have? What do these lines of symmetry have in common?
   (**b**) What is the order of rotational symmetry about the centre of a circle?

2. From Fig. 11.19, find
   (**a**) the number of planes of symmetry of the cylinder,
   (**b**) the order of rotational symmetry about the axis of symmetry.

From the above Investigation, we see that a cylinder has an infinite number of planes of symmetry. It also has an infinite order of rotational symmetry about an axis of symmetry of the cylinder. In Book 3, we will learn that this is also true for right cones and spheres. What do these solids have in common?

## Thinking Time

In a cuboid with rectangular faces, a plane joining the vertices $WXYZ$ cuts the cuboid diagonally as shown in Fig. 11.20(**a**).

(**a**) Explain why $WXYZ$ is not a plane of symmetry of the cuboid.

An axis of rotation joins the vertices $X$ and $Z$ of the cuboid as shown in Fig. 11.20(**b**).

(**b**) Explain if the cuboid has rotational symmetry about the axis.

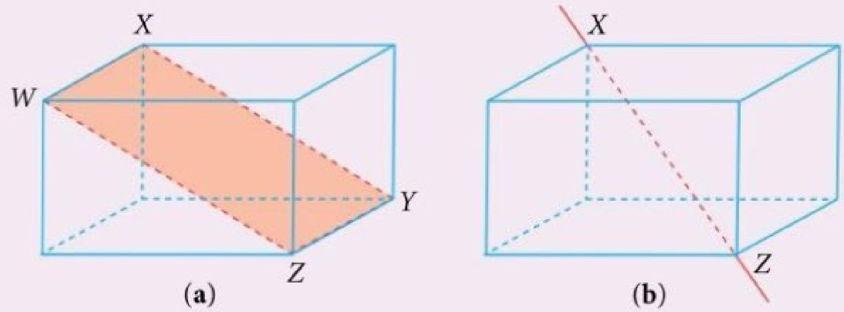
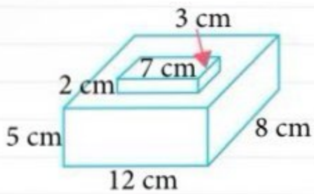



Fig. 11.20

Basic | Intermediate | Advanced

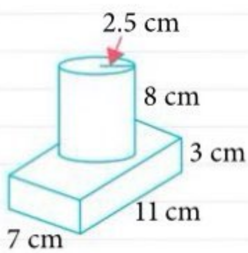
## Exercise 11D

**1.** A solid is made up of a cuboid with dimensions 7 cm by 3 cm by 2 cm, and another bigger cuboid with dimensions 12 cm by 8 cm by 5 cm.



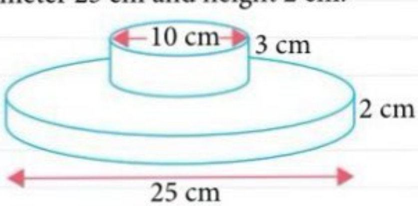

Find
**(i)** the volume,
**(ii)** the total surface area,
of the solid.

**2.** A solid consists of a cylinder of a base radius of 2.5 cm and a height of 8 cm, and a cuboid with dimensions 11 cm by 7 cm by 3 cm.



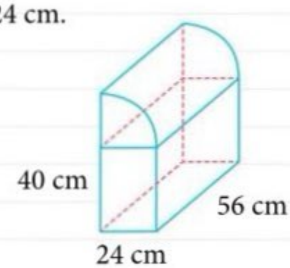

Find
**(i)** the volume,
**(ii)** the total surface area,
of the solid.

**3.** A solid is made up of a cylinder of diameter 10 cm and height 3 cm, and another bigger cylinder of diameter 25 cm and height 2 cm.



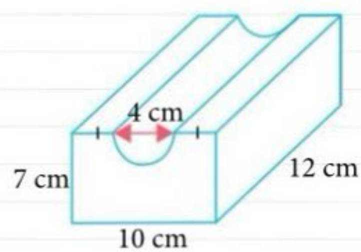

Find
**(i)** the volume,
**(ii)** the total surface area,
of the solid.

**4.** The figure shows a glass block made up of a rectangular prism with dimensions 56 cm by 24 cm by 40 cm and one-quarter of a cylinder with a base radius of 24 cm.



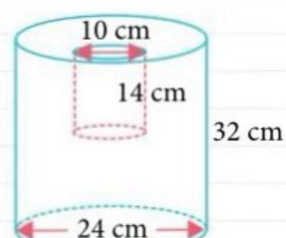

Find
**(i)** the volume,
**(ii)** the total surface area,
of the glass block.

**5.** The figure shows a solid cuboid of dimensions 12 cm by 10 cm by 7 cm, with a half-cylinder of diameter 4 cm horizontally carved out of it.




Find
**(i)** the volume,
**(ii)** the total surface area,
of the solid.

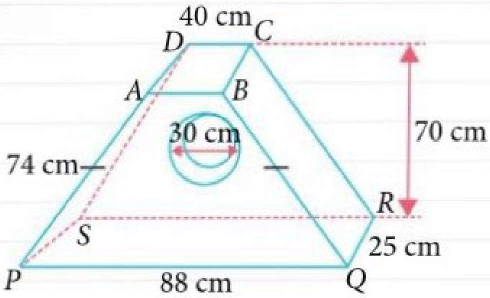
**6.** The figure shows a solid cylinder of diameter 24 cm and height 32 cm. A cylinder of diameter 10 cm and height 14 cm is removed from the original cylinder.




**(i)** Find the volume of the remaining solid.
The remaining solid is to be painted on all its surfaces.
**(ii)** Find the area that will be covered in paint.

## Exercise 11D

7. The figure shows a solid trapezoidal prism with a cylindrical hole of diameter 30 cm.




Given that $AB = DC = 40$ cm, $PQ = SR = 88$ cm, $PS = QR = 25$ cm, $AP = BQ = 74$ cm and the height of the solid is 70 cm, find

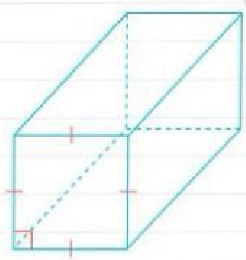
**(i)** the volume,
**(ii)** the total surface area,

of the solid.

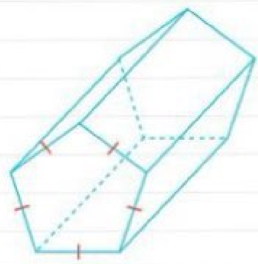
8. For each of the following prisms,

**(i)** state the total number of planes of symmetry,
**(ii)** identify the axes of symmetry and state the order of symmetry about each axis.

**(a)**



**(b)**

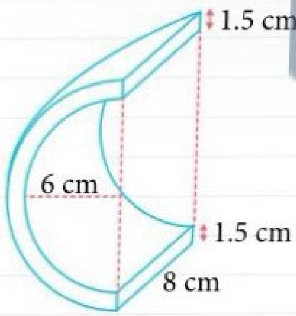


9. A C-shaped solid with an internal radius of 6 cm and a uniform thickness of 1.5 cm has a height of 8 cm.

Find

**(i)** the volume,
**(ii)** the total surface area,

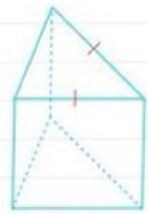
of the solid.



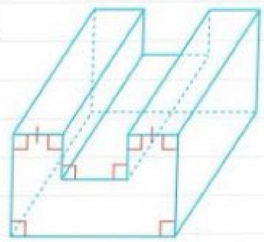

10. Copy each of the following prisms and

**(i)** sketch the planes and axes of symmetry of the prism,
**(ii)** state the order of rotational symmetry about each axis of symmetry.

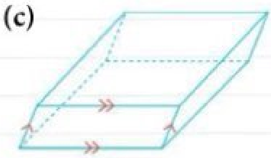
**(a)**



**(b)**



**(c)**



## Looking Back

A **measure** allows us to quantify a geometrical attribute of an object for study. In order for us to make sense of this quantity, standard units of measurement are used to make comparisons more convenient.

In this chapter, we have looked at two different measures: volume and surface area. Throughout history, geometry was developed to solve practical problems because many real-world objects can be **modelled** using geometrical figures. For example, the ancient Egyptians needed to know the amount of building materials required to construct the pyramids. Hence, different ways to compute or estimate the volumes or surface areas were developed. Mathematics offers a precise and concise way of doing this when certain attributes are known. The formulae we have learnt in this chapter form the foundation for finding the volumes and surface areas of other more complicated solids.

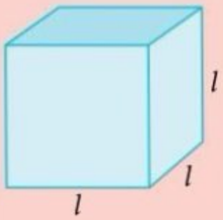
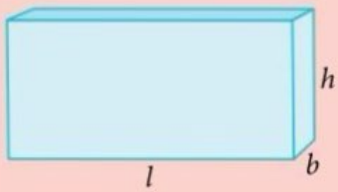
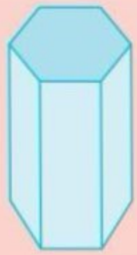
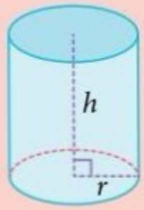
## Summary

1. **Conversion of units**

   1 ml = 1 $cm^3$
   1 $l$ = 1000 ml = 1000 $cm^3$
   1 $m^3$ = 1 000 000 $cm^3$

2. A **prism** has a uniform polygonal cross section. A cube is an example of a prism and a cylinder is a non-example of a prism.
   - Give another example and another non-example of a prism.

3. **Volume, total surface area and symmetry of solids**

| Name | Cube | Cuboid | Prism | Closed cylinder |
|---|---|---|---|---|
| **Figure** |  |  |  |  |
| **Volume** | $l^3$ | $l \times b \times h$ | Area of cross section × height<br>= base area × height | $\pi r^2 h$ |
| **Total surface area** | $6l^2$ | $2(lb + lh + bh)$ | Total area of the lateral faces<br>+ 2 × base area<br>= perimeter of the base<br>× height + 2 × base area | $2\pi r^2 + 2\pi rh$ |
| **Number of planes of symmetry** | 9 | 3<br>(for cuboids with rectangular faces) | $n + 1$<br>(for right prisms with regular $n$-sided<br>polygonal base) | infinite |
| **Number of axes of symmetry** | 13 | 3<br>(for cuboids with rectangular faces) | $n + 1$<br>(for right prisms with regular<br>$n$-sided polygonal base) | infinite |

- Think of an example of a right prism with regular polygonal base. Identify the axes of symmetry and state the order of rotational symmetry about each axis.

# CHAPTER 12

## Introduction to Set Notation and Probability

Is it going to rain today? Which football team is likely to win the match tonight? What are my chances of being selected to represent the school in the basketball match? These questions are just some of the many scenarios in real life where we encounter situations involving chance and uncertainty. But how do we know whether something is going to happen or not? How do we 'measure' that likelihood? In this chapter, we will begin our study of an interesting topic in mathematics: probability, which allows us to measure or quantify the likelihood of an event happening. In other words, we are going to assign a number to tell us how likely or unlikely an event is going to occur.

### Learning Outcomes

What will we learn in this chapter?

- What sets and Venn diagrams are
- How to use set language and notations to describe relationships in a variety of contexts
- How to use Venn diagrams to represent a set, the universal set, the complement of a set, and a subset
- What probability is
- What the sample space of a probability experiment is
- How to find the probability of a single event
- How to solve problems involving the probability of single events
- Why understanding probability can be useful in helping us make decisions about uncertain events

## Introductory Problem

We frequently make statements such as:

- "It is *unlikely* that I will go to the concert."
- "I am *certain* that I saw Sara across the street yesterday."
- "There is a 50 : 50 *chance* of our school winning the National Inter-School Basketball Championship."
- "It is *likely* that I will go out later in the afternoon."
- "I cannot *predict* whether I will obtain a 'six' in my next roll of the die."
- "It will *probably* rain today."

Often, we are uncertain whether an **event** will occur (this also implies that the event has not happened yet).
We can discuss the chance of occurrence of an event that *has not occurred*.
If an event has already happened, it is pointless to say that it has a 100% chance of occurrence.

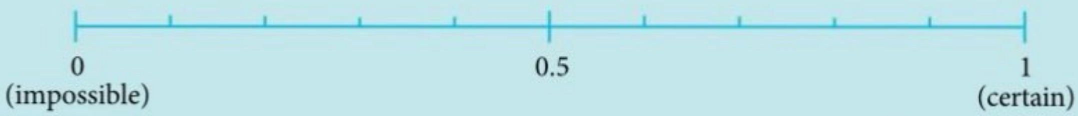
**1.** Each of the following events $A$ to $D$ may or may not happen.

Event $A$: Today is Thursday, so it will be Friday tomorrow.
Event $B$: It will rain in Pakistan tomorrow.
Event $C$: The sun will rise from the west tomorrow.
Event $D$: I will obtain a 'tail' when I toss an ordinary coin.

Mark each letter on the line below to show the likelihood of them occurring.


**2.** We can use values between 0 and 1 inclusive to measure the chance of an event occurring, where an impossible event takes on the value 0 and a certain event takes on the value 1.
If there is a 50 : 50 chance (or 50% chance) that an event will occur, what value does it take?

**3.** Think of any four events. Write down the events and mark out each of them on the number line below based on the estimated chance of occurrence.




The measure of chance, which takes on values between 0 and 1 inclusive, is known as **probability**.
In this chapter, we will learn how to calculate the probability of a single event. Let us first learn a type of mathematical notation called the **set notation**. This **notation** will allow us to express ideas in probability in a concise manner.

# 12.1 Sets and set notations

## A. Sets

In everyday life, we often encounter a collection of objects such as a pile of books, a bunch of keys, a team of players and a school of dolphins. In English, we use different terms such as 'pile', 'bunch', 'team' and 'school' to describe different collections of objects.

In mathematics, we use the term 'set' to describe any collection of *well-defined* and *distinct* objects.

For example, if we let $S$ be the set of all the vowels of the English alphabet, we can list all the members (or what we call **elements**) of $S$ in *set notation* like this:

$S = \{a, e, i, o, u\}$.

Since the letter 'a' is an element of $S$, we write: $a \in S$, where the notation $\in$ denotes 'is an element of'.

Since the letter 'b' is not an element of $S$, we write: $b \notin S$, where the notation $\notin$ denotes 'is not an element of'.

Since there are 5 elements belonging to $S$, we write: $\mathbf{n(S)} = 5$, where the notation $n(S)$ denotes 'the number of elements belonging to the set $S$'.

**Big Idea**

**Notations**

By convention, we use capital letters to represent a set, e.g. $S$, and small letters to represent the elements in a set, e.g. e. Sometimes, the elements can be in capital letters.

We also use other notations such as $\in$ and $\notin$ to represent relationships in a concise and precise manner.

In this chapter, we will learn a few more set notations.

$\in$ denotes 'is an element of'.

$\notin$ denotes 'is not an element of'.

$n(S)$ denotes 'the number of elements belonging to the set $S$'.

**Practise Now 1A**

Similar and Further Questions

**Exercise 12A**

Questions 1, 5, 6, 9(a)–(d)

1. $A$ is the set of even positive integers less than 10.
   **(i)** List all the elements of $A$ in set notation.
   **(ii)** State whether each of the following statements is true or false.
   **(a)** $8 \in A$ **(b)** $7 \notin A$
   **(c)** $10 \in A$ **(d)** $0 \notin A$
   **(iii)** Using the notation $\in$ or $\notin$, describe whether each of the following numbers is an element of, or is not an element of, $A$.
   **(a)** 2 **(b)** 5
   **(c)** 11 **(d)** 6
   **(iv)** State the value of $n(A)$.
2. Given that $B = \{3, 6, 9, 12, 15, \ldots, 30\}$, find the value of $n(B)$.

**Information**

The set of all positive multiples of 3, i.e. $\{3, 6, 9, 12, 15, \ldots\}$, contains an infinite number of elements and it is called an *infinite* set. On the other hand, $B = \{3, 6, 9, 12, 15, \ldots, 30\}$ is a *finite* set.

## Class Discussion

**Well-defined and distinct objects in a set**

1. Let $H$ be the collection of all the tall students in a class. Is $H$ a set?
   Hint: A set is a collection of *well-defined* objects. Is $H$ well-defined?
2. Let $T$ be a collection of 2 identical pens. How should we list the elements of $T$?
   $\{P, P\}$, $\{P\}$ or $\{P_1, P_2\}$?
   Hint: How many elements does $T$ have? A set is a collection of *distinct* objects. Are the elements of $\{P, P\}$ distinct?
3. Let $S$ be the set of letters used to form the word 'CLEVER'. How should we list the elements?
   Hint: Are there repeated letters in the word 'CLEVER'?

In general, a set is **not** any collection of objects. The objects in a set must be *well-defined* and *distinct*.
In Question 2 of the above Class Discussion, we cannot write $T = \{P, P\}$ because the elements of $\{P, P\}$ are not distinct. As there are 2 distinct elements belonging to the set $T$ (the 2 identical pens are distinct), we have to write $T = \{P_1, P_2\}$.
In Question 3 of the above Class Discussion, the same letter 'E' is used twice when forming the word 'CLEVER'. Therefore, $S = \{C, L, E, V, R\}$.

**Attention**

Although the 2 pens are identical, they are *still distinct*. To understand this, consider a pair of identical twins, Nadia and Sara. Nadia and Sara are identical. However, Nadia is not Sara, and Sara is not Nadia, i.e. each of them is distinct.

## B. Describing a set

There are a few ways to describe a set.

- Describing a set *in words*, e.g. $S$ is the set of positive even integers less than 10.
- *Listing* all the elements of a set in set notation, e.g. $S = \{2, 4, 6, 8\}$.
- *Describing* the elements of a set in set notation, e.g. $S = \{x : x \text{ is a positive even integer less than } 10\}$.
  We read this as "$x$ is such that $x$ is a positive even integer less than 10."

**Attention**

Do *not* write {$x$ is a positive even integer less than 10} or {primes}. These are not proper set notations.

## C. Equal sets

Two sets $A$ and $B$ are **equal** if they contain exactly the same elements. We write $A = B$. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 1, 3\}$, then all the elements of $A$ and of $B$ are the same, i.e. $A = B$, although the order of the elements of $A$ is different from that of $B$.

**Worked Example 1**

### Listing elements in set notation

It is given that $A = \{x : x \text{ is a positive integer such that } 2 \leqslant x < 11\}$ and
$B = \{x : x \text{ is a positive integer between 3 and 11 inclusive}\}$.

**(i)** List all the elements of $A$ and of $B$ in set notation.
**(ii)** Is $n(A) = n(B)$? Explain.
**(iii)** Is $A = B$? Explain.

**Solution**

**(i)** $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$B = \{3, 4, 5, 6, 7, 8, 9, 10, 11\}$

**(ii)** Yes, $n(A) = n(B)$ because $A$ and $B$ have the same number of elements, which is 9.

**(iii)** No, $A \neq B$ because $A$ and $B$ have different elements, e.g. $2 \in A$ but $2 \notin B$.

**Problem-solving Tip**

**(i)** If an integer is *between* 3 and 11, the integer cannot be equal to 3 or 11.
If an integer is between 3 and 11 *inclusive*, the integer can be 3 or 11.

**Information**

For ease of reference, we should list the elements
**(i)** in ascending order for numbers,
**(ii)** in alphabetical order for letters, or
**(iii)** according to the given order.
E.g. if $S$ is the set of letters used to form the word 'MATH', listing the elements according to the given order is often clearer and more convenient: $S = \{M, A, T, H\}$.

**Practise Now 1B**

Similar and Further Questions
**Exercise 12A**
Questions 2(a)–(d), 3(a)–(d), 7, 10(a)–(d)

It is given that $C = \{x : x \text{ is a positive integer between 10 and 18}\}$ and
$D = \{x : x \text{ is a positive integer such that } 10 \leqslant x < 18\}$.

**(i)** List all the elements of $C$ and of $D$ in set notation.
**(ii)** Is $n(C) = n(D)$? Explain.
**(iii)** Is $C = D$? Explain.

**Big Idea**

**Measures**
The number of elements belonging to a set is a measure of the size of the set. When comparing the sizes of two sets, we usually state if one set has more or fewer elements than the other set.

