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think! NEW SYLLABUS MATHEMATICS

8th Edition

For
Cambridge O Level and Cambridge IGCSE Mathematics

3

Dr Yeap Ban Har • Dr Joseph B. W. Yeo • Dr Choy Ban Heng
Sharon Lee • Teh Keng Seng • Wong Lai Fong • Wong-Ng Siew Hiong

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PREFACE

think! Mathematics is a series of textbooks specially designed to provide students valuable **learning experiences** by engaging their minds and hearts as they learn mathematics.

The features of this textbook series reflect the important shifts towards the development of 21st century competencies and a greater appreciation of mathematics, as articulated in the Singapore mathematics curriculum and other international curricula. Every chapter begins with a Chapter Opener and an Introductory Problem to motivate the development of the key concepts in the topic. The Chapter Opener gives a coherent overview of the **big ideas** that will frame the study of the topic, while the Introductory Problem positions problem solving at the heart of learning mathematics. Two key considerations guide the development of every chapter – seeing mathematics as a tool and as a discipline. Opportunities to engage in Investigation, Class Discussion, Thinking Time, Journal Writing and Performance Tasks are woven throughout the textbook to enhance students' learning experiences. Stories, songs, videos and puzzles serve to arouse interest and pique curiosity. Real-life examples serve to influence students to appreciate the beauty and usefulness of mathematics in their surroundings.

Underpinning the writing of this textbook series is the belief that all students can learn and appreciate mathematics. Worked Examples are carefully selected, questions in the Reflection section prompt students to reflect on their learning, and problems are of varying difficulty levels to ensure a high