**Thinking Time**

**(a)** If $A$ and $B$ are two sets such that $A = B$, is $n(A) = n(B)$? Explain.
**(b)** If $A$ and $B$ are two sets such that $n(A) = n(B)$, is $A = B$? If not, give a counterexample.

## D. Empty sets

Consider the sets $A = \{0, 1, 2\}$, $B = \{0\}$ and $C = \{\ \}$.
The set $A$ contains 3 elements: 0, 1 and 2.
The set $B$ contains 1 element: 0.
The set $C$ does not contain any elements, so $C$ is called an **empty set** (or a **null set**).
We use the symbol **Ø** (pronounced as 'phi') to denote an empty set, i.e. $C = \varnothing$.

**Attention**

The set $\{\varnothing\}$ is not an empty set. It is a set containing one element, which is the symbol $\varnothing$.

**Worked Example 2**

### Identifying empty set

It is given that $A$ is the set of vowels used to form the word 'RHYTHM',

$B = \{x : x \text{ is a positive integer less than } 1\}$, and

$C = \{0\}$.

**(i)** Which sets are empty sets? Write the empty sets in set notation.

**(ii)** Are $A$ and $B$ equal sets? Explain.

**(iii)** Are $B$ and $C$ equal sets? Explain.

**Solution**

**(i)** $A$ and $B$ are empty sets, i.e. $A = \varnothing$ and $B = \varnothing$.

**(ii)** $A$ and $B$ are equal sets because both of them are empty sets.

**(iii)** $B$ and $C$ are not equal sets because $B$ is an empty set whereas $C$ is not an empty set.

**Problem-solving Tip**

**(i)** In general, we do not write $A = \{\ \}$ since we already have the symbol $\varnothing$ to denote an empty set.

**(ii)** *All empty sets are equal*, regardless of the context.

**Practise Now 2**

Similar and Further Questions
Exercise 12A
Questions 4(a)–(d), 8, 11(a)–(d)

It is given that $P$ is the set of consonants used to form the word 'IAO',

$Q = \{x : x \text{ is a prime number less than } 2\}$, and

$R = \{\varnothing\}$.

**(i)** Which sets are empty sets? Write the empty sets in set notation.

**(ii)** Are $P$ and $Q$ equal sets? Explain.

**(iii)** Are $Q$ and $R$ equal sets? Explain.

**Problem-solving Tip**

**(i)** There are 26 letters in the English alphabet. Five of them are vowels, namely 'a', 'e', 'i', 'o' and 'u'. The rest are consonants.

## Reflection

1. How do I determine whether two sets are equal?
2. What have I learnt in this section that I am still unclear of?

Basic | Intermediate | Advanced

## Exercise 12A

**1.** $A$ is the set of odd positive integers less than 11.

**(i)** List all the elements of $A$ in set notation.

**(ii)** State whether each of the following statements is true or false.

**(a)** $1 \in A$ **(b)** $4 \notin A$

**(c)** $0 \in A$ **(d)** $11 \notin A$

**(iii)** State the value of $n(A)$.

**2.** For each of the following sets, list all its elements and state the number of elements in set notation.

**(a)** $B = \{x : x \text{ is a positive integer between 1 and 10}\}$

**(b)** $C = \{x : x \text{ is a negative integer between } -10 \text{ and } -1 \text{ inclusive}\}$

**(c)** $D = \{x : x \text{ is a positive even integer such that } -2 < x \leqslant 12\}$

**(d)** $E$ is the set of vowels used to form the word 'HAPPY'.

## Exercise 12A

3. For each of the following sets, list all its elements and state the number of elements in set notation.
   (a) $F = \{x : x \text{ is a colour of the rainbow}\}$
   (b) $G = \{x : x \text{ is a public holiday in Pakistan}\}$
   (c) $H$ is the set of consonants used to form the word 'SYMMETRY'.
   OPEN (d) $J = \{x : x \text{ is a teacher teaching my current class}\}$

4. List all the elements of each of the following sets in set notation. If it is an empty set, write the empty set in set notation.
   (a) $K$ is the set of odd numbers that are multiples of 2.
   (b) $L = \{x : x \text{ is an even prime number}\}$
   (c) $M$ is the set of quadrilaterals with 5 vertices each.
   (d) $N = \{x : x \text{ is a month of the year with more than 31 days}\}$

5. $P$ is the set of days in a week.
   (i) List all the elements of $P$ in set notation.
   (ii) Using the notation $\in$ or $\notin$, describe whether each of the following is an element of, or is not an element of, $P$.
   (a) Tuesday (b) Sunday
   (c) March (d) Holiday
   (iii) State the value of $n(P)$.

6. $Q$ is the set of perfect squares more than 1 and less than 50.
   (i) Is $10 \in Q$?
   (ii) List all the elements of $Q$ in set notation.
   (iii) State the value of $n(Q)$.

7. It is given that $R = \{0, 2, 4, 6, 8, \ldots\}$ and $S = \{0, 2, 4, 6, 8\}$.
   (i) Describe set $R$ and set $S$ in words.
   (ii) Is $n(R) = n(S)$? Explain.
   (iii) Is $R = S$? Explain.

8. It is given that $T = \{x : x \text{ is a perfect square between 10 and 15}\}$,
   $U = \{x : x \text{ is a positive integer less than 5 that is both a perfect square and a perfect cube}\}$, and
   $V = \{0\}$.
   (i) Which sets are empty sets? Write the empty sets in set notation.
   (ii) Are $T$ and $U$ equal sets? Explain.
   (iii) Are $U$ and $V$ equal sets? Explain.

9. State whether each of the following statements is true or false. If it is false, explain why.
   (a) $c \notin \{c, a, r\}$ (b) $car \in \{c, a, r\}$
   (c) $\{c\} \in \{c, a, r\}$ (d) $\{c, a, r\} = 3$

10. Describe the elements of each of the following sets in set notation.
   (a) $X = \{2, 3, 5, 7, 11, 13, \ldots\}$
   (b) $Y = \{0, 4, 8, 12, 16, \ldots\}$
   (c) $Z = \{1, 2, 3, 4, 6, 12\}$
   OPEN (d) $W$ is the set of students in my current class who wear spectacles.

11. State whether each of the following statements is true or false. If it is false, explain why.
   (a) $\{0\} = \varnothing$ (b) $\varnothing = \{\ \}$
   (c) $\{\varnothing\}$ is an empty set. (d) $n(\varnothing) = 0$

# 12.2 Venn diagrams, universal sets, complements of sets and subsets

## A. Venn diagrams, universal sets and complements of sets

We can represent a set using a **Venn diagram** as shown in Fig. 12.1.

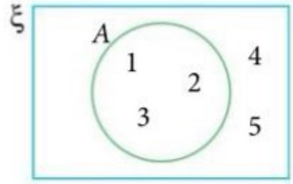



Fig. 12.1

In Fig. 12.1, the rectangle represents the set of all the elements that are under consideration for this particular situation, i.e. $\{1, 2, 3, 4, 5\}$. This is called the **universal set** and is denoted by the symbol $\xi$, i.e. $\xi = \{1, 2, 3, 4, 5\}$.
The circle represents the set $A = \{1, 2, 3\}$.
We observe that the elements 4 and 5 are outside the circle but inside the rectangle, i.e. $4 \notin A$ and $5 \notin A$. The set of all the elements belonging to $\xi$ but *not* to $A$ is called the **complement** of the set $A$, and is denoted by $A'$ (pronounced as 'A prime'), i.e. $A' = \{4, 5\}$.

**Big Idea**

**Diagrams**
A Venn diagram is a succinct visual representation of sets and their elements. At a glance, we can see clearly that the set $A$ contains the elements 1, 2 and 3, but not 4 and 5.

**Attention**

When drawing a Venn diagram,
- do not put commas between the elements,
- do not write the elements too close together,
- write the elements inside the set, but label the set $\xi$ and the set $A$ outside the rectangle and circle respectively.

**Worked Example 3**

### Universal set and complement of a set

It is given that $\xi = \{x : x \text{ is an integer between 1 and 10 inclusive}\}$ and
$A = \{x : x \text{ is a perfect square}\}$.

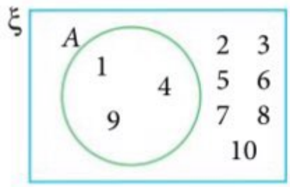
**(i)** List all the elements of the universal set and of the set $A$ in set notation.
**(ii)** Draw a Venn diagram to represent the sets $\xi$ and $A$.
**(iii)** From the Venn diagram, list all the elements of $A'$ in set notation.
**(iv)** Describe the set $A'$ in words.
**(v)** State the values of $n(\xi)$, $n(A)$ and $n(A')$.
**(vi)** Is $n(A) + n(A') = n(\xi)$? Explain.

**Solution**

**(i)** $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 4, 9\}$

**(ii)**




**(iii)** $A' = \{2, 3, 5, 6, 7, 8, 10\}$

**(iv)** $A'$ is the set of integers between 1 and 10 inclusive which are not perfect squares.

**(v)** $n(\xi) = 10$, $n(A) = 3$ and $n(A') = 7$

**(vi)** Yes. Since $A$ and $A'$ contain all the elements of $\xi$, and $A$ and $A'$ do not contain any common elements, $n(A) + n(A') = n(\xi)$.

**Problem-solving Tip**

**(i)** The set $A$ does *not* contain all the perfect squares because all the elements under consideration for this particular situation (i.e. belonging to the universal set) are only the integers between 1 and 10 inclusive.

**Practise Now 3**

Similar and Further Questions
**Exercise 12B**
Questions 1, 2, 6, 7, 11

It is given that $\xi = \{x : x \text{ is an integer between 1 and 13 inclusive}\}$ and

$B = \{x : x \text{ is a prime number}\}$.

**(i)** List all the elements of the universal set and of the set $B$ in set notation.

**(ii)** Draw a Venn diagram to represent the sets $\xi$ and $B$.

**(iii)** From the Venn diagram, list all the elements of $B'$ in set notation.

**(iv)** Describe the set $B'$ in words.

**(v)** State the values of $n(\xi)$, $n(B)$ and $n(B')$.

**(vi)** Is $n(B) + n(B') = n(\xi)$? Explain.

**Thinking Time**

**(i)** Given that $A = \{1, 2, 3\}$, can we find $A'$ if we do not define what the universal set $\xi$ is? Explain.

**(ii)** For any set $S$, is it always true that $n(S) + n(S') = n(\xi)$? Explain.

## B. Subsets

Consider the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$.
How can we draw a Venn diagram to represent the sets $A$ and $B$ such that we do not repeat the common elements? Since all the elements in a set are distinct (i.e. we *cannot* write the same element more than once), we can draw the Venn diagram as shown in Fig. 12.2.

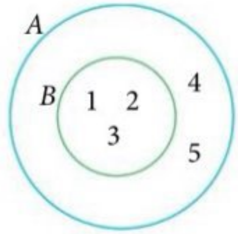



Fig. 12.2

**Big Idea**

**Diagrams**
A Venn diagram can be used to display the relationship between sets clearly. At a glance, we can see that $B$ is a subset of $A$ because $B$ is completely inside $A$.

We observe that $B$ is *completely inside* $A$, i.e. every element of $B$ is an element of $A$, and $B \neq A$. We say that $B$ is a *subset* of $A$.

Now, consider the sets $A = \{1, 2, 3, 4, 5\}$ and $C = \{1, 2, 3, 4, 5\}$. Can we draw the set $C$ completely inside the set $A$? Yes, we can draw the two sets as shown in Fig. 12.3.

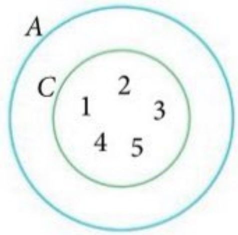



Fig. 12.3

**Attention**

In Fig. 12.3, we can also draw $A$ completely inside $C$, so $A$ is also a subset of $C$. In other words, if $C$ is a subset of $A$ and $A$ is also a subset of C, then $A = C$.

Since $C$ is *completely inside* $A$, i.e. every element of $C$ is an element of $A$, then $C$ is also a *subset* of $A$.
In this case, $C = A$ (equal sets), unlike $B \neq A$.

We call both $B$ and $C$ **subsets** of $A$ and we write $B \subseteq A$ and $C \subseteq A$.
In addition, $B$ is also a **proper subset** of $A$ (because $B \neq A$, unlike $C = A$) and we write $B \subset A$.

If $B$ is a **subset** of $A$, then every element of $B$ is an element of $A$.
Similarly, if every element of $B$ is an element of $A$, then $B$ is a subset of $A$ and we write $B \subseteq A$.
If $B$ is a subset of $A$, there are two cases: either
**(i)** $B \subset A$ ($B$ is a *proper subset* of $A$, i.e. every element of $B$ is an element of $A$ and $B \neq A$), or
**(ii)** $B = A$ ($B$ and $A$ are *equal sets*, i.e. every element of $B$ is an element of $A$ and vice versa).
Note that $\subseteq$ denotes 'is a subset of',
$\nsubseteq$ denotes 'is not a subset of',
$\subset$ denotes 'is a proper subset of',
$\not\subset$ denotes 'is not a proper subset of'.

**Attention**

You can think of $\subseteq$ (subset) as consisting of two notations:
$\subset$ (proper subset) and
$=$ (equal sets).

**Worked Example 4**

### Subset and proper subset

It is given that $X$ is the set of letters used to form the word 'SINGAPORE' and $Y$ is the set of letters used to form the word 'PANS'.

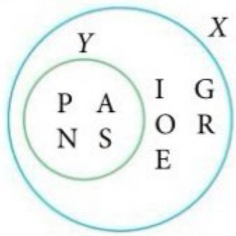
**(i)** Draw a Venn diagram to represent the sets $X$ and $Y$.

**(ii)** Is $Y \subset X$? Explain.

**(iii)** Is $X \not\subseteq Y$? Explain.

**Solution**

**(i)**




**(ii)** Yes, $Y \subset X$ because every element of $Y$ is an element of $X$ and $Y \neq X$.

**(iii)** Yes, $X \not\subseteq Y$ because there are elements in $X$ (e.g. I) that are not elements of $Y$.

**Problem-solving Tip**

**(ii)** In explaining whether one set is a proper subset of another, we should use more precise language as shown. We seldom say that one set is completely inside another set.

**Practise Now 4A**

Similar and Further Questions
**Exercise 12B**
Questions 3, 4(a)–(d), 8, 9, 12

1. It is given that $C = \{1, 2, 3, 4, 5, 6, 7\}$ and
   $D = \{1, 3, 5, 7\}$.
   **(i)** Draw a Venn diagram to represent the sets $C$ and $D$.
   **(ii)** Is $D \subset C$? Explain.
   **(iii)** Is $C \not\subseteq D$? Explain.

2. State whether each of the following statements is true or false.
   **(a)** $\{2, 5\} \subset \{2, 3, 5, 7\}$ **(b)** $\{2, 3, 5, 7\} \not\subset \{7, 5, 3, 2\}$
   **(c)** $\{2, 5\} \subseteq \{5, 2\}$ **(d)** $\{2, 5\} \not\subseteq \{2, 3, 5, 7\}$

3. It is given that $P = \{x : x \text{ is an integer such that } 0 < x \leqslant 13\}$,
   $Q = \{x : x \text{ is a prime number less than } 13\}$, and
   $R = \{x : x \text{ is a positive integer not more than } 13\}$.
   **(i)** List all the elements of $P$ and of $Q$ in set notation.
   **(ii)** Is $Q \subset P$ or $P \subset Q$? Explain.
   **(iii)** List all the elements of $R$ in set notation.
   **(iv)** Is $P \subseteq R$ or $R \subseteq P$? Explain.

**Class Discussion**

### Understanding subset

1. Is a subset a set?
2. If $A$ is a proper subset of $B$, is $A$ also a subset of $B$?
3. If $C$ is a subset of $D$, is $C$ also a proper subset of $D$?
4. If $E$ is a subset of $F$, then every element of $E$ is an element of $F$.
   Is every element of $F$ also an element of $E$?

5. For each of the Venn diagrams shown in Fig. 12.4, is $P$ a proper subset of $Q$ or vice versa?

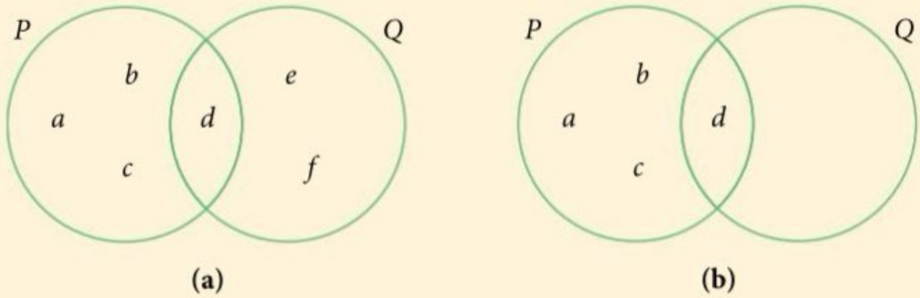



Fig. 12.4

## C. Listing all the subsets of a set

Consider the set $S = \{1, 2\}$. What are all the subsets of $S$?

Firstly, it is obvious that $\{1\}$ and $\{2\}$ are subsets of $S$ as shown in Fig. 12.5(**a**) and (**b**).

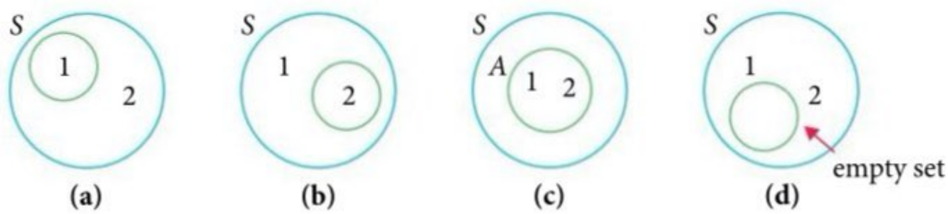



Fig. 12.5

In Fig. 12.5(**c**), we observe that we can draw a set $A = \{1, 2\}$ inside the set $S$. Thus $\{1, 2\}$ is also a subset of $S$, i.e. *S is a subset of itself*.

In Fig. 12.5(**d**), we observe that we can draw the empty set completely inside $S$.

Thus the empty set $\varnothing$ is also a subset of $S$.

Therefore, all the subsets of $S = \{1, 2\}$ are $\varnothing$, $\{1\}$, $\{2\}$ and $\{1, 2\}$.

**Practise Now 4B**

Similar and Further Questions

**Exercise 12B**

Questions 5(a)–(d), 10(a)–(d), 13

List all the subsets of each of the following sets in set notation.

**(a)** $S = \{7, 8\}$

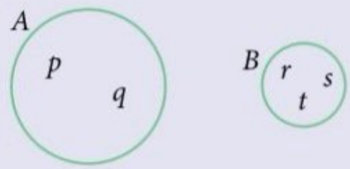
**(b)** $T = \{a, b, c\}$

**Attention**

There is no need to draw any Venn diagrams when listing all the subsets of a set.

## Reflection

1. David drew a Venn diagram showing two sets $A = \{p, q\}$ and $B = \{r, s, t\}$.



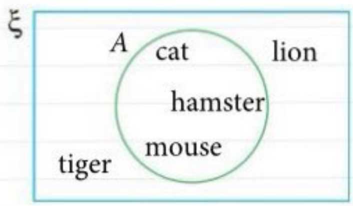

Bernard said that the circle representing the set $B$ should be larger than the circle representing the set $A$ because $B$ contains more elements than $A$. David replied that it does not matter. Who do you agree with? Why?

2. How do I explain the difference between a subset and a proper subset?
3. What have I learnt in this section that I am still unclear of?

Advanced | Intermediate | Basic

## Exercise 12B

**1.** The Venn diagram shows the elements of $\xi$ and of $A$.

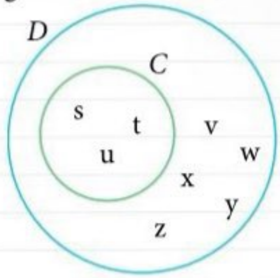



**(i)** List all the elements of the universal set and of the set $A$ in set notation.
**(ii)** List all the elements of $A'$ in set notation.

**2.** It is given that $\xi = \{x : x \text{ is an integer between } 1 \text{ and } 10 \text{ inclusive}\}$ and $B = \{x : x \text{ is an even number}\}$.

**(i)** List all the elements of the universal set and of the set $B$ in set notation.
**(ii)** Draw a Venn diagram to represent the sets $\xi$ and $B$.
**(iii)** From the Venn diagram, list all the elements of $B'$ in set notation.
**(iv)** Describe the set $B'$ in words.

**3.** The Venn diagram shows the elements of $C$ and of $D$.




**(i)** List all the elements of $C$ and of $D$ in set notation.
**(ii)** Is $C$ a proper subset of $D$? Explain.

**4.** State whether each of the following statements is true or false.
**(a)** $\{b, c\} \subset \{a, b, c, d\}$
**(b)** $\{a, b, c, d\} \not\subset \{d, c, b, a\}$
**(c)** $\{b, c\} \subseteq \{c, b\}$
**(d)** $\{b, c\} \not\subseteq \{a, b, c, d\}$

**5.** List all the subsets of each of the following sets in set notation.
**(a)** $E = \{a, b\}$
**(b)** $F = \{\text{Singapore, Malaysia}\}$
**(c)** $G = \{14, 16\}$
**(d)** $H = \{7\}$

Basic Intermediate Advanced

## Exercise 12B

6. It is given that $\xi = \{x : x \text{ is an integer between 0 and } 10\}$ and $J = \{x : x \text{ is not a prime number}\}$.
   (i) List all the elements of $\xi$, of $J$ and of $J'$ in set notation.
   (ii) Describe the set $J'$ in words.
   (iii) State the values of $n(\xi)$, $n(J)$ and $n(J')$.
   (iv) Is $n(J) + n(J') = n(\xi)$? Explain.

7. It is given that $\xi$ is the universal set containing the first 10 letters of the English alphabet and $K$ is the set of consonants.
   (i) List all the elements of $\xi$, $K$ and $K'$ in set notation.
   (ii) Describe the set $K'$ in words.
   (iii) State the values of $n(\xi)$, $n(K)$ and $n(K')$.
   (iv) Is $n(K) + n(K') = n(\xi)$? Explain.

8. It is given that $L = \{-2, -1, 0, 1, 2, 3\}$ and $M = \{0, 1, 2, 3\}$.
   (i) Draw a Venn diagram to represent the sets $L$ and $M$.
   (ii) Is $M \subset L$? Explain.
   (iii) Is $L \not\subseteq M$? Explain.

9. It is given that $N = \{x : x \text{ is an integer such that } 0 < x \leqslant 20\}$,
   $P = \{x : x \text{ is a positive multiple of 4 that is less than 20}\}$, and
   $Q = \{x : x \text{ is a positive integer less than 21}\}$.
   (i) List all the elements of $N$ and of $P$ in set notation.
   (ii) Is $N \subset P$ or $P \subset N$? Explain.
   (iii) List all the elements of $Q$ in set notation.
   (iv) Is $N \subseteq Q$ or $Q \subseteq N$? Explain.

10. List all the subsets of each of the following sets in set notation.
   (a) $R = \{7, 8, 9\}$
   (b) $S = \{x : x \text{ is a prime number less than } 7\}$
   (c) $T = \{a, b, c, d\}$
   (d) $U = \{x : x \text{ is a letter used to form the word 'UNION'}\}$

11. It is given that $\xi = \{x : x \text{ is a positive integer less than } 21\}$ and $V = \{x : x \text{ is a number that is a multiple of } 3\}$.
   (i) List all the elements of $V'$ in set notation.
   (ii) Describe the elements of $V'$ in set notation.

12. It is given that $W = \{x : x \text{ is a rational number}\}$ and $X = \{x : x \text{ is an integer}\}$.
   (i) Is $X \subset W$? Explain.
   (ii) Is $W \not\subset X$? Explain.

13. If $Y$ is a set such that $n(Y) = a$, express the number of subsets of $Y$ in terms of $a$.

# 12.3 Probability experiment and sample space

In the **Introductory Problem**, we have seen how the values 0 to 1 inclusive are used to measure the chance of an event occurring. This measure is known as **probability**. Before we can compute the probability of an event, we need to learn how to list all the possible outcomes of a **probability experiment** — a process or operation whose outcomes cannot be predicted with certainty.

In other words, the outcome of a probability experiment depends on chance. Consider what happens when tossing a coin. We cannot predict with certainty the outcome of the toss. Hence, tossing a coin is an example of a probability experiment where we have two possible outcomes: a 'head' or a 'tail'. A set comprising all the **possible outcomes** of a probability experiment is called the **sample space**. In the case of tossing a coin, the sample space is a 'head' and a 'tail'. What is the sample space when a die is rolled?

Table 12.1 shows some examples of probability experiments and their possible outcomes.

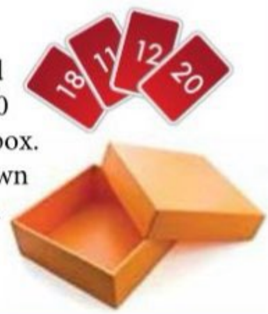
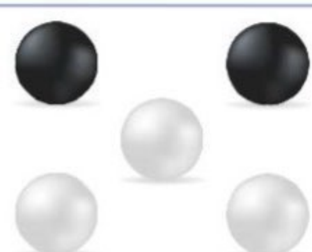
| Probability experiment | Possible outcomes |
|---|---|
|  **Tossing a coin** |  **Head**  **Tail** |
|  **Rolling a die** |  |
| Ten identical cards numbered 11, 12, 13, …, 20 are placed in a box. One card is drawn at random from the box.  |   |
| Two black balls and three white balls of the same size are placed in a bag. One ball is drawn at random from the bag.  |  |

Table 12.1

**Attention**

The plural form of 'die' is 'dice.' Unless otherwise stated, a 'die' will always refer to an ordinary 6-sided die.

**Information**

Ancient Roman coins had the head of the emperor on one side and this side was referred to as 'head'. The use of 'tail' is believed to have originated based on opposites, in terms of the body parts of animals.

Worked Example 5

## Listing sample space

A die is rolled. Write down the sample space and state the total number of possible outcomes.

**Solution**

Sample space = {1, 2, 3, 4, 5, 6}

Total number of possible outcomes = 6

> **Attention**
> We use a pair of braces { } to enclose the possible outcomes when listing the sample space.

**Practise Now 5**

Similar and Further Questions
Exercise 12C
Questions 1, 2(a), (b)

A spinner is divided into five equal sectors of different colours.
When the spinner is spun, the colour of the sector on which the pointer lands is noted.
Write down the sample space and state the total number of possible outcomes.


> **Attention**
> We assume that the spinner will not stop at the line in between any two adjacent sectors.

Worked Example 6

## Listing more complex sample space

For each of the following experiments, write down the sample space and state the total number of possible outcomes.

**(a)** Drawing a ball at random from a bag containing two identical black balls and three identical white balls.

**(b)** Choosing a two-digit number at random.

**Solution**

**(a)** Let $B_1$ and $B_2$ represent the two black balls; and $W_1$, $W_2$ and $W_3$ represent the three white balls.
Sample space = $\{B_1, B_2, W_1, W_2, W_3\}$
Total number of possible outcomes = 5

**(b)** Sample space = {10, 11, 12, 13, …, 99}
Total number of possible outcomes
= number of integers from 1 to 99 – number of integers from 1 to 9
= 99 – 9
= 90

> **Problem-solving Tip**
> **(a)** There is a difference between drawing the 1st or the 2nd black ball. So we use $B_1$ and $B_2$ to differentiate between them. Similarly, for the three white balls, we represent them using $W_1$, $W_2$ and $W_3$.
> **(b)** We use an ellipsis '…' when there are too many outcomes to list. Here, we list the first 4 outcomes so that the reader can interpret that they are consecutive integers, followed by '…' and the final outcome at the end.

**Practise Now 6**

Similar and Further Questions
Exercise 12C
Questions 2(c)–(e)

For each of the following experiments, write down the sample space and state the total number of possible outcomes.

**(a)** Drawing a marble at random from a bag containing five identical blue marbles and four identical red marbles.

**(b)** Picking a letter at random from a box containing identical cards with letters that spell the word 'NATIONAL'.

**(c)** Selecting a receipt at random from a receipt book with running serial numbers from 357 to 389.

Problem-solving Tip

**(b)** Are the two 'N's in the word 'NATIONAL' the same 'N' or are they distinct?

# 12.4 Probability of single events

In the previous section, we have learnt that there are two possible outcomes 'head' and 'tail' when tossing a coin. If the chance of obtaining a 'head' is the same as the chance of obtaining a 'tail', we say that the two outcomes are *equally likely to occur*, and the coin is **fair** or **unbiased**.

If the event is to obtain a 'head', then the **favourable outcome** is 'head' and we write event = {head}, or we can let $E$ be the event and H be 'head' and write $E = \{H\}$.

Attention

Can a coin be unfair or biased? Discuss.

Since there are two possible outcomes that are equally likely to occur, then the chance of obtaining a 'head' is 1 out of 2. We say that the **probability** of obtaining a 'head' is $\frac{1}{2}$.

What is the probability of obtaining a 'tail'?

In general, in a probability experiment with equally likely outcomes, the probability, $P(E)$, of an event $E$ happening is given by:

$$P(E) = \frac{\text{number of favourable outcomes for event } E}{\text{total number of possible outcomes}}.$$

Big Idea

Measures

The probability of the occurrence of an event is a *measure* of the chance or likelihood of the event occurring. It can only take on values between 0 and 1 inclusive. It cannot be negative or greater than 1. Why?

We say that the above is a theoretical approach to finding the probability of an event.

In the example of tossing a coin, notice that the event of obtaining a head, $E = \{H\}$, is a set of favourable outcome(s). It is a subset of the sample space = {H, T}.

Thus, the probability of event $E$ can also be written as $P(E) = \frac{n(E)}{n(\text{sample space})}$.

**Worked Example 7**

### Finding probability of single event

A card is drawn at random from a box containing 12 cards numbered 1, 2, 3, 4, …, 12.

Find the probability of drawing

**(i)** a '7', **(ii)** a perfect square,

**(iii)** a negative number, **(iv)** a number less than 13.

**Solution**

Total number of possible outcomes = 12

**(i)** P(drawing a '7') = $\frac{1}{12}$

**(ii)** There are 3 perfect squares from 1 to 12, i.e. 1, 4 and 9.

$$\text{P(drawing a perfect square)} = \frac{3}{12} = \frac{1}{4}$$

**(iii)** There are no negative numbers from 1 to 12.

$$\text{P(drawing a negative number)} = \frac{0}{12} = 0$$

**(iv)** All the 12 numbers from 1 to 12 are less than 13.

$$\text{P(drawing a number less than 13)} = \frac{12}{12} = 1$$

**Attention**

The probability of an *impossible* event happening is 0, while the probability of a *certain* event occurring is 1. However, the converse is not true: you will learn of the exceptions at higher levels.

**Practise Now 7**

Similar and Further Questions
**Exercise 12C**
Questions 3, 4, 10, 17–19

A ball is drawn at random from a bag containing some balls numbered 10, 11, 12, 13, …, 24. Find the probability of drawing

**(i)** a '21', **(ii)** an odd number,
**(iii)** a prime number, **(iv)** a perfect cube.

**Recall**

**(iii)** A **prime number** is a positive integer that has *exactly 2 different factors*, 1 and itself.

**Worked Example 8**

### Solving probability problems involving playing cards

A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing

**(i)** a black card, **(ii)** a red Ace,
**(iii)** a diamond, **(iv)** a card which is not a diamond.

**Attention**

There are 4 suits in a standard pack of 52 playing cards, i.e. club ♣, diamond ♦, heart ♥ and spade ♠.

Each suit has 13 cards, i.e. Ace, 2, 3, 4, …, 10, Jack, Queen and King.

All the clubs and spades are black in colour.

All the diamonds and hearts are red in colour.

All the Jack, Queen and King cards are also known as picture cards.

**Solution**

Total number of possible outcomes = 52

**(i)** There are 26 black cards in the pack.

$$\text{P(drawing a black card)} = \frac{26}{52} = \frac{1}{2}$$

**(ii)** There are 2 red Aces in the pack, i.e. the Ace of hearts and the Ace of diamonds.

$$\text{P(drawing a red Ace)} = \frac{2}{52} = \frac{1}{26}$$

**(iii)** There are 13 diamonds in the pack.

$$\text{P(drawing a diamond)} = \frac{13}{52} = \frac{1}{4}$$

**(iv) Method 1:**

Since there are 13 diamonds in the pack, then there are $52 - 13 = 39$ cards which are not diamonds.

$$\text{P(drawing a card which is not a diamond)} = \frac{39}{52} = \frac{3}{4}$$

**Method 2:**

Since a card is either a diamond or not a diamond, then

P(drawing a diamond) + P(drawing a card which is not a diamond) = 1

$\therefore$ P(drawing a card which is not a diamond)

$= 1 - \text{P(drawing a diamond)}$

$= 1 - \frac{1}{4}$

$= \frac{3}{4}$

**Reflection**

**(iv)** Which method do you prefer? Why?

**Practise Now 8**

Similar and Further Questions
**Exercise 12C**
Questions 5, 11–13

A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing

**(i)** a red card,
**(ii)** an Ace,
**(iii)** the three of clubs,
**(iv)** a card which is not the three of clubs.

From Worked Example 8(**iv**) and Practise Now 8(**iv**), we have observed that for any event $E$,

$$P(\text{not } E) = 1 - P(E)$$

**Attention**

Since $E$ is a set of favourable outcomes, the set "not $E$" is complement of set $E$. Hence,

n(not $E$) = n(sample space) – n($E$), and

$$P(\text{not } E) = \frac{n(\text{not } E)}{n(\text{sample space})} = \frac{n(\text{sample space}) - n(E)}{n(\text{sample space})} = 1 - P(E).$$

**Worked Example 9**

## Solving probability problems involving letters of the English alphabet

A letter is chosen at random from the word 'MATHEMATICS'. Find the probability that the letter is

**(i)** an 'A', **(ii)** a vowel, **(iii)** not a vowel.

**Solution**

Total number of letters = 11

**(i)** There are two 'A's.

$$\text{P(an 'A' is chosen)} = \frac{2}{11}$$

**(ii)** There are 4 vowels, i.e. 2 'A's, 1 'E' and 1 'I'.

$$\text{P(letter chosen is a vowel)} = \frac{4}{11}$$

**(iii)** P(letter chosen is not a vowel) = 1 – P(letter chosen is a vowel)

$$= 1 - \frac{4}{11}$$

$$= \frac{7}{11}$$

**Problem-solving Tip**

**(i)** Are the two 'A's in the word 'MATHEMATICS' the same 'A' or are they distinct?
**(ii)** There are 26 letters in the English alphabet. Five of them are vowels, namely 'a', 'e', 'i', 'o', and 'u'.

**Practise Now 9**

Similar and Further Questions
**Exercise 12C**
Questions 6–9, 14–16, 20

1. A letter is chosen at random from the word 'CHILDREN'. Find the probability that the letter is
   **(i)** a 'D', **(ii)** a consonant, **(iii)** not a consonant.

2. A marble is drawn at random from a bag containing 9 red marbles, 6 yellow marbles, 4 purple marbles and 5 blue marbles. Find the probability of drawing
   **(i)** a purple marble, **(ii)** a red or a blue marble,
   **(iii)** a white marble, **(iv)** a marble that is not white.

3. A box contains 24 balls, some of which are red, some of which are green and the rest are blue. The probabilities of drawing a red ball and a green ball at random from the box are $\frac{1}{3}$ and $\frac{1}{6}$ respectively. Find the number of blue balls in the box.

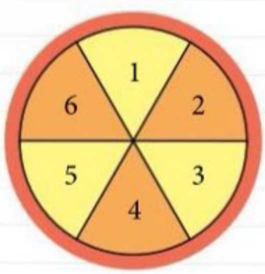
## Reflection

1. Can I explain the terms sample space, possible outcomes and favourable outcomes?
2. What is the range of the probability of an event occurring?

Basic Intermediate Advanced

## Exercise 12C

1. A dart board is divided into 6 equal sectors. When a dart lands on it, the number of the sector on which it lands is noted. Write down the sample space and state the total number of possible outcomes.




2. For each of the following experiments, write down the sample space and state the total number of possible outcomes.
   **(a)** Tossing a fair tetrahedral die with faces labelled 2, 3, 4 and 5 respectively.
   **(b)** Drawing a card at random from a box containing ten identical cards labelled A, B, C, D, E, F, G, H, I, J.
   **(c)** Drawing a disc at random from a bag containing 5 identical red discs, 3 identical blue discs and 2 identical green discs.
   **(d)** Picking a letter at random from a box containing identical cards with letters that spell the word 'TEACHER'.
   **(e)** Choosing a three-digit number at random.

3. An 8-sided fair die with faces labelled 2, 3, 3, 4, 7, 7, 7 and 9 is rolled once. Find the probability of getting
   **(i)** a '7',
   **(ii)** a '3' or a '4',
   **(iii)** a number less than 10,
   **(iv)** a number which is not '2'.

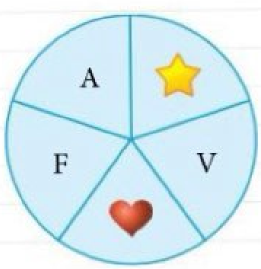
## Exercise 12C

4. A card is drawn at random from a box containing some cards numbered 10, 11, 12, 13, …, 22. Find the probability of drawing
   (i) an even number,
   (ii) a number between 13 and 19 inclusive,
   Hint: A number between *a* and *b* inclusive includes the numbers *a* and *b*.
   (iii) a composite number that is less than 18,
   Hint: A composite number is a positive integer with more than two different factors.
   (iv) a number greater than 22,
   (v) a number that is divisible by 4.

5. A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing
   (i) the Ace of spades, (ii) a heart or a club,
   (iii) a picture card, (iv) a non-picture card.

6. Each of the letters of the word 'PROBABILITY' is written on a card. All the cards are well-shuffled and placed face down on a table. A card is turned over at random. Find the probability that the card shows
   (i) the letter 'A', (ii) the letter 'B',
   (iii) a vowel, (iv) a consonant.

7. A spinner is divided into 5 equal sectors. When the spinner is spun, what is the probability that the pointer will stop at a sector whose label is
   (i) ❤ ?
   (ii) a letter of the English alphabet?
   (iii) a vowel?
   (iv) a consonant?




8. 8 girls with brown hair, 3 girls with blonde hair, 1 girl with red hair, 11 boys with brown hair, 4 boys with blonde hair and 3 boys with red hair. If a student is chosen at random to take part in a survey, find the probability that the student
   (i) is a girl,
   (ii) does not have brown hair,
   (iii) is not a boy with red hair,
   (iv) has black hair.

9. Li Ting has five novels and five comic books in her bag. Three of her books are in Japanese, two of which are comic books. The rest of her books are in English. If a book is chosen at random from her bag, find the probability of choosing
   (i) a book in Japanese,
   (ii) a novel which is in English.

10. A two-digit number is chosen at random. Find the probability that the number is
    (i) less than 20, (ii) a perfect square.

11. Two Joker cards are added to a standard pack of 52 playing cards. A card is then drawn at random from the 54 cards. Find the probability of drawing
    (i) a red card, (ii) a two,
    (iii) a Joker, (iv) a Queen or a King.
    Note: A Joker card is neither a black nor a red card.

12. All the clubs are removed from a standard pack of 52 playing cards. A card is drawn at random from the remaining cards. Find the probability of drawing
    (i) a black card,
    (ii) a diamond,
    (iii) a picture card,
    (iv) a card which is not an Ace.

## Exercise 12C

**13.** Raju wakes up in the morning and notices that his digital clock reads 07 25.


*A B*

After noon, he looks at the clock again. What is the probability that

**(i)** the number in column *A* is a 4?
**(ii)** the number in column *B* is an 8?
**(iii)** the number in column *A* is less than 6?
**(iv)** the number in column *B* is greater than 5?

**14.** A class of 38 students went on a short trip to Bangkok. Of the 18 boys, 6 of them checked in their luggage at the airport. 8 of the girls did not check in their luggage. If a student is chosen at random, find the probability that the student

**(i)** is a girl who did not check in her luggage,
**(ii)** checked in his/her luggage.

**15. (a)** A class has 16 boys and 24 girls. Of the 16 boys, 3 are left-handed. Of the 24 girls, 2 are left-handed. If a student is chosen at random to clean the whiteboard, find the probability that the student is

**(i)** a boy,
**(ii)** left-handed.

**(b)** The student chosen to clean the whiteboard in part **(a)** is a girl who is right-handed. Another student is selected at random from the remaining students to collect assignments. Find the probability that the student is

**(i)** a boy who is left-handed,
**(ii)** a girl who is right-handed.

**16.** There are a total of 117 pairs of socks in a clothes bin. Each pair of socks is placed in a bag. The probabilities of selecting a yellow pair of socks and a grey pair of socks at random from the bin are $\frac{2}{9}$ and $\frac{3}{13}$ respectively. Find the number of pairs of socks in the bin which are

**(i)** yellow,
**(ii)** neither yellow nor grey.

**17.** An IQ test consists of 80 multiple-choice questions. A question is selected at random. Find the probability that the question number

**(i)** contains only a single digit,
**(ii)** is greater than 67,
**(iii)** contains exactly one '7',
**(iv)** is divisible by both 2 and 5.

**18.** Each of the numbers 2, 3, 5 and 7 is written on a card. Two of the cards are drawn at random to form a two-digit number. Find the probability that the two-digit number is

**(i)** divisible by 4,
**(ii)** a prime number.

**19.** A biased tetrahedral die with faces labelled 1, 2, 3 and 4 is rolled once. The chance of getting a '3' is twice that of getting a '1'. The chance of getting a '2' is thrice that of getting a '3'. There is an equal chance of getting a '2' and a '4'. Find the probability of getting a prime number.

**20.** A two-digit number is selected at random from a set of consecutive positive integers. The probability that this number is not divisible by 8 is $\frac{33}{38}$. Find a possible sample space.

## 12.5 Further examples of probability of single events

In all the previous problems on finding the probability of a single event, we had to count the numbers of possible outcomes and favourable outcomes in order to compute using:

$$P(E) = \frac{\text{number of favourable outcomes for event } E}{\text{total number of possible outcomes}}.$$

What if the outcomes cannot be counted? Let us compare the two spinners below:

(a)

(b)

Fig. 12.6

Since all the sectors in Fig. 12.6(**a**) are equal in size, all the five possible outcomes are *equally likely to occur*.

However, in Fig. 12.6(**b**), the sectors are of different sizes. So the four possible outcomes are *not equally likely to occur* and we need to take into account the area of each sector.

In this case, the measure of all the possible outcomes can be given by the area of the circle, and the measure of the outcome 'Green' can be given by the area of the green sector.

Therefore, $P(\text{Green}) = \frac{\text{area of green sector}}{\text{area of circle}} = \frac{1}{2}$.

In general, to compute the probability of an event $E$, we have:

$$P(E) = \frac{\text{measure of favourable outcomes for event } E}{\text{measure of all possible outcomes}}.$$

Note that for both spinners, the measure can be given by the area of the sector or of the circle. However, for Fig. 12.6(**a**), we are able to simplify this to the number of sectors since the sectors have equal areas.

Worked Example 10

## Solving probability problems involving angles of sectors

A circle is divided into sectors of different colours. A point is selected at random in the circle. Find the probability that the point lies in the

(i) red sector,
(ii) blue sector,
(iii) black sector.


**Attention**

We assume that the point selected will not lie on the line in between any two adjacent sectors.

### Solution

(i) P(point selected lies in the red sector)

$$= \frac{\text{area of red sector}}{\text{area of circle}}$$

$$= \frac{\text{angle of red sector}}{\text{angle of circle}}$$

$$= \frac{90°}{360°}$$

$$= \frac{1}{4}$$

**Problem-solving Tip**

(i) The area of a sector is proportional to its angle. Therefore, $\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{angle of sector}}{\text{angle of circle}}$, and the angle of the circle is 360°.
Visually, we can also see that the red sector makes up $\frac{1}{4}$ of the circle.

(ii) Angle of the blue sector $= 360° - 180° - 90° - 45°$
$= 45°$

P(point selected lies in the blue sector) $= \frac{\text{area of blue sector}}{\text{area of circle}}$

$$= \frac{\text{angle of blue sector}}{\text{angle of circle}}$$

$$= \frac{45°}{360°}$$

$$= \frac{1}{8}$$

(iii) P(point selected lies in the black sector) = 0

**Problem-solving Tip**

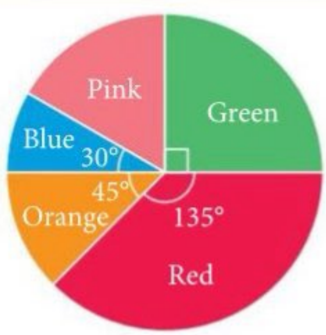
(iii) Since there is no black sector, the event is impossible and so its probability is 0.

**Practise Now 10**

Similar and Further Questions
Exercise 12D
Questions 1, 2, 9

A circle is divided into sectors of different colours.
A point is selected at random in the circle.
Find the probability that the point lies in the

(i) red sector,
(ii) pink sector,
(iii) yellow sector,
(iv) blue or orange sector.



Worked Example 11

## Solving probability problems involving algebra

A box contains $x$ red marbles, $(x + 3)$ yellow marbles and $(4x - 15)$ blue marbles.

**(i)** Find an expression, in terms of $x$, for the total number of marbles in the box.

**(ii)** A marble is drawn at random from the box. Write down an expression, in terms of $x$, for the probability that the marble is blue.

**(iii)** Given that the probability in part **(ii)** is $\frac{1}{2}$, find the value of $x$.

### Solution

**(i)** Total number of marbles $= x + (x + 3) + (4x - 15)$
$= 6x - 12$

**(ii)** P(drawing a blue marble) $= \frac{4x-15}{6x-12}$

**(iii)** Given that $\frac{4x-15}{6x-12} = \frac{1}{2}$,

$$\begin{aligned} 2(4x - 15) &= 6x - 12 \\ 8x - 30 &= 6x - 12 \\ 2x &= 18 \\ x &= 9 \end{aligned}$$

Practise Now 11

Similar and Further Questions
**Exercise 12D**
Questions 3–8, 10, 11

1. There are 12 green balls and $(x + 2)$ yellow balls in a box.

   **(i)** Find an expression, in terms of $x$, for the total number of balls in the box.

   **(ii)** A ball is drawn at random from the box. Write down an expression, in terms of $x$, for the probability that the ball is yellow.

   **(iii)** Given that the probability in part **(ii)** is $\frac{2}{5}$, find the value of $x$.

2. There are 28 boys and 25 girls in a school hall. After $y$ girls leave the hall, the probability of selecting a girl at random becomes $\frac{3}{7}$. Find the value of $y$.

# 12.6 Experimental approach to finding probability

In Sections 12.4 and 12.5, we have learnt how to calculate the probability of a single event by finding the measures of the possible and favourable outcomes. In *theory*, when we toss a coin, the probability of obtaining a 'head' is $\frac{1}{2}$ because there are only two possible outcomes 'head' and 'tail', and one favourable outcome 'head':

$$\text{P(Head)} = \frac{\text{number of favourable outcomes for event } E}{\text{total number of possible outcomes}} = \frac{1}{2}.$$

What if we toss a coin a number of times and record the number of times we get a 'head'? Will the probability obtained through an experimental approach be equal to the probability of $\frac{1}{2}$ obtained through a theoretical approach? For example, if we toss a coin 10 times, will we always get 5 'heads' and 5 'tails'?

## Investigation — Tossing a coin

**1.** Toss a coin 20 times.

**(i)** Record the outcome of each toss in the following table.

| Outcome | Tally | Number of 'heads' or 'tails' for 20 tosses | Fraction of obtaining a 'head' or a 'tail' |
|---|---|---|---|
| Head | | | |
| Tail | | | |

Table 12.2

**(ii)** Compare your results with those of your classmates. Are they the same? Why or why not?

**2. (i)** In groups of 4 or 5, add and record the total number of 'heads' obtained by you and your group members. Repeat for the total number of 'tails'. Compute the fraction of obtaining a 'head' or a 'tail'.

| Outcome for ___ tosses | Total number of 'heads' or 'tails' | Fraction of obtaining a 'head' or a 'tail' |
|---|---|---|
| Head | | |
| Tail | | |

Table 12.3

**(ii)** As a class, add and record the total number of 'heads' obtained by all students. Repeat for the total number of 'tails'. Compute the fraction of obtaining a 'head' or a 'tail'.

| Outcome for ___ tosses | Total number of 'heads' or 'tails' | Fraction of obtaining a 'head' or a 'tail' |
|---|---|---|
| Head | | |
| Tail | | |

Table 12.4

**3.** Look at the last column in the three tables. Do the fractions of obtaining a 'head' or a 'tail' approach the theoretical value of $\frac{1}{2}$ when there are more tosses?

4. If we toss a coin 1000 times, would we expect to obtain *exactly* 500 'heads' and *exactly* 500 'tails'? Explain your answer.

Fig. 12.7 shows a line graph of the fraction of obtaining a 'head' against the number of tosses for another coin-tossing experiment.

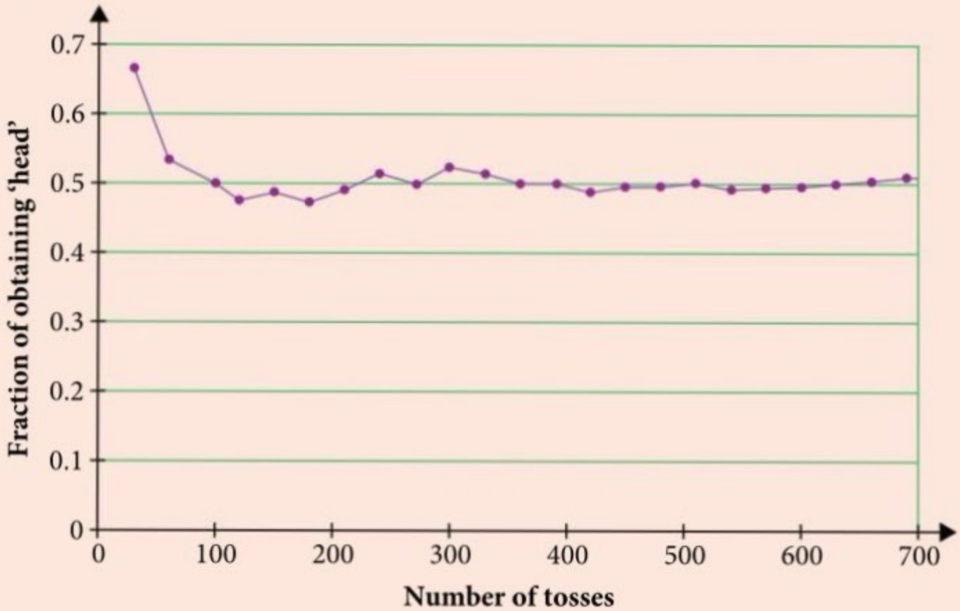



Fig. 12.7

5. We observe from Fig. 12.7 that the fraction of obtaining a 'head' approaches the theoretical value of $\frac{1}{2}$ when the number of tosses increases.

   (i) Does this mean that for fewer tosses, the fraction of obtaining a 'head' will never be $\frac{1}{2}$?

   (ii) Does this mean that for more tosses, the difference between the fraction of obtaining a 'head' and $\frac{1}{2}$ will become smaller and smaller?

The fraction of obtaining a 'head' is called the *relative frequency* of obtaining a 'head', i.e.

$$\text{Relative frequency} = \frac{\text{number of occurrences}}{\text{total number of trials}}.$$

From the above Investigation, we have learnt the following:

- The *relative frequency* of obtaining a 'head' in a probability experiment is not always equal to the **theoretical value** of $\frac{1}{2}$.
- When the number of tosses increases, the relative frequency of obtaining a 'head' will generally *approach* the theoretical value of $\frac{1}{2}$, but the relative frequency *may also deviate* further from $\frac{1}{2}$ for some large number of tosses.

**Information**

**Coin-toss champions**

- Count Buffon (1707–1788) tossed a coin 4040 times and obtained 2048 'heads', so relative frequency = 0.5069.
- John Kerrich (1903–1985) tossed a coin 10 000 times and obtained 5067 'heads', so relative frequency = 0.5067.
- Karl Pearson (1857–1936) tossed a coin 24 000 times and obtained 12 012 'heads', so relative frequency = 0.5005.

As the number of trials increases, the relative frequency approaches the theoretical value of 0.5.

- On the other hand, when the number of tosses is small, it is *still possible* for the relative frequency of obtaini a 'head' to be equal to the theoretical value of $\frac{1}{2}$, e.g. see the point (100, 0.5) in Fig. 12.7.

Although the relative frequency is not always equal to the theoretical value of $\frac{1}{2}$, we can still use theoretical values of probability to *estimate* the number of times we can *expect* to obtain the favourable outcome (e.g. 'head') from the total number of trials. This is the **expected frequency** of the event (e.g. of obtaining a 'head'):

Expected frequency of event = probability of occurrence × total number of trials.

## Reflection

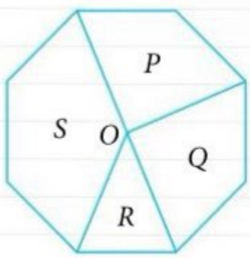
1. What is the difference between probabilities obtained through a theoretical approach and an experimental approach?
2. What have I learnt in this section or chapter that I am still unclear of?

Basic | Intermediate | Advanced

## Exercise 12D

1. A survey is conducted to find out which of the four fruits, apple, papaya, guava and mango, students in a class prefer. The pie chart shows the results of the survey. A student is selected at random. Find the probability that the student prefers
   (i) apple,
   (ii) mango,
   (iii) papaya or guava.


2. A regular octagon is divided into 4 regions, where $O$ is its centre. A point is selected at random in the octagon. Find the probability that the point lies in
   (i) region $R$,
   (ii) region $S$,
   (iii) region $P$ or $Q$.



Advanced Intermediate Basic

## Exercise 12D

3. There are 15 girls and $x$ boys at a school parade square.
   **(i)** Write down an expression, in terms of $x$, for the total number of students at the school parade square.
   **(ii)** A student is selected at random. Write down an expression, in terms of $x$, for the probability that the student is a girl.
   **(iii)** Given that the probability in part **(ii)** is $\frac{1}{5}$, find the value of $x$.

4. Santa Claus has $(3h + 11)$ red presents and $(h + 5)$ white presents in his sack. Albert selects a present at random from the sack. Given that the probability that he obtains a red present is $\frac{19}{26}$, find the value of $h$.

5. Some patients participated in a clinical trial for a new drug to treat osteoporosis. A patient is selected at random. The probability that the patient had no change in his bone mass density is $\frac{7}{13}$, the probability that he had a slight reduction in his bone mass density is $\frac{1}{k}$ and the probability that he had a significant reduction in his bone mass density is $\frac{1}{2k}$. Find the value of $k$.

6. A carton contains 15 toothbrushes, of which $p$ have soft bristles. After 5 more toothbrushes with soft bristles are added to the carton, the probability of drawing a toothbrush with soft bristles becomes $\frac{3}{4}$. Find the value of $p$.

7. There are 23 boys and 35 girls on the school's track and field team. After $q$ boys and $(q + 4)$ girls graduate at the end of this year, the probability of selecting a boy at random to represent the school for an event becomes $\frac{2}{5}$. Find the value of $q$.

8. OPEN A bottle contains 2 red marbles, 8 green marbles and 10 blue marbles. $x$ red marbles are added. What is a possible value of $x$ such that the probability of drawing a red marble is now at least $\frac{1}{2}$?

9. $ABCD$ is a square. $M$ is the midpoint of $AB$ and $N$ is the midpoint of $AD$. A point is selected at random in the square. Find the probability that the point lies in the triangle $MAN$.

10. A bag contains 40 balls, some of which are red, some of which are yellow and the rest are black. The probabilities of drawing a red ball and a yellow ball at random from the bag are $\frac{1}{4}$ and $\frac{2}{5}$ respectively.
    **(i)** Find the probability of drawing a black ball at random from the bag.

    $(2x + 1)$ red balls and $(x + 2)$ yellow balls are added to the bag while $(x - 3)$ black balls are removed from the bag. The probability of drawing a yellow ball at random from the bag is now $\frac{3}{7}$. Find
    **(ii)** an expression, in terms of $x$, for the total number of balls in the bag now,
    **(iii)** the number of yellow balls in the bag now.

11. There are 50 students in an auditorium, of which $2x$ are boys and $y$ are girls. After $(y - 6)$ boys leave the auditorium and $(2x - 5)$ girls enter the auditorium, the probability of selecting a girl at random becomes $\frac{9}{13}$. Find the value of $x$ and of $y$.

## Looking Back

In this chapter, we begin to explore the notion of probability, a **measure** of chance, likelihood and uncertainty. In mathematics, we usually deal with certainty and look to obtain exact answers. As you encounter questions in this exciting and highly practical topic of probability, you may feel uncertain (no pun intended) at times about your own thinking. What you need to understand is that the study of probability is a testimony of the power of mathematics in its ability to quantify even the occurrence of uncertain events. Here, we learn the basics of determining the probability of an event occurring by considering the concept of equally likely outcomes together with two important assignments of probability: P(impossible event) = 0, and P(certain event) = 1.

In Book 4, we will investigate how we can use probability to find the likelihood of the occurrence of more complex events.

## Summary

1. A set is a collection of **well-defined** and **distinct** objects. Each object belonging to a set is called an **element**.
   - $\in$ denotes 'is an element of'
   - $\notin$ denotes 'is not an element of'
   - $n(S)$ denotes 'the number of elements belonging to the set $S$'

2. There are a few ways to describe a set. We can
   - use *words*, e.g. $S$ is the set of positive even integers less than 10;
   - *list* all the elements of a set in set notation, e.g. $S = \{2, 4, 6, 8\}$;
   - *describe* the elements of a set in set notation, e.g. $S = \{x : x \text{ is a positive even integer less than } 10\}$.

3. Two sets $A$ and $B$ are **equal** if they contain exactly the same elements. We write $A = B$.
   The **empty** (or **null**) set is the set that contains no elements. It is denoted by the symbol $\varnothing$.
   - Give an example of two sets $A$ and $B$ which are equal. Is $n(A) = n(B)$? Explain.
   - Give an example of an empty set $C$. State the value of $n(C)$.

4. A **Venn diagram** can be used to represent the relationships between sets.

5. **(a)** The **universal set** is the set of all elements that are under consideration for a particular situation. It is denoted by the symbol $\xi$.
   **(b)** The **complement** of a set $A$ is the set of all the elements belonging to $\xi$ but *not* to $A$. It is denoted by $A'$.
   **(c)** $B$ is a **subset** of $A$ if every element of $B$ is an element of $A$. We write $B \subseteq A$.
   There are two cases: either
   **(i)** $B \subset A$ ($B$ is a *proper subset* of $A$, i.e. every element of $B$ is an element of $A$ and $B \neq A$), or
   **(ii)** $B = A$ ($B$ and $A$ are *equal sets*, i.e. every element of $B$ is an element of $A$ and vice versa).
   Note that $\subseteq$ denotes 'is a subset of',
   $\nsubseteq$ denotes 'is not a subset of',
   $\subset$ denotes 'is a proper subset of',
   $\not\subset$ denotes 'is not a proper subset of'.
   - It is given that $\xi = \{1, 2, 3, 4, 5, 6, 7\}$ and
     $A = \{2, 3, 5, 7\}$.
     Draw a Venn diagram to represent the sets $\xi$ and $A$, and list all the elements of $A'$ in set notation.
   - It is given that $\xi = \{1, 2, 3, 4, 5, 6, 7\}$,
     $A = \{2, 3, 5, 7\}$, and
     $B = \{3, 5, 7\}$.
     Draw a Venn diagram to represent the sets $\xi$, $A$ and $B$.

## Summary

6. **Probability** is a measure of chance.
7. A **sample space** is the collection of all the possible outcomes of a probability experiment.
8. In a probability experiment with *equally likely outcomes*, the probability, $P(E)$, of an event $E$ happening is given by:

$$P(E) = \frac{\text{number of favourable outcomes for event } E}{\text{total number of possible outcomes}}$$

If the outcomes cannot be counted, then the probability, $P(E)$, of an event $E$ happening is given by:

$$P(E) = \frac{\text{measure of favourable outcomes for event } E}{\text{measure of all possible outcomes}}$$

- Give an example of a probability experiment where the outcomes cannot be counted. How would you measure the possible and favourable outcomes?

9. For any event $E$, $0 \leqslant P(E) \leqslant 1$.
If $E$ is an *impossible* event, then it will *never* occur and $P(E) = 0$.
If $E$ is a *certain* event, then it will *definitely* occur and $P(E) = 1$.
   - Give an example of an impossible event and an example of a certain event.
10. For any event $E$, $P(\text{not } E) = 1 - P(E)$.
   - Give an example of when this formula is useful.

# Answer Keys

## Chapter 1 Linear Functions and Graphs

### Practise Now 1B

1. (i) 5 (ii) −1
2. (i) $-\frac{2}{5}$ (ii) $\frac{4}{5}$

### Practise Now 1C

1. (ii) $2\frac{1}{2}$

### Practise Now 2

1. (a) Gradient = 4, $y$-intercept = −3
   (b) Gradient = −1, $y$-intercept = 7
   (c) Gradient = 1, $y$-intercept = $\frac{5}{2}$
   (d) Gradient = −0.5, $y$-intercept = 7.16
   (e) Gradient = 6, $y$-intercept = 0
   (f) Gradient = 0, $y$-intercept = 6
2. (a) $y = 3x + 5$
   (b) $y = -7x - 2$
   (c) $y = x - \frac{2}{3}$
   (d) $y = 7.69 - x$
   (e) $y = -\frac{1}{2}x$
   (f) $y = -\frac{1}{2}$

### Practise Now 3

(a) 2 (b) $-\frac{1}{2}$
(c) $\frac{3}{4}$ (d) −6

### Practise Now 4

(i) (a) \$42 (b) \$66
(c) \$102
(ii) 42, 66, 102
(iv) $y = 12x + 6$
(v) (a) \$90 (b) \$162

### Practise Now 5

(i) 0 (iii) $y = 4.2x$
(iv) (a) KRW 1260
(b) PKR 12
(c) KRW 1134

### Practise Now 6

(i) 20 minutes (ii) 9 km
(iii) (a) $\frac{9}{10}$ (b) 0
(c) $-\frac{4}{5}$ (d) 0
(e) $-\frac{5}{7}$

### Exercise 1A

1. $A(-4, -3)$, $B(-2, 4)$, $C(3, 4)$, $D(4, 2)$, $E(1, 1)$, $F(3, -3)$
3. (i) 17 (ii) −3
4. (i) −3 (ii) 10
5. (a) Rectangle
   (b) Rhombus
   (c) Isosceles triangle
   (d) Quadrilateral
   (e) Trapezium
6. 18 units$^2$
7. The points lie on a straight line.
8. (a) (i) $-1\frac{2}{3}$ (ii) $1\frac{1}{3}$
   (b) (i) 1 (ii) $-\frac{3}{4}$
9. −5 and 7

### Exercise 1B

1. (ii) They are parallel lines.
2. (ii) They are parallel lines.
3. (ii) They are parallel lines.
4. (a) Gradient = 3, $y$-intercept = 7
   (b) Gradient = −1, $y$-intercept = −1
   (c) Gradient = 6, $y$-intercept = 6
   (d) Gradient = −4, $y$-intercept = $\frac{10}{3}$
   (e) Gradient = 0.2, $y$-intercept = 0
   (f) Gradient = 0, $y$-intercept = −11
5. (a) $y = 2x + 4$
   (b) $y = -2x - 4$
   (c) $y = x - \frac{1}{5}$
   (d) $y = 3.78 - x$
   (e) $y = -\frac{2}{3}x$
   (f) $y = -\frac{2}{3}$
6. (a) $m = 1, c = 0$
   (b) $m = 1, c = -1$
   (c) $m = -2, c = 8$
   (d) $m = -\frac{1}{4}, c = 4$
   (e) $m = \frac{3}{2}, c = 3$
   (f) $m = -3, c = -3$
   (g) $m = -\frac{4}{3}, c = 4$
   (h) $m = \frac{4}{5}, c = 4$
7. Line 1: 0; Line 2: undefined
9. (ii) $a = 2, b = 12, c = 1.5$
11. (i) −1; 0; 1; $-2\frac{1}{2}$; undefined
    (ii) $y = -x - 2$; $y = x + 2$; $y = -2\frac{1}{2}x + 3\frac{3}{4}$
12. 0; −3; undefined; $\frac{1}{2}$

### Exercise 1C

1. (i) (a) \$4 (b) \$11
   (c) \$21
   (ii) 4, 11, 21
   (iv) $y = 2x + 3$
   (v) (a) \$18 (b) \$40
2. (i) (a) 15 km
   (b) 39 km
   (ii) \$11.60
3. (ii) $d = 0.02p$
   (iii) \$4.70
4. (i) 100, 200, 300, 400
   (ii) There is a fixed overhead cost of \$50.
   (iii) \$390
   (iv) 72
5. (i) \$0.80
   (ii) \$3.80
   (iii) Company B
   (iv) Company B
   (v) Company A
6. (i) 18, 36, 72
   (ii) $y = 3.6x$
   (iii) No
7. (i) 1000 hours
   (ii) 50 km
   (iii) (a) 50 (b) 0
   (c) 60
8. (i) 40 km
   (ii) $1\frac{1}{2}$ hours
   (iii) (a) 20 (b) $-13\frac{1}{3}$
   (c) −20
   (iv) (a) 20 km/h
   (b) 0 km/h
   (c) $13\frac{1}{3}$ km/h

## Chapter 2 Linear Graphs and Simultaneous Linear Equations

### Practise Now 1A

(a) $y = 1, y = -3.5$

### Practise Now 1B

(a) $x = 4, x = -1.2$

### Practise Now 1C

(a) 7 (c) 4
(d) (ii) 2.5

### Practise Now 3

1. $x = 1, y = 2$
2. $x = -1, y = 2$

### Practise Now 4

1. (a) $x = 4, y = 1$
   (b) $x = 3, y = -1$
   (c) $x = \frac{1}{2}, y = -\frac{35}{18}$
   (d) $x = \frac{9}{2}, y = \frac{1}{5}$
2. $x = -3, y = 5$

### Practise Now 5

1. (a) $x = 3, y = 4$
   (b) $x = -1, y = 3$
2. (a) $x = 1, y = -2$
   (b) $x = 1, y = -3$

### Practise Now 6

$x = 10, y = 3$

### Practise Now 7

1. $x = -1, y = 2$
2. $x = 2, y = -1$

### Practise Now 8

(a) $x = 9, y = 15$
(b) $x = -3, y = 6$

### Practise Now 9

1. 13.5, 22.5
2. 34°, 146°
3. 30 cm

### Practise Now 10

1. 10 years old, 40 years old
2. $76

### Practise Now 11

$\frac{7}{9}$

### Practise Now 12

65

### Introductory Problem Revisited

$240, $160

### Exercise 2A

1. (a) $y = 6, y = -2$
2. (a) $x = 0.5, x = -2$
4. (i) $p = -0.5, q = 4.5$
   (iii) $-3$
   (iv) (b) 3.5
5. (i) $y = 2x - 4$
   (ii) $p = -2; q = 3.5$
   (iii) $r = 1, -2$
6. (i) $-5, -3, 1$
   (iii) (b) 1.25 units$^2$

### Exercise 2B

1. (a) $x = -1, y = -3$
   (b) $x = -5, y = -2$
   (c) $x = 3, y = 1$
   (d) $x = 0, y = 2$
   (e) $x = 5, y = 3$
   (f) $x = 3, y = -4$
2. (a) $x = 4, y = 2$
   (b) $x = 1, y = -1$
   (c) $x = 2.6, y = -2.6$
   (d) $x = -1.5, y = -0.5$
3. (a) (i) $-7, 9, 17$
   (b) (i) 0, 2, 3
   (c) $x = -4, y = 1$
4. (a) Infinite number of solutions
   (b) No solution
   (c) Infinite number of solutions
   (d) No solution
5. (a) No solution
   (b) Infinite number of solutions

### Exercise 2C

1. (a) $x = 8, y = 8$
   (b) $x = 12, y = 7$
   (c) $x = 1, y = \frac{1}{4}$
   (d) $x = 3, y = 2$
   (e) $x = 1, y = 2$
   (f) $x = 1, y = 1$
   (g) $x = \frac{3}{2}, y = -\frac{3}{2}$
   (h) $x = -\frac{3}{2}, y = 2$
   (i) $a = -7, b = -13$
   (j) $c = 2, d = \frac{1}{2}$
   (k) $f = 1, h = -\frac{1}{2}$
   (l) $j = \frac{10}{3}, k = -3$
2. (a) $x = 3, y = 2$
   (b) $x = 4, y = -5$
   (c) $x = -1, y = 2$
   (d) $x = -8, y = 19$
   (e) $x = 4, y = 5$
   (f) $x = -2, y = -4$
3. (a) $x = 3, y = 1$
   (b) $x = 7, y = -11$
   (c) $x = 1, y = 2$
   (d) $x = 3, y = -1$
   (e) $x = 2, y = 3$
   (f) $x = \frac{13}{3}, y = -\frac{1}{3}$
4. (a) $x = 6, y = 1$
   (b) $x = 1, y = 3$
   (c) $x = -1, y = -1$
   (d) $x = 3, y = 10$
   (e) $x = 1, y = 2$
   (f) $x = 5, y = -1$
   (g) $x = \frac{22}{7}, y = \frac{41}{7}$
   (h) $x = 0, y = \frac{3}{2}$
5. (a) $x = 0.75, y = -0.25$
   (b) $x = 3, y = 5$
   (c) $x = 2, y = -1.5$
   (d) $x = 0.5, y = 1.5$
6. (a) $x = 2, y = 1$
   (b) $x = \frac{13}{7}, y = 10$
   (c) $x = 1, y = 1$
   (d) $x = 2, y = -1$
   (e) $x = 2, y = -2$
   (f) $x = -6, y = 2$
7. (a) $x = 5, y = 6$
   (b) $x = 1, y = -1$
   (c) $x = 6, y = -4$
   (d) $x = 13, y = 11$
8. (a) $x = -4, y = 4$
   (b) $x = 4, y = -3$
   (c) $x = \frac{2}{3}, y = 0$
   (d) $x = 1, y = -2$
   (e) $x = -3, y = 5$
   (f) $x = \frac{6}{11}, y = -\frac{19}{11}$
9. (a) $x = 15, y = -5$
   (b) $x = -2, y = 11$
   (c) $x = 9, y = 4$
   (d) $x = 3, y = 6$
10. $p = 1, q = -2$
11. $p = 2, q = 3$
12. 19 m, 6 seconds
13. (a) $x = -1, y = 3$
    (b) $x = 2, y = 1$
    (c) $x = 1, y = 4$
    (d) $x = -2, y = \frac{1}{2}$

### Exercise 2D

1. 25, 113
2. 5, 15
3. 8, 40
4. $15, $27
5. $2, $2.40
6. $\frac{47}{7}, \frac{48}{7}$
7. 46°, 74°
8. 21 cm
9. 875 cm$^2$
10. 32 cm
11. Polar bear: 7 years old, Giant panda: 6 years old
12. $69
13. 7, 20
14. 10
15. $32, $48
16. $13 000, $12 000
17. $\frac{3}{5}$
18. 6, 14
19. $0.45, $1.50
20. (i) 4000 (ii) $5
21. 72

## Chapter 3 Linear Inequalities

### Practise Now 1

(a) $x > 6$ (b) $x \leqslant -5$
(c) $x \leqslant 3$ (d) $x > -2.5$

### Practise Now 2

(a) $x \geqslant 10$ (b) $y < \frac{1}{2}$

### Practise Now 3

$x > 14$
(i) 17 (ii) 27

### Practise Now 4

1. (a) $x \geqslant -\frac{6}{5}$ (b) $y < -\frac{1}{3}$
   (c) $z \leqslant -\frac{11}{2}$
2. 4

### Practise Now 5

(a) $x < \frac{3}{2}$ (b) $x < -\frac{5}{2}$
(c) $x \geqslant 6$ (d) $x \geqslant \frac{9}{2}$

### Practise Now 6

12

### Practise Now 7

60

### Introductory Problem Revisited

(a) $x \geqslant 65$ (b) No

### Practise Now 8

13

### Practise Now 9

$-1 \leqslant x \leqslant 5$

### Practise Now 10

1. No solution
2. $-13 < y \leqslant 7$

### Practise Now 11

38 20-cent coins and 10 50-cent coins

### Practise Now 13

2. $2y \leqslant -x + 4$; $y + 2x + 1 \geqslant 0$; $y > x - 4$
3. (i) $p > 2q$; $p + q > 5$; $p < 8$
   (iii) $p = 6$, $q = 0$

### Exercise 3A

1. (a) $<$ (b) $<$
   (c) $>$ (d) $>$
   (e) $\leqslant$ (f) $\leqslant$
2. (a) $x < 7$ (b) $x \leqslant -\frac{21}{2}$
   (c) $y \geqslant 5$ (d) $y > -4$
   (e) $a < 1$ (f) $b \geqslant 7$
   (g) $c < -2$ (h) $d \geqslant 0$
3. (a) $\frac{8}{7}$ (b) 2
   (c) 33
4. (a) $a > \frac{7}{2}$ (b) $b < -\frac{2}{5}$
   (c) $c \geqslant -\frac{3}{2}$ (d) $d \leqslant 0$
5. $x \leqslant \frac{9}{2}$
   (i) 4 (ii) 4
6. (a) $e \leqslant 6$ (b) $f > \frac{1}{4}$
   (c) $g < 3$ (d) $h \geqslant \frac{5}{4}$
   (e) $a \geqslant \frac{3}{2}$ (f) $b < \frac{7}{20}$
7. (a) $p > \frac{16}{11}$ (b) $q \leqslant -\frac{33}{40}$
   (c) $r \leqslant \frac{127}{30}$ (d) $s < \frac{29}{125}$
8. (a) $x < -\frac{7}{2}$
   (b) $x < \frac{8}{7}$
   (c) $x \geqslant -6$
   (d) $x \geqslant 1$
9. (a) Always true
   (b) Always true
   (c) Always true
   (d) Sometimes true
   (e) Sometimes true
   (f) Never true

### Exercise 3B

1. 19
2. 14
3. 87
4. 15 625
5. 18
6. 11
7. A if $\leqslant$ 10 days, B if $>$ 10 days

### Exercise 3C

1. (a) $-2 \leqslant x \leqslant 7$
   (b) $-\frac{4}{3} < x < 5$
2. (a) $-2, -1, 0$
   (b) 3, 4, 5, 6, 7, 8
3. (a) $x \geqslant 2$
   (b) $3 < x \leqslant 5$
   (c) $-9 < x < -3$
4. Any integer between 121 and 157 inclusive.
7. (a) $y \leqslant -x + 1$; $y < x - 2$; $y \leqslant -2$
   (b) $y - 3x + 3 < 0$; $x \leqslant 1$; $y > -6$
8. 29, 31, 37, 41, 43, 47
9. 18
10. (a) $\frac{7}{2} \leqslant a \leqslant \frac{13}{3}$
    (b) $1 < b < 4$
    (c) $\frac{4}{3} < c < 6$
    (d) $0 \leqslant d < 3$
11. (a) $-8 \leqslant a \leqslant 4$
    (b) $b \leqslant -6$
    (c) $\frac{5}{6} \leqslant c < 1$
    (d) $-2 \leqslant d < \frac{25}{3}$
12. $180
13. (a) $y \leqslant -x + 1$; $y > -2$; $3y \geqslant 5x - 15$
    (b) $y \leqslant -x + 1$; $y < -2$; $3y \leqslant 5x - 15$
14. $a = 1$, $b = 4$
15. (a) $a = 1$, $b = 5$, $c = 4$
    (b) $a = 1$, $b = -1$, $c = 6$
    (c) $a = 3$, $b = 6$, $c = 4$
16. No

## Chapter 4 Expansion and Factorisation of Algebraic Expressions

### Practise Now 1

(a) $-5x^2$ (b) $-9x^2$
(c) $-10y^2$ (d) $-13y^2$
(e) $-12w^2$ (f) $-23w^2$
(g) $29x^2$ (h) $-10x^2$

### Practise Now 2

(a) $-7x^2 - 4$
(b) $2x^2 - 5x$
(c) $-y^2 + 4xy$
(d) $-12y^2 - 9y + 3xy$
(e) $4a^2 - 9b^2 - 4$
(f) $19h^2 + 4k^2 - 3hk$

### Practise Now 3

(a) $2x + 10$
(b) $-18x + 3y$
(c) $5 + 2ab - 3ac$
(d) $-4 + 14ax + 12ay$

### Practise Now 4

(a) $-13xy - 7xz$
(b) $-26pq - 6pr - 12qr$

### Practise Now 5

(a) $8ax + 7ay + 8bx + 7by$
(b) $10cx + 18cy + 5dx + 9dy$
(c) $5ax - 10ay + 2x - 4y$
(d) $18ac - 6ad + 15bc - 5bd$
(e) $2cx + 3dx - 8cy - 12dy$
(f) $21ax + 14bx - 3a - 2b$
(g) $18pr - 24ps - 15qr + 20qs$
(h) $14px - 6py - 63qx + 27qy$
(i) $21ac - 9ad + 35bc - 15bd$
(j) $-12r + 8rt + 20ru - 9s + 6st + 15su$

### Practise Now 6

(a) $12ad - ac - bc + 4bd$
(b) $-y^2 + 3xy + xz + 3yz$
(c) $pr + 7ps + 8qr - 9qs$
(d) $2h^2 + 8km$

### Practise Now 7

(a) $8x^2 + 12x$
(b) $-11a^2 + 44a$
(c) $-15x^2 - 20x$
(d) $-12n^2 + 29n$

### Practise Now 8

(a) $7x^2 - 7x - 6$
(b) $-2x^2 + x + 20$
(c) $27y^2 - 17y - 8$
(d) $-4k^2 - 57k$

### Practise Now 9

(a) $xy^2z + x^3y - x^2y^2$
(b) $2h^2km + h^2m - h^2m^2$

### Practise Now 10

(a) $12x^2 + 41x + 35$
(b) $9x^2 + 50x - 24$
(c) $-21y^2 - 71y + 22$
(d) $21k^2 - 47k + 20$

### Practise Now 11

(a) $-2x^2 + 25x - 8$
(b) $-19y^2 + 167y - 141$

### Practise Now 12

1. $10x^2 - 33xy - 7y^2$
2. $10v^2 - 7vw - 21w^2$

### Practise Now 13

(a) $3x^3 + 11x^2 + 12x + 4$
(b) $2p^3 + 3p^2q - 5pq^2 - 6q^3$

### Practise Now 14

(a) $4(3x + 2)$
(b) $7(3 + 5a)$
(c) $-5(3x + 5)$
(d) $-4(2 + 5p)$
(e) $3a(4y - 9x)$
(f) $-6x(7y + 2z)$
(g) $18p(2 - 3q + r)$
(h) $-3z(3 + 8b + 5c)$

### Practise Now 15A

(a) $2x(5x + 4)$
(b) $5a(2a - 3)$
(c) $-7b(7 + 4b)$
(d) $2\pi r(r + h)$
(e) $yz^2(x^2z - 1)$
(f) $c^2d^2(d + c - 1)$

### Practise Now 15B

(a) $(x+1)(x+5)$
(b) $(x+5)(x-3)$
(c) $(x+3)(x-4)$
(d) $(x+7)(x-2)$
(e) $(y-2)(y-6)$
(f) $(y-1)(y-5)$
(g) $(z+2)(z+6)$
(h) $(z+1)(z-8)$

### Introductory Problem Revisited

$(x+8)$ cm

### Practise Now 15C

(a) $(2x+3)(x+2)$
(b) $(3x-2)(x+4)$
(c) $(3y-4)(2y-1)$
(d) $(1-2x)(x+7)$ or $-(2x-1)(x+7)$
(e) $-(3-x)^2$ or $(x-3)(3-x)$
(f) $(6x-5)(3-x)$ or $(5-6x)(x-3)$ or $-(6x-5)(x-3)$
(g) $2(2x+1)(x-2)$
(h) $-(5a+2)(a+3)$

### Practise Now 16

2. No

### Practise Now 17

(a) $(x-2y)(x+4y)$
(b) $(x+3y)(x-5y)$
(c) $(6x+5y)(x+y)$
(d) $3(2x-3y)(x-2y)$
(e) $(a-3b)(2b-a)$ or $(3b-a)(a-2b)$ or $-(a-3b)(a-2b)$
(f) $-2(c-3d)^2$ or $2(c-3d)(3d-c)$
(g) $3p(2q-5r)(q-7r)$
(h) $(3xy-8)(xy+2)$

### Practise Now 18

(a) $a(a+3)(a+2)$
(b) $b(b+5)(b-1)$
(c) $c(2c-1)(c+4)$
(d) $d(2d+3)(d-6)$
(e) $e(3e-2)(e-1)$
(f) $f(4f-1)(f-4)$
(g) $g(2g+1)(2g-3)$
(h) $h(3h-1)(2h-1)$

### Practise Now 19

(a) $(a+2d)(b+c)$
(b) $(3p+7s)(q+r)$
(c) $(3a-4b)(2x+5y)$
(d) $(h+6k)(3p-2q)$

### Practise Now 20

(a) $(3x-4)$ and $(2y-5)$
(b) $(3a-7)$ and $(2b-3c)$

### Practise Now 21

(a) $(x+y)(x-3)$
(b) $(3w-4)(5w-2z)$

### Practise Now 22

(a) $x^2+12x+36$
(b) $16y^2+24y+9$
(c) $49+42a+9a^2$
(d) $\frac{1}{4}x^2+8x+64$
(e) $4x^2+12xy+9y^2$
(f) $25a^2+20ab+4b^2$

### Practise Now 23

(a) $x^2-8x+16$
(b) $25y^2-30y+9$
(c) $64-32a+4a^2$
(d) $\frac{4}{9}x^2-8x+36$
(e) $b^2-6ab+9a^2$
(f) $9a^2-24ab+16b^2$

### Practise Now 24

(a) $x^2-9$ (b) $25y^2-16$
(c) $9-4a^2$ (d) $64-\frac{1}{16}x^2$
(e) $4x^2-49y^2$ (f) $36b^2-a^2$

### Practise Now 25

(a) 10 609 (b) 1 002 001
(c) 2401 (d) 38 809
(e) 39 975 (f) 639 996

### Practise Now 26

1. 86
2. 194

### Practise Now 27

(ii) $2n+3$
(iii) $4n^2+4n+1$; $4n^2+12n+9$

### Practise Now 28

(a) $(x+5)^2$ (b) N.A.
(c) $(3y+4)^2$ (d) $\left(6a+\frac{2}{3}\right)^2$
(e) $(5a+4b)^2$ (f) N.A.

### Practise Now 29

(a) $2(2x-7)^2$ (b) $\frac{1}{3}(2t-3)^2$
(c) $\left(1-\frac{1}{3}q\right)^2$ (d) $\left(\frac{4}{5}-3n\right)^2$
(e) $(5x-y)^2$ (f) N.A.

### Practise Now 30

(a) $(9x+4)(9x-4)$
(b) $(3+5y)(3-5y)$
(c) N.A.
(d) $4(a+4b)(a-4b)$
(e) $2\left(\frac{2}{5}b+3a\right)\left(\frac{2}{5}b-3a\right)$
(f) $8(x+2)(2x-3)$

### Practise Now 31

(a) 10 600 (b) 44 400
(c) −11 400 (d) 39 400

### Practise Now 32

1. (i) $(x+2y)(x-2y)$
(ii) $x=3$, $y=1$
2. (a) $(x+3)(x-3)$
(b) 47, 53

### Exercise 4A

1. (a) $-9x^2$ (b) $-12x^2$
(c) $9y^2$ (d) $-16y^2$
(e) $-13e^2$ (f) $-31f^2$
(g) $g^2$ (h) $-13h^2$
2. (a) $-10x^2-9$
(b) $-x^2-2x$
(c) $15y^2+19z-8yz$
(d) $4y^2+9xy$
(e) $12w^2-3w-8$
(f) $-h^2-16k^2+4hk$
3. (a) $10x+10$
(b) $-12x+8y$
(c) $8xy-8x$
(d) $-27xy+18xz$
(e) $2+15a-33ab$
(f) $-5-6cd-9ce$
(g) $7-42pq+18pr$
(h) $11+96st+56su$
4. (a) $xy+10xz$
(b) $10ab-34ac$
(c) $9de-20df$
(d) $-25hk-12hm$
5. (a) $4ax+9ay+4bx+9by$
(b) $25ce+10cf+5de+2df$
(c) $7mn-21mp+3n-9p$
(d) $21tv+12tw-49uv-28uw$
(e) $2ax-12ay-bx+6by$
(f) $-3hq-21hr+5kq+35kr$
6. (a) $ac-6ad-bc-3bd$
(b) $8xy+7bx-30ay-35ab$
(c) $8pq-ps-12qr+15rs$
(d) $18hk+24km-9mn-3hn$
7. (a) $13ab+3ac-6bc$
(b) $10dh-32df-56fh$
(c) $65mn-20kn$
(d) $28wy-42wx+36xy$
8. (a) $ax+3bx+x+9ay+27by+9y$
(b) $14p-2pr+10ps+35q-5qr+25qs$
(c) $44mt-33mu-110m-48nt+36nu+120n$
(d) $10vw+45wx+30wz+28vy+126xy+84yz$
9. (a) $5c^2+4ab+18ac-3bc$
(b) $15wy-3wx-23xz+3yz$
(c) $-22q^2+35pr+50pq-qr$
(d) $-169h^2-100hm-118km+39hk$
10. $143ab-230b+103bd-11ac-60c+21cd-66ad-165a+8bc$
11. No

### Exercise 4B

1. (a) $15a^2-20a$
(b) $-24b^2-40b$
(c) $15n^2-10n$
(d) $m^2+m$
2. (a) $5a^2+23a+12$
(b) $-18b^2+30b+18$
(c) $c^2-5c$
(d) $-24d^2+20d$
3. (a) $-6a^2-9ab^2$
(b) $-8c^3+20c^2d$

(c) $-7hk^2 + 3h^2k$
(d) $5x^2y - 20xy^2z$

4. (a) $6k^2 + 19km$
(b) $-6n^2 + 9np$
(c) $-3t^2w - tw^2$
(d) $-5x^2y^2$

5. (a) $x^2 + 10x + 21$
(b) $12y^2 + 23y + 5$
(c) $t^2 - 7t - 8$
(d) $v^2 - 12v + 35$

6. (a) $x^2 + 7xy + 6y^2$
(b) $x^2 - 2xy - 15y^2$
(c) $3c^2 + cd - 14d^2$
(d) $5h^2 + 32hk - 21k^2$

7. (a) $a^3 + 6a^2 + 11a + 6$
(b) $b^3 + 2b^2 - 19b - 20$
(c) $6m^3 - 11m^2n - mn^2 + 6n^3$
(d) $12x^3 - 91x^2y + 118xy^2 - 24y^3$

8. (a) $14a^2 - 25a - 12$
(b) $-10b^2 + 12b - 3$
(c) $-3c^2 + 29c$
(d) $7d^2 - 12d$
(e) $6f^2 - 41f$
(f) $-13h^2 - h$

9. (a) $39x^3y^2 - 13x^2y^2$
(b) $96m^2n - 8mn^2w + 56mn^3$
(c) $6p^2 + 2p^3q^2 + 14pqr^3$
(d) $-7s^3t + 28s^2t^3 + 21s^3tu^3$

10. (a) $2x^2z - 5x^2yz + 4x^3z + 3xz^2$
(b) $ab^3 + abc^2 + 3ab^3c$

11. (a) $6a^2 - 15a - 9$
(b) $25b^2 + 25b - 14$
(c) $28c^2 - 75c + 50$
(d) $-6d^2 - 13d + 70$
(e) $-17f^2 + f + 16$
(f) $27h^2 - 201h + 190$

12. (a) $x^2 + 4x + 8$
(b) $2y^2 + 16y - 7$
(c) $11t^2 - 23t - 18$
(d) $3w^2 - 10w - 12$

13. (a) $-2a^2 + 5a + 4$
(b) $4b^2 - 21b - 35$
(c) $7c^2 + 27c - 51$
(d) $7d$

14. (a) $x^3 + 5x^2 + 2x + 10$
(b) $2x^2 + 7xy - 4x - 15y^2 + 6y$
(c) $x^3 + 3x^2 + 3x + 2$
(d) $-3x^3 + 12x^2 - 13x + 12$

15. (a) $8x^2 - 25xy - 12y^2$
(b) $-3x^2 - 8xy + 3y^2$
(c) $16x^2 - 11xy - 12y^2$
(d) $35x^2 + 49xy - 22y^2$

16. (a) $3x^3 + 20x^2 + 31x + 10$
(b) $-8x^3 - 68x + 2$
(c) $10m^3 + 8m^2n - n^3$
(d) $6x^3 - 8x^2y - 16xy^2 + 18y^3$

17. $10x^2 + 3x - 45$, $4x^2 + 50x$; $-x^2 + 40x + 10$, $15x^2 + 13x - 55$

18. $\$(28x^2 - 12x - 25)$

## Exercise 4C

1. (a) $8(x + 8)$
(b) $-3(4p + 9q)$
(c) $4a(4w + 5v)$
(d) $4b(d - 9c)$
(e) $7x(2y - 1 + 3z)$
(f) $-u(8t + 4 + 11s)$

2. (a) $4x(x + 4)$
(b) $6y(3y - 1)$
(c) $3x(13y - 5xz)$
(d) $-2\pi y^3(4x + 5)$

3. (a) $(a + 1)(a + 8)$
(b) $(b + 3)(b + 5)$
(c) $(c - 4)(c - 5)$
(d) $(d - 2)(d - 14)$
(e) $(f - 2)(f + 8)$
(f) $(h - 10)(h + 12)$
(g) $(k + 2)(k - 6)$
(h) $(m + 1)(m - 21)$

4. (a) $(3n + 7)(n + 1)$
(b) $(2p + 1)(2p + 3)$
(c) $(3q - 4)(2q - 3)$
(d) $(4r - 3)(r - 1)$
(e) $(4s - 5)(2s + 3)$
(f) $(6t - 5)(t + 4)$
(g) $(2u + 3)(2u - 7)$
(h) $(9w + 13)(2w - 3)$

5. (a) $(a - b)(a + 4b)$
(b) $(c + 3d)(c - 7d)$
(c) $(2h - 3k)(h + 5k)$
(d) $(3m + 2n)(m - 6n)$

6. (a) $a(a + 4)(a + 1)$
(b) $b(3b + 1)(b - 3)$
(c) $c(6b - 5)(c - 1)$
(d) $d(3d - 2)(2d - 3)$

7. (a) $-xy^2(z^2 + xy)$
(b) $2a^2b^2(6b + 3a - 1)$
(c) $2\pi p(5pr - 10pq - 7qr^3)$
(d) $\frac{1}{3}v^2w(9vw - 54tw^2 + t)$

8. (a) $(7 - a)(a + 5)$ or $-(a - 7)(a + 5)$
(b) $(1 - 3b)(b - 25)$ or $(3b - 1)(25 - b)$ or $-(3b - 1)(b - 25)$
(c) $2(2c + 1)(c + 2)$
(d) $5(d - 5)(d - 24)$
(e) $4(2f - 5)(f + 3)$
(f) $3(8h + 3)(h - 1)$
(g) $2(5 - k)(2k + 3)$ or $-2(k - 5)(2k + 3)$
(h) $5(7m - 6)(m + 1)$

9. No

10. (a) $3(p + 2q)(p + 3q)$
(b) $t(2r - 5s)(r - 2s)$
(c) $(xy - 3)(xy + 5)$
(d) $(4xy + 5)(3xy - 8)$
(e) $2z(2xy - 3)(xy - 4)$

11. (a) $2x(x + 2)(x + 1)$
(b) $-3p(p + 3)(p - 2)$

12. (a) $(a + b)(x - z)$
(b) $(c + 2d)(9d - 2c)$

13. (a) $\frac{1}{9}(4p - 3)(p + 3)$
(b) $-0.2r(16q - 3)(4q + 1)$ or $0.2r(3 - 16q)(4q + 1)$

14. No

15. (i) $\frac{1}{2}(7x + 10)(2x - 17)$
(ii) $x = 10$; speed = 80 km/h, time = 1.5 h

## Exercise 4D

1. (a) $(x + 3)(y + 4)$
(b) $(a + 4)(x - 5)$
(c) $(4c - 3)(3y + 5)$
(d) $(4a + b)(x + 3y)$
(e) $(2x + z)(3y - 2)$
(f) $(d + f)(y - z)$

2. (a) $(2x - 3)(y - 4)$
(b) $(2x - 5)(3y - 2)$
(c) $(5 - 7p)(2 - q)$
(d) $(k - h)(x - y)$
(e) $(2a - c)(b - 3d)$
(f) $2(4m - n)(3x + y)$

3. No

4. (a) $(x + 2y)(1 + y)$
(b) $(x + 2y)(x - 3)$
(c) $(x + 2y)(3x - 4z)$
(d) $xy(y - 5)(x +$

5. (a) $24(y - 5x^2)(6p + q)$
(b) $2(x + 2y)(2y - x)(10y - 5x +$

6. (i) $a = 1, b = 5, c = 2$
(ii) 10 m by 4 m by 5 m

## Exercise 4E

1. (a) $a^2 + 8a + 16$
(b) $9b^2 + 12b + 4$
(c) $c^2 + 8cd + 16d^2$
(d) $81h^2 + 36hk + 4k^2$
(e) $9a^2 + 24ab + 16b^2$
(f) $4b^2 + 12ab + 9a^2$

2. (a) $m^2 - 18m + 81$
(b) $25n^2 - 40n + 16$
(c) $81 - 90p + 25p^2$
(d) $9q^2 - 48qr + 64r^2$
(e) $9a^2 - 24ab + 16b^2$
(f) $25b^2 - 30ab + 9a^2$

3. (a) $s^2 - 25$ (b) $w^2 - 100x^2$
(c) $4t^2 - 121$ (d) $49 - 4u^2$

4. (a) 1 447 209
(b) 795 664
(c) 159 991
(d) 3 999 996

5. 56

6. 40

7. (a) $\frac{1}{25}a^2 + \frac{6}{5}ab + 9b^2$
(b) $\frac{1}{4}c^2 + \frac{2}{3}cd + \frac{4}{9}d^2$

8. (a) $\frac{9}{4}h^2 - 15hk + 25k^2$
(b) $\frac{36}{25}m^2 + \frac{36}{5}mn + 9n^2$

9. (a) $25 - 36p^2$
(b) $81r^2 - \frac{16}{25}q^2$
(c) $\frac{t^2}{9} - \frac{s^2}{4}$
(d) $u^4 - 16$

10. (a) $x^2 + 24x + 84$
(b) $23x^2 + 8xy - 57y^2$

11. 6

12. 25

13. $\frac{1}{256}x^4 - \frac{1}{625}y^4$

14. (i) $4q^2$ (ii) 400

15. (i) $a^2$ (ii) 9

16. (ii) $2m + 2$
(iii) $4m^2$; $4m^2 + 8m + 4$

17. **(ii)** $2m + 3$
**(iii)** $4m^2 + 4m + 1$;
$4m^2 + 12m + 9$

### Exercise 4F

1. **(a)** $(a + 7)^2$ **(b)** $(2b + 1)^2$
**(c)** $(c + d)^2$ **(d)** $(2h + 5k)^2$
**(e)** $(3a + 5b)^2$
2. **(a)** $(m - 5)^2$ **(b)** $(13n - 2)^2$
**(c)** $(9 - 10p)^2$
**(d)** $(7q - 3r)^2$
3. **(a)** $(s + 12)(s - 12)$
**(b)** $(6t + 5)(6t - 5)$
**(c)** $(15 + 7u)(15 - 7u)$
**(d)** $(7w + 9x)(7w - 9x)$
4. **(a)** 1800 **(b)** −680
**(c)** 54 **(d)** −8200
5. **(a)** $3(a + 2)^2$
**(b)** $\left(5b+\frac{1}{2}c\right)^2$
**(c)** N.A.
**(d)** $\left(\frac{4}{7}w+\frac{1}{5}v\right)^2$
**(e)** $(h^2 + k)^2$
**(f)** N.A.
6. **(i)** $(x + 2)$ cm
**(ii)** $(x^3 + 6x^2 + 12x + 8)$ cm$^3$
7. **(a)** $4(3m - 2n)^2$
**(b)** N.A.
**(c)** $\frac{1}{3}(p-q)^2$
**(d)** $\left(4r-\frac{1}{8}s\right)^2$
**(e)** $(5 - tu)^2$
**(f)** $3\left(5w-\frac{1}{4}z\right)^2$
8. **(a)** $2(4a + 7b)(4a - 7b)$
**(b)** $\left(c+\frac{1}{2}d\right)\left(c-\frac{1}{2}d\right)$
**(c)** $(m + 8n^2)(m - 8n^2)$
**(d)** N.A.
**(e)** $\left(\frac{3h}{10}+4k\right)\left(\frac{3h}{10}-4k\right)$
**(f)** $15\left(2p+\frac{1}{8}q\right)\left(2p-\frac{1}{8}q\right)$
9. **(a)** $a(a + 6)$
**(b)** $-(5b + 19)(5b + 11)$
**(c)** $(c + d + 2)(c - d - 2)$
**(d)** $(2h - 1 + 2k)(2h - 1 - 2k)$
**(e)** $3(x - 6)(3x + 8)$
**(f)** $4p$
10. 38 000
11. **(i)** $(x + 2y)(x - 2y)$
**(ii)** $x = 7, y = 3$
12. **(a)** $-(11x + 7)(7x + 11)$
**(b)** $(4x + 1 + 3y)(4x + 1 - 3y)$
**(c)** $(2x + y - 2)(2x - y + 2)$
**(d)** $13(x + y + 1)(x + y - 1)$
13. **(a)** $(x + 11)(x - 11)$
**(b)** 79, 101

## Chapter 5 Number Patterns

### Practise Now 1

1. **(a)** 28, 33 **(b)** −50, −56
**(c)** 1215, 3645
**(d)** −18, 6
2. **(a)** 22, 29 **(b)** 15, 11

### Practise Now 2

**(i)** 23 **(ii)** 58

### Practise Now 3

1. **(a)** $T_n = 4n + 1$
**(b)** $T_n = 5n + 2$
**(c)** $T_n = 6n - 4$
**(d)** $T_n = 3n - 2$
2. **(i)** 30, 38 **(ii)** $8n - 18$
**(iii)** 382

### Practise Now 4

1. **(a)** $T_n = n^2 + n + 1$
**(b)** $T_n = \frac{n^2}{2} + 2n - 1$
**(c)** $T_n = -2n^2 - 4n$
**(d)** $T_n = -\frac{n^2}{4}$
2. **(i)** 17
**(ii)** $T_n = 2n^2 - 4n + 19$

### Practise Now 5

1. **(a)** $T_n = n^3 + 2n^2 + 3n + 4$
**(b)** $T_n = -2n^3 + 4n^2$
**(c)** $T_n = -n^2 + 6n + 2$
**(d)** $T_n = n^3 + 0.5n^2 + 7n - 4$
2. **(ii)** $T_n = -1.5n^3 + 0.5n^2 + 2n - 1$
**(iii)** −4921

### Practise Now 6

1. **(a)** $T_n = 2^{(n-1)}$
**(b)** $T_n = \left(\frac{1}{2}\right)^{(n-1)}$
**(c)** $T_n = -3\left(\frac{3}{2}\right)^{(n-1)}$
**(d)** $T_n = -0.2^{(n-1)}$
2. **(i)** $\frac{3}{4}$
**(ii)** $T_n = \left(\frac{3}{2}\right)[2^{(n-1)}]$
**(iii)** 384

### Practise Now 7

1. **(ii)** $2 + 5 \times 4 = 22$;
$2 + 6 \times 4 = 26$;
$2 + n \times 4 = 4n + 2$
**(iii)** 8082 **(iv)** No
2. **(i)** $72 = 8 \times 9$
**(ii)** 10 **(iii)** Yes
3. **(ii)** 5, 15; 6, 21; $n$, $\frac{1}{2}n(n+1)$
**(iii)** 5050 **(iv)** No

### Practise Now 8

**(i)** 6, 12; 7, 14; $n + 1$, $2n + 2$
**(ii)** 54; 110 **(iii)** 59; 60
**(iv)** Possible values:
$a = 5$ and $b = 7$

### Exercise 5A

1. **(a)** 39, 44 **(b)** 40, 32
**(c)** 384, 768 **(d)** 50, 25
**(e)** 16, −4 **(f)** −288, 576
**(g)** −87, −94 **(h)** −50, −40
2. **(i)** 15 **(ii)** 21
**(iii)** 105
3. **(a)** $T_n = 6n + 1$
**(b)** $T_n = 3n - 7$
**(c)** $T_n = 7n + 53$
**(d)** $T_n = -3n + 17$
4. **(i)** 42, 49 **(ii)** $7n$
**(iii)** 735
5. **(i)** 30, 34 **(ii)** $4n + 6$
**(iii)** 806
6. **(a)** 7, 2, −3
**(b)** **(i)** $7n - 2$
**(ii)** 103
**(iii)** 32
7. **(ii)** $T_n = 2n^2 - 3n + 1$
**(iii)** 319
8. **(ii)** $T_n = n^3 - 2n^2 + n + 4$
**(iii)** 8404
9. **(i)** $a = 3$; $r = 16$
**(iii)** 768, 12 288, 196 608
10. **(a)** 9, 15 **(b)** 12,
**(c)** −33, −32 **(d)** 88, 85
11. **(i)** 459 **(ii)** Yes
12. **(i)** 216, 343 **(ii)** $n^3$
**(iii)** 15
13. **(a)** **(i)** −38, −45
**(ii)** $T_n = -7n + 4$
**(b)** **(i)** $T_n = -7n - 86$
**(ii)** Yes
14. **(a)** 3, 9, 19, 33
**(b)** **(i)** $T_n = 2n^2 - 1$
**(ii)** 2887
15. **(i)** $a = 4$; $b = -12$; $c = 9$
**(ii)** 9, 25, 49
16. **(a)** **(i)** $a = 2$; $d = 3$
**(ii)** 13, 42, 103, 208
**(b)** **(i)** $T_n = 2n^3 - 2n^2 + n - 1$
**(ii)** 832 574
17. **(a)** $T_n = 3[4^{(n-1)}]$
**(b)** **(i)** $T_n = 3[4^{(n-1)}] + n^2$
**(ii)** 786 532
18. **(a)** −67, −131
**(b)** 8, 13
**(c)** 144, 196
**(d)** −216, 343

### Exercise 5B

1. **(ii)** 2, 3, 4, 5, $n - 1$
**(iii)** 29
2. **(i)** 16, 25, 36, 49, $(n + 1)^2$
**(ii)** 441 **(iii)** 10
**(iv)** No
3. **(i)** 5; 6; 11
**(ii)** **(a)** $n + 1$
**(b)** $2n + 1$
**(iii)** 65 **(iv)** Yes
4. **(i)** $54 = 6 \times 9$
**(ii)** 13 **(iii)** No
5. **(i)** $1 + 3 + 5 + 7 + 9 + 11$
$= 36 = 6^2 = (5 + 1)^2$
**(ii)** $a = 25$, $c = 13$, $d = 12$
**(iii)** No
6. **(i)** 89 760; 91 520;
$1760n + 88\,000$
**(ii)** 19 years
7. **(i)** 1 5 10 10 5 1
**(ii)** $1 + 5 + 10 + 10 + 5 + 1$
$= 32 = 2^5$; $2^{n-1}$
**(iii)** $2^n - 1$
**(iv)** No

8. (i) 7, 12; 8, 14; $n + 2$, $2n + 2$
   (ii) 23; 48
   (iii) Possible values: $j = 13$ and $k = 18$; 33

## Chapter 6 Financial Transactions

### Practise Now 1A

1. (i) PKR 230 400; PKR 108 000
   (ii) 25%
2. (i) 250
3. (ii) 625

### Practise Now 1B

1. Number of pens, 45; number of pencils, 27; number of notebooks, 54
2. (i) 210
   (ii) 55 toys per hour

### Practise Now 1C

1. (a) $33\frac{1}{3}\%$ or 33.3%
   (b) 20%
2. (a) PKR 142 240
   (b) $73 696

### Practise Now 2

1. (a) PKR 213 000
   (b) $0.14

### Practise Now 3

1. 12%
2. $564

### Practise Now 4

1. (i) PKR 122 000
   (ii) No
2. (i) $200 (ii) $153

### Practise Now 5

1. PKR 22 420
2. $544

### Practise Now 6

1. PKR 706
2. PKR 2500

### Practise Now 7

1. PKR 280 000
2. $725 000

### Practise Now 8

1. PKR 10 000
2. 3.5% per annum

### Practise Now 9

1. PKR 22 360
2. PKR 25 480

### Practise Now 10

(i) PKR 11 305
(ii) PKR 639 540
(iii) PKR 155 040

### Practise Now 11

1. $24 750; $174 750
2. 4 years

### Practise Now 12

1. PKR 6465
2. (a) $60.60 (b) $61.16
3. 3%

### Practise Now 13

1. PKR 3710
2. PKR 50 200

### Practise Now 14

1. $7308

### Practise Now 15

1. Wife, PKR 36 000; each daughter, PKR 42 000; each son, PKR 84 000
2. PKR 770 000

### Practise Now 16

1. (i) 3 : 1 : 6
   (ii) Imran, PKR 24 030; Cheryl, PKR 8010; Joyce, PKR 48 060
2. $2040

### Exercise 6A

1. (a) $5; Profit of 12.5%; Profit of $11\frac{1}{9}\%$ or 11.1%
   (b) $120; Loss of 20%; Loss of 25%
   (c) PKR 91 520; PKR 3520; Profit of $3\frac{11}{13}\%$ or 3.85%
   (d) PKR 4402; PKR 1278; Loss of $29\frac{1}{31}\%$ or 29.0%
   (e) $24; $4.14; Profit of 14.7%
   (f) $545; $38.15; Loss of 7.53%
2. 25%
3. (i) PKR 620 000
   (ii) PKR 837 000
4. $18.75
5. 20%
6. PKR 2816
7. (i) $700 (ii) $651
8. PKR 23 600
9. PKR 291 000
10. (a) $16 250 (b) $480 000
11. $33\frac{1}{3}\%$ or 33.3%
12. PKR 38 400
13. $0.99
14. 12%
15. Possible cost price = $10, selling price = $13
16. (i) PKR 484 000
    (ii) No
17. (i) $500 (ii) $361.20
18. PKR 876
19. $18 000
20. There is no difference.
21. PKR 2250

### Exercise 6B

1. PKR 17 360
2. PKR 48 750
3. (i) $134.87
   (ii) $3716.80
   (iii) $516.80
4. PKR 53 760
5. 2.5 years
6. PKR 12 986
7. PKR 450 000
8. PKR 99 838
9. 1600
10. 2.5
11. (i) PKR 9400
12. (i) PKR 21 074
    (ii) PKR 20 991
13. 3.01%
14. $2264
15. 32 000.00

### Exercise 6C

1. PKR 10 000
2. PKR 3550
3. (a) PKR 2125
   (b) PKR 311 250
   (c) PKR 1 095 000
   (d) PKR 3 007 500
4. PKR 117 369
5. Wife, PKR 31 200; each daughter, PKR 54 600; each son, PKR 109 200
6. PKR 67 000
7. $1267.50
8. PKR 290 160
9. PKR 5 140 000, PKR 3 855 000, PKR 3 855 000
10. $150 500, $236 500, $107 500
11. $105 500
12. Nadia, PKR 260 800; Joyce, PKR 391 200; Waseem, PKR 326 000
13. (i) PKR 5 100 000
    (ii) Ken, PKR 1 275 000; Shaha, PKR 2 125 000; David, PKR 1 700 000

## Chapter 7 Direct and Inverse Proportions

### Practise Now 1

(a) $15.95 (b) 8 kg

### Practise Now 2

1. (i) $y = 5x$ (ii) 50
   (iii) 12
2. 17.5
3. 8, 9.5; 24, 42

### Practise Now 3

(i) $C = \frac{5}{3}d$ (ii) $75
(iii) 72 km

### Practise Now 4

(i) $8280
(ii) 58
(iii) $C = 41n + 5000$
(iv) No

### Practise Now 5

(a) $y$ and $x^2$ (b) $\sqrt{y}$ and $x^3$

### Practise Now 6

1. (i) $y = 2x^2$ (ii) 50
   (iii) $\pm 4$
2. 84
3. 2.5, 7; 36, 225

### Practise Now 7

(i) $l = 24.8T^2$ (ii) 15.9 cm
(iii) 1.20 seconds

### Practise Now 8

40 minutes

### Practise Now 9

(a) 10.5 hours (b) 21 minutes

### Introductory Problem Revisited

11

### Practise Now 10

1. (i) $\frac{5}{4}$ (ii) $y = \frac{10}{x}$
   (iii) 1
2. 6
3. 1, 5; 8, $\frac{4}{3}$

### Practise Now 11

(i) 2 A (ii) 2 Ω

### Practise Now 12

(a) $y$ and $x^2$
(b) $y^2$ and $\sqrt[3]{x}$
(c) $y$ and $x + 2$

### Practise Now 13

1. (i) $\frac{1}{2}$ (ii) $y = \frac{32}{x^2}$
   (iii) $\pm 2$
2. 3.6
3. $\frac{1}{4}$, 36; 4, 2

### Practise Now 14

(i) 1.6 N (ii) 1.26 m

### Exercise 7A

1. (i) $41\frac{2}{3}$ kg or 41.7 kg
   (ii) 72
2. (i) 1.25 m (ii) 32
3. (i) $x = 1.5y$ (ii) 9
   (iii) 8
4. (i) $Q = 7P$ (ii) 35
   (iii) 6
5. (a) \$60 (b) \$ $\frac{ac}{b}$
6. $3\frac{3}{5}$ kg or 3.6 kg
7. 4.5
8. 4
9. (a) 36, 44; 1, 5
   (b) 8, 9.5; 2.4, 6.6
10. (i) $y = 4x$
11. (i) $z = 8y$
12. (i) $F = 9.8m$ (ii) 137.2
    (iii) 22
13. (i) $P = 2.5T$ (ii) 60
    (iii) 4.8
14. (i) $V = 1.5R$ (ii) 22.5
    (iii) 10
15. 95 tonnes
16. (i) \$1360 (ii) 135
    (iii) $D = 8n + 600$
    (iv) No

### Exercise 7B

1. (i) $x = 4y^3$ (ii) 864
   (iii) 3
2. (i) $z^2 = 2w$ (ii) $\pm 6$
   (iii) 12.5
3. (i) 2 (ii) 3
4. (a) $y$ and $x^2$
   (b) $y$ and $\sqrt{x}$
   (c) $y^2$ and $x^3$
   (d) $p^3$ and $q^2$
5. $\pm 27$
6. $-3$ or 5
7. 5, 7; 81, 192
8. 0.5, 1.8; 0.016, 0.686
9. (i) $L = 2.5\sqrt{N}$
   (ii) 5 cm
   (iii) 36 hours
10. 16
11. $4a$
12. 800%
13. (i) $x^3$

### Exercise 7C

1. (b), (c) and (e)
2. 16 days
3. (i) 8 (ii) $x = \frac{200}{y}$
   (iii) 0.5
4. (i) $Q = \frac{1}{2P}$ (ii) 0.1
   (iii) 2.5
5. 5
6. (i) 840 (ii) 40 days
7. 2 days
8. 0.5
9. 5
10. (a) 0.5, 8; 6, 4.8
    (b) 4.5, 14.4; 12, 1.44
11. (i) 600 kHz (ii) 375 m
12. (i) $t = \frac{24}{N}$ (ii) 4 hours
    (iii) 32
13. $4\frac{14}{19}$ minutes or
    4.74 minutes
14. 36 minutes
15. 70
16. $R = \frac{96}{C}$

### Exercise 7D

1. (i) 6.25 (ii) $x = \frac{400}{y^3}$
   (iii) 5
2. (i) $z = \frac{27}{\sqrt{w}}$ (ii) 6.75
   (iii) 81
3. (a) $y$ and $x^2$
   (b) $y$ and $\sqrt{x}$
   (c) $y^2$ and $x^3$
   (d) $n$ and $m - 1$
   (e) $q$ and $(p + 1)^2$
4. $\frac{10}{3}$
5. $\pm 2.5$
6. 10, 20; 10, 1.25
7. (i) $F = \frac{k}{d^2}$,
   where $k$ is a constant
   (ii) 80 N
8. (i) 20 cm (ii) 12 cm
9. 6
10. $\frac{b}{9}$
11. 0.04
12. No

## Chapter 8 Congruence and Similarity

### Practise Now 1

*A* and *H*; *B* and *E*; *C* and *F*; *D*, *G* and *I*

### Practise Now 2

(i) 5 (ii) $DC$
(iii) $AD$, 2 (iv) $BC$, 5.3
(v) $\angle ABC$, 90

### Practise Now 3

(a) No
(b) $\triangle DEF \equiv \triangle TSU$
(c) No

### Practise Now 4

(a) (i) 38° (ii) 28°
    (iii) 28° (iv) 27 cm
    (v) 9 cm
(b) $AC \parallel ED$

### Practise Now 5

(a) No (b) Yes

### Practise Now 6

1. $x = 30, y = 4.2$
2. $w = 60, x = 100, y = 7.2, z = 4.5$

### Practise Now 7

1. $a = 30, b = 7.5$
2. $x = 72, y = 9.125$

### Practise Now 8

5 m

### Practise Now 9

1. 18 cm, 30 cm
2. 7.5 cm, 8 cm
3. 4.5 m

### Practise Now 10A

1. (i) 3.125 m (ii) 1.36 cm
2. (i) 268 m (ii) 26.8 cm

### Practise Now 10B

(i) 1 : 150 (ii) 12 m

### Practise Now 11

1. (i) 32.5 km (ii) 5 cm
   (iii) $\frac{1}{500\,000}$
2. (i) 1 km (ii) 29 cm

### Practise Now 12

1. (i) 12 km² (ii) 4.5 cm²
2. (i) 1 : 20 000
   (ii) 0.56 km²

### Exercise 8A

1. $A$ and $F$; $B$ and $J$; $C$ and $E$; $D$ and $G$; $I$ and $K$
2. (i) 3.5 cm (ii) $VZ$
   (iii) $WX$, 3.5 cm
   (iv) $ZY$, 2.1 cm
   (v) $YX$, 2 cm
   (vi) $\angle VWX$, 90
3. $EF = 3.4$ cm, $GH = 2.4$ cm, $\angle FEH = 100°$, $\angle FGH = 75°$, $MN = 5$ cm, $OL = 3$ cm, $\angle LMN = 65°$, $\angle NOL = 120°$
4. (a) $\triangle ABC \equiv \triangle PQR$
   (b) $\triangle DEF \equiv \triangle TSU$
   (c) No
5. (i) 56° (ii) 16 cm
6. (i) 75° (ii) 10.9 cm
7. (i) 6 cm (ii) 96°

### Exercise 8B

1. (a) $x = 90, y = 35, z = 55$
   (b) $x = 28, y = 34$
   (c) $x = 7.2, y = 10.8$
   (d) $x = 9.6, y = 5\frac{5}{6}$ or 5.83
2. (a) No (b) Yes
3. (a) $x = 95, y = 52, z = 4.8$
   (b) $x = 80, y = 10.5$
4. $x = 16, y = 1.875$
5. $x = 100, y = 270, z = 100$
6. $x = 56, y = 6$
7. $x = 67, y = 10.5$
8. 90 cm
9. 8 m
10. 50 cm$^2$
11. (a) (i) $4y$ m (ii) 1.5 m
    (b) $\frac{hk}{h+k}$ m
12. (i) 4 m

### Exercise 8C

1. 10 cm, 3.5 cm
2. (i) 2 (ii) 8 cm, 7 cm
3. (i) 4.5 m by 3.75 m
   (ii) 6.75 m$^2$ (iii) 78.75 m$^2$
4. (i) 1 cm to 9 km
   (ii) 50.4 km
5. (i) 1300 km (ii) $\frac{1}{5\ 000\ 000}$
6. (i) 1.1 km (ii) 0.5 cm
7. (i) 320 km$^2$ (ii) 2 cm$^2$
8. 4 cm, 9 cm
9. 8 cm
10. (i) 1 cm to 0.25 m
    (ii) 17 cm
11. (i) 273 m (ii) 22.8 cm
12. (i) 26.3 km (ii) 15.0 cm
13. (i) 1 : 12 500
    (ii) 312.5 m
14. (i) 21.9 km$^2$
    (ii) 0.505 cm$^2$
15. (i) 1 : 50 000
    (ii) 56 cm (iii) 3 km$^2$
16. (i) $\frac{1}{6\ 000\ 000}$
    (ii) 312 km
    (iii) \$82.80
    (iv) 2 hours 42 minutes
    (v) 42 km/h
17. 50

## Chapter 9 Pythagoras' Theorem

### Practise Now 1A

(a) $AB$ (b) $DE$
(c) $PQ$

### Practise Now 1B

1. 10 m
2. 25 cm

### Practise Now 2

1. 9 cm
2. 21 m

### Practise Now 3

1. (a) (i) 4 cm (ii) 8.54 cm
   (b) $C$
2. (i) 60 cm (ii) 40.5 cm
3. (i) 13.9 cm (ii) 13.4 cm

### Practise Now 4

1. 27.8 m
2. Yes

### Practise Now 5

8.68 m

### Practise Now 6

No

### Practise Now 7

(i) 20.8 km (ii) 36.6 km

### Introductory Problem Revisited

3.09 m

### Practise Now 8

1. (a) No (b) Yes; $\angle R$
2. (ii) 46.1 m

### Exercise 9A

1. (a) $a = 29$ (b) $b = 37$
   (c) $c = 15.6$ (d) $d = 37.0$
2. 17 cm
3. 8.67 m
4. (a) $a = 36$ (b) $b = 12.8$
   (c) $c = 7.33$ (d) $d = 20.0$
5. 56 cm
6. 8.98 m
7. (i) 28 cm (ii) 10.8 cm
8. Possible lengths: 5 cm, 3 cm
9. (a) 5.73 m (b) $H$
10. (a) $a = 22.6$ (b) $b = 21.9$
    (c) $c = 15.1$ (d) $d = 28$
    (e) $e = 44.6$
11. (a) $a = 15, b = 61.8$
    (b) $c = 4.28, d = 30.3$
    (c) $e = 4.29, f = 14.7$
    (d) $g = 32.1$
12. (i) 11.4 m (ii) 111 m$^2$
13. 9.85 cm
14. 669 m$^2$

### Exercise 9B

1. 50.3 m
2. 70.7 m
3. 58.3 m
4. 4.66 m
5. 24 inches
6. 17.5 m
7. 3
8. 31.1 cm
9. 13 cm
10. (i) 4.34 m (ii) 1.72 m
11. 60
12. (i) Possible area: 108 m$^2$
    (ii) \$5940
13. (i) 9.22 cm (ii) 5.86 cm
14. (i) 93 m$^2$ (ii) 26.6 m
    (iii) 12.4 m
15. (a) (i) 33 cm (i[illegible]
    (b) 1089 cm$^2$, 1386 cm$^2$
    (c) (i) 44 cm (ii) 838 cm$^2$
    (d) Circle
16. 10.2 km

### Exercise 9C

1. (a) Yes; $\angle B$ (b) No
   (c) No (d) Yes; $\angle O$
3. No

## Chapter 10 Trigonometric Ratios

### Practise Now 1A

1. (i) $AC$ (ii) $BC$
   (iii) $AB$
2. (i) $\frac{3}{5}$ (ii) $\frac{4}{5}$
   (iii) $\frac{3}{4}$ (iv) $\frac{4}{5}$
   (v) $\frac{3}{5}$ (vi) $\frac{4}{3}$
3. (a) (i) $\frac{a}{c}$ (ii) $\frac{b}{c}$
   (iii) $\frac{a}{b}$ (iv) $\frac{b}{c}$
   (v) $\frac{a}{c}$ (vi) $\frac{b}{a}$
   (b) $a = 5, b = 5, c = 7.07$

### Practise Now 1B

(a) 0.914 (b) 3.63
(c) 0.952 (d) 3.78
(e) 15.1 (f) 0.360

### Practise Now 2

1. (a) $a = 5.5$ (b) $r = 9.45$
2. (a) $z = 9.25$ (b) $a = 12.3$
3. (a) $y = 17.1$ (b) $p = 26.5$

### Practise Now 3

1. (i) 20.5 m (ii) 10.4 m
2. (i) 6.39 m (ii) 6.62 m
   (iii) 12.4 m (iv) 8.28 m
   (v) 33.7 m (vi) 59.3 m$^2$

### Practise Now 4A

(a) 51.3° (b) 69.5°
(c) 50.9°

### Practise Now 4B

(a) $x = 35.4$ (b) $y = 64.3$
(c) $z = 57.2$

### Practise Now 5

1. (i) 23.3° (ii) 8.29 m
2. (i) 36.9° (ii) 3.75 cm

### Practise Now 6

1. 3.85 m
2. 53.1 m
3. 20.0 m

### Practise Now 7

28.4°

### Practise Now 8

9.12 m

### Practise Now 9

1.88 m

### Introductory Problem Revisited

163 m

### Exercise 10A

1. (a) (i) $PQ$ (ii) $PR$
   (iii) $QR$
   (b) (i) $XY$ (ii) $XZ$
   (iii) $YZ$
2. (a) (i) $\frac{5}{13}$ (ii) $\frac{12}{13}$
   (iii) $\frac{5}{12}$ (iv) $\frac{12}{13}$
   (v) $\frac{5}{13}$ (vi) $\frac{12}{5}$
   (b) (i) $\frac{24}{25}$ (ii) $\frac{7}{25}$
   (iii) $\frac{24}{7}$ (iv) $\frac{7}{25}$
   (v) $\frac{24}{25}$ (vi) $\frac{7}{24}$
3. (i) $\frac{y}{z}$ (ii) $\frac{x}{z}$
   (iii) $\frac{y}{x}$ (iv) $\frac{x}{z}$
   (v) $\frac{y}{z}$ (vi) $\frac{x}{y}$
4. (a) 1.07 (b) 0.967
   (c) 0.864 (d) 1.23
   (e) 2.35 (f) 0.285
5. (a) (i) $\frac{c}{a}$ (ii) $\frac{b}{a}$
   (iii) $\frac{c}{b}$ (iv) $\frac{b}{a}$
   (v) $\frac{c}{a}$ (vi) $\frac{b}{c}$
   (b) $a = 4.24$, $b = 3$, $c = 3$
6. (a) 4.66 (b) 1.53
   (c) 1.13 (d) −1.41
   (e) 0.723 (f) 0.657
   (g) 2.01 (h) 0.910

### Exercise 10B

1. (a) $a = 13.8$ (b) $b = 37.5$
2. (a) $a = 10.9$ (b) $b = 35.1$
3. (a) $a = 7.44$ (b) $b = 6.77$
4. (a) $a = 6.71$, $b = 9.95$
   (b) $c = 11.7$, $d = 10.9$
   (c) $e = 6.81$, $f = 9.76$
   (d) $g = 22.6$, $h = 24.3$
5. (i) 7.38 m (ii) 10.9 m
6. (i) 21.7 cm (ii) 37.5 cm
   (iii) 68.3 cm
7. (i) 537 m (ii) 13 200 m$^2$
8. $\frac{2}{9}$

### Exercise 10C

1. (a) 31.8° (b) 43.5°
   (c) 68.7°
2. (a) $a = 27.5$ (b) $b = 54.0$
   (c) $c = 67.8$ (d) $d = 28.4$
   (e) $e = 41.4$ (f) $f = 41.8$
   (g) $g = 48.7$ (h) $h = 41.8$
   (i) $i = 51.9$
3. (i) 64.6° (ii) 6.32 m
4. (i) 27.9° (ii) 8.04 cm
5. (i) 17.1° (ii) 7.50 m
6. 76.1°
7. (i) 71.8° (ii) 20.4 m
8. 19.3°

### Exercise 10D

1. 21.2 m
2. 15.1 m
3. 72.2 m
4. 52.9°
5. (i) 4.33 m (ii) 2.5 m
6. 17.0°
7. 7.90 m
8. 20.6 m
9. 50.0°
10. 1.53 cm
11. 50.9 cm
12. (i) 38.6 m (ii) 34.7 m
    (iii) 39.7°
13. 13.2 m

## Chapter 11 Volume, Surface Area and Symmetry of Prisms and Cylinders

### Practise Now 1

(a) (i) 10 000 000 cm$^3$
(ii) 10 000 000 ml
(b) (i) 0.165 m$^3$ (ii) 165 $l$

### Practise Now 2

1. (i) 52 cm (ii) 4446
   (iii) Yes
2. 22 cm

### Practise Now 3

3.024 m$^3$

### Practise Now 4

1. (i) 400 cm$^3$ (ii) 340 cm$^2$
2. (i) 1.152 $l$ (ii) 544 cm$^2$
3. 54 cm$^2$

### Practise Now 5

1. 160 m$^3$
2. 4.5

### Practise Now 6

(i) 162 cm$^3$ (ii) 180 cm$^2$

### Practise Now 7

1. 5730 cm$^3$
2. 8.84 cm

### Practise Now 8

1. 49.9 $l$
2. 520 minutes

### Practise Now 9

1. (i) 297 cm$^2$ (ii) 47 : 54
2. (ii) 158 cm$^2$ (iii) 358 cm$^2$

### Practise Now 10

1. (i) 5600 cm$^3$ (ii) 1920 cm$^2$
2. (i) 406 cm$^3$ (ii) 445 cm$^2$

### Practise Now 11

(a) 13 (b) 6

### Exercise 11A

1. (a) (i) 4 000 000 cm$^3$
   (ii) 500 000 cm$^3$
   (b) (i) 0.25 m$^3$
   (ii) 0.0678 m$^3$
2. (a) (i) 840 000 cm$^3$
   (ii) 840 000 ml
   (b) (i) 0.002 56 m$^3$
   (ii) 2.56 $l$
3. (i) 16 cm (ii) 105
4. (a) (i) 480 cm$^3$
   (ii) 376 cm$^2$
   (b) (i) 420 cm$^3$
   (ii) 358 cm$^2$
   (c) (i) 115 200 mm$^3$
   (ii) 27 360 mm$^2$
   (d) (i) $7\frac{1}{2}$ cm$^3$
   (ii) $41\frac{1}{2}$ cm$^2$
   (e) (i) $\frac{21}{64}$ cm$^3$
   (ii) $3\frac{43}{160}$ cm$^2$
   (f) (i) 4.095 cm$^3$
   (ii) 19.26 cm$^2$
5. (a) 2160 mm$^3$, 1284 mm$^2$
   (b) 8 cm, 158 cm$^2$
   (c) 2.5 cm, 89.5 cm$^2$
   (d) 8 m, 432 m$^2$
6. (a) 0.190 m (b) 756
7. 4.1 m
8. 5.352 m$^3$
9. 6480 cm$^3$
10. (i) 456 000
    (ii) $25 080 000
    (iii) $836
11. (i) 4.8 $l$ (ii) 0.142 m$^2$
12. (i) 112 $l$ (ii) 1.16 m$^2$
13. 726 cm$^2$
14. 614.125 cm$^3$
15. 0.0495 m$^3$
16. 138.6 $l$
17. (i) 5 cm (ii) 540 cm$^3$
    (iii) 18 (iv) No
18. (i) 26 m$^2$, 78 m$^3$; 25 m$^2$, 75 m$^3$; 36 m$^2$, 64.8 m$^3$
    (ii) No

### Exercise 11B

1. (a) 369 840 cm$^3$
   (b) 16 644 cm$^3$
   (c) 960 cm$^3$ (d) 1152 cm$^3$
   (e) 770 cm$^3$ (f) 4725 cm$^3$
2. (a) 6 cm$^2$, 42 cm$^3$
   (b) 14 cm, 693 cm$^3$

(c) 32 cm, 240 cm$^2$
(d) 400 cm, 95.94 cm$^2$

3. 102 480 m$^3$

4. (a) (i) 180 cm$^3$
(ii) 264 cm$^2$
(b) (i) 153 cm$^3$
(ii) 250 cm$^2$

5. (i) 2000 m$^3$ (ii) 1490.25 m$^2$

### Exercise 11C

1. 18.8 *l*
2. 82.9 cm
3. 7.42 *l*
4. (a) (i) 1850 cm$^3$
(ii) 836 cm$^2$
(b) (i) 4.52 m$^3$
(ii) 17.3 m$^2$
(c) (i) 44 500 mm$^3$
(ii) 7350 mm$^2$
5. (a) 8.00 cm, 4.00 cm, 453 cm$^2$
(b) 28.0 cm, 14.0 cm, 2990 cm$^2$
(c) 2 cm, 42.0 cm, 553 cm$^2$
(d) 8 m, 21.0 m, 629 m$^2$
6. 19 244.75 cm$^2$
7. 4895
8. 144 m
9. 106 cm$^3$
10. 2280 *l*
11. 26 minutes
12. (i) 3744 cm$^3$ (ii) 16.5 cm
(iii) 1110 cm$^2$
13. (i) 725 cm$^2$ (ii) $\frac{5}{8}$
14. 40 minutes
15. (i) 23.8 cm (ii) 3750 cm$^2$
16. 834 cm$^2$

### Exercise 11D

1. (i) 522 cm$^3$ (ii) 432 cm$^2$
2. (i) 388 cm$^3$ (ii) 388 cm$^2$
3. (i) 1220 cm$^3$ (ii) 1230 cm$^2$
4. (i) 79 100 cm$^3$
(ii) 12 100 cm$^2$
5. (i) 765 cm$^3$ (ii) 563 cm$^2$
6. (i) 13 400 cm$^3$
(ii) 3760 cm$^2$
7. (i) 94 300 cm$^3$
(ii) 16 800 cm$^2$
8. (a) (i) 9
(b) (i) 6
9. (i) 254 cm$^3$ (ii) 427 cm$^2$

## Chapter 12 Introduction to Set Notation and Probability

### Practise Now 1A

1. (i) $A = \{2, 4, 6, 8\}$
(ii) (a) True
(b) True
(c) False
(d) True
(iii) (a) $2 \in A$
(b) $5 \notin A$
(c) $11 \notin A$
(d) $6 \in A$
(iv) 4
2. 10

### Practise Now 1B

(i) $C = \{11, 12, 13, 14, 15, 16, 17\}$, $D = \{10, 11, 12, 13, 14, 15, 16, 17\}$
(ii) No (iii) No

### Practise Now 2

(i) $P$ and $Q$; $P = \emptyset$ and $Q = \emptyset$
(ii) Yes (iii) No

### Practise Now 3

(i) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, $B = \{2, 3, 5, 7, 11, 13\}$
(iii) $B' = \{1, 4, 6, 8, 9, 10, 12\}$
(iv) $B'$ is the set of integers between 1 and 13 inclusive which are not prime numbers.
(v) 13, 6, 7 (vi) Yes

### Practise Now 4A

1. (ii) Yes (iii) Yes
2. (a) True (b) True
(c) True (d) False
3. (i) $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, $Q = \{2, 3, 5, 7, 11\}$
(ii) $Q \subset P$
(iii) $R = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$
(iv) $P \subseteq R$ and $R \subseteq P$

### Practise Now 4B

(a) $\emptyset$, {7}, {8}, {7, 8}
(b) $\emptyset$, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}

### Practise Now 5

{Orange, Purple, Green, Blue, Red}; 5

### Practise Now 6

(a) $\{B_1, B_2, B_3, B_4, B_5, R_1, R_2, R_3, R_4\}$; 9
(b) $\{N_1, A_1, T, I, O, N_2, A_2, L\}$; 8
(c) {357, 358, 359, 360, …, 389}; 33

### Practise Now 7

(i) $\frac{1}{15}$ (ii) $\frac{7}{15}$
(iii) $\frac{1}{3}$ (iv) 0

### Practise Now 8

(i) $\frac{1}{2}$ (ii) $\frac{1}{13}$
(iii) $\frac{1}{52}$ (iv) $\frac{51}{52}$

### Practise Now 9

1. (i) $\frac{1}{8}$ (ii) $\frac{3}{4}$
(iii) $\frac{1}{4}$
2. (i) $\frac{1}{6}$ (ii) $\frac{7}{12}$
(iii) 0 (iv) 1
3. 12

### Practise Now 10

(i) $\frac{3}{8}$ (ii) $\frac{1}{6}$
(iii) 0 (iv) $\frac{5}{24}$

### Practise Now 11

1. (i) $14 + x$ (ii) $\frac{x+2}{14+x}$
(iii) 6
2. 4

### Exercise 12A

1. (i) $A = \{1, 3, 5, 7, 9\}$
(ii) (a) True
(b) True
(c) False
(d) True
(iii) 5
2. (a) $B = \{2, 3, 4$ [illegible] $6,$ [illegible] $8, 9\}$; $n(B) = 8$
(b) $C = \{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1\}$; $n(C) = 10$
(c) $D = \{2, 4, 6, 8, 10, 12\}$; $n(D) = 6$
(d) $E = \{A\}$; $n(E) = 1$
3. (a) $F$ = {red, orange, yellow, green, blue, indigo, violet}; $n(F) = 7$
(b) $G$ = {Kashmir Solidarity Day, Pakistan Day, Eid al-Fitr, Labour Day, Eid al-Adha, Ashura, Independence Day of Pakistan, Mawlid, Iqbal Day, Christmas Day, Quaid-e-Azam's Birthday}; $n(G) = 11$
(c) $H$ = {S, Y, M, T, R, Y}; $n(H) = 6$
4. (a) $K = \emptyset$ (b) $L = \{2\}$
(c) $M = \emptyset$ (d) $N = \emptyset$
5. (i) $P$ = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}
(ii) (a) Tuesday $\in P$
(b) Sunday $\in P$
(c) March $\notin P$
(d) Holiday $\notin P$
(iii) 7
6. (i) No
(ii) $Q = \{4, 9, 16, 25, 36, 49\}$
(iii) 6
7. (i) $R$ is the set of non-negative even integers.
$S$ is the set of non-negative even integers less than 10.
(ii) No (iii) No
8. (i) $T$; $T = \emptyset$
(ii) No (iii) No
9. (a) False (b) False
(c) False (d) False
10. (a) $X = \{x : x$ is a prime number$\}$
(b) $Y = \{x : x$ is a non-negative multiple of 4$\}$

(c) $Z = \{x : x$ is a positive integer and a factor of $12\}$

11. (a) False (b) True
(c) False (d) True

## Exercise 12B

1. (i) ξ = {cat, hamster, lion, mouse, tiger},
$A$ = {cat, hamster, mouse}
(ii) $A'$ = {lion, tiger}
2. (i) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$B = \{2, 4, 6, 8, 10\}$
(iii) $B' = \{1, 3, 5, 7, 9\}$
(iv) $B'$ is the set of integers between 1 and 10 inclusive which are odd numbers.
3. (i) $C$ = {s, t, u},
$D$ = {s, t, u, v, w, x, y, z}
(ii) Yes
4. (a) True (b) True
(c) True (d) False
5. (a) Ø, {a}, {b}, {a, b}
(b) Ø, {Singapore}, {Malaysia}, {Singapore, Malaysia}
(c) Ø, {14}, {16}, {14, 16}
(d) Ø, {7}
6. (i) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
$J = \{1, 4, 6, 8, 9\}$,
$J' = \{2, 3, 5, 7\}$
(ii) $J'$ is the set of integers between 0 and 10 which are prime numbers.
(iii) 9, 5, 4 (iv) Yes
7. (i) ξ = {a, b, c, d, e, f, g, h, i, j},
$K$ = {b, c, d, f, g, h, j},
$K'$ = {a, e, i}
(ii) $K'$ is the set of the first 10 letters of the English alphabet which are vowels.
(iii) 10, 7, 3 (iv) Yes
8. (ii) Yes (iii) Yes
9. (i) $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$,
$P = \{4, 8, 12, 16\}$
(ii) $P \subset N$
(iii) $Q = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$
(iv) $N \subseteq Q$ and $Q \subseteq N$
10. (a) Ø, {7}, {8}, {9}, {7, 8}, {7, 9}, {8, 9}, {7, 8, 9}
(b) Ø, {2}, {3}, {5}, {2, 3}, {2, 5}, {3, 5}, {2, 3, 5}
(c) Ø, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b, c, d}
(d) Ø, {I}, {N}, {O}, {U}, {I, N}, {I, O}, {I, U}, {N, O}, {N, U}, {O, U}, {I, N, O}, {I, N, U}, {I, O, U}, {N, O, U}, {I, N, O, U}
11. (i) $V' = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20\}$
(ii) $V' = \{x : x$ is an integer that is not a multiple of 3 such that $0 < x < 21\}$
12. (i) Yes (ii) Yes
13. $2^a$

## Exercise 12C

1. {1, 2, 3, 4, 5, 6}; 6
2. (a) {2, 3, 4, 5}; 4
(b) {A, B, C, D, E, F, G, H, I, J}; 10
(c) $\{R_1, R_2, R_3, R_4, R_5, B_1, B_2, B_3, G_1, G_2\}$; 10
(d) $\{T, E_1, A, C, H, E_2, R\}$; 7
(e) {100, 101, 102, 103, …, 999}; 900
3. (i) $\frac{3}{8}$ (ii) $\frac{3}{8}$
(iii) 1 (iv) $\frac{7}{8}$
4. (i) $\frac{7}{13}$ (ii) $\frac{7}{13}$
(iii) $\frac{5}{13}$ (iv) 0
(v) $\frac{3}{13}$
5. (i) $\frac{1}{52}$ (ii) $\frac{1}{2}$
(iii) $\frac{3}{13}$ (iv) $\frac{10}{13}$
6. (i) $\frac{1}{11}$ (ii) $\frac{2}{11}$
(iii) $\frac{4}{11}$ (iv) $\frac{7}{11}$
7. (i) $\frac{1}{5}$ (ii) $\frac{3}{5}$
(iii) $\frac{1}{5}$ (iv) $\frac{2}{5}$
8. (i) $\frac{2}{5}$ (ii) $\frac{11}{30}$
(iii) $\frac{9}{10}$ (iv) 0
9. (i) $\frac{3}{10}$ (ii) $\frac{2}{5}$
10. (i) $\frac{1}{9}$ (ii) $\frac{1}{15}$
11. (i) $\frac{13}{27}$ (ii) $\frac{2}{27}$
(iii) $\frac{1}{27}$ (iv) $\frac{4}{27}$
12. (i) $\frac{1}{3}$ (ii) $\frac{1}{3}$
(iii) $\frac{3}{13}$ (iv) $\frac{12}{13}$
13. (i) $\frac{1}{6}$ (ii) $\frac{1}{10}$
(iii) 1 (iv) $\frac{2}{5}$
14. (i) $\frac{4}{19}$ (ii) $\frac{9}{19}$
15. (a) (i) $\frac{2}{5}$ (ii) $\frac{1}{8}$
(b) (i) $\frac{1}{13}$ (ii) $\frac{7}{13}$
16. (i) 26 (ii) 64
17. (i) $\frac{9}{80}$ (ii) $\frac{13}{80}$
(iii) $\frac{1}{5}$ (iv) $\frac{1}{10}$
18. (i) $\frac{1}{4}$ (ii) $\frac{1}{3}$
19. $\frac{8}{15}$
20. {16, 17, 18, 19, …, 53}

## Exercise 12D

1. (i) $\frac{5}{12}$ (ii) $\frac{5}{24}$
(iii) $\frac{3}{8}$
2. (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$
(iii) $\frac{1}{2}$
3. (i) $15 + x$ (ii) $\frac{15}{15+x}$
(iii) 60
4. 9
5. $\frac{13}{4}$
6. 10
7. 7
8. 18
9. $\frac{1}{8}$
10. (i) $\frac{7}{20}$ (ii) $2x + 46$
(iii) 30
11. $x = 16, y = 18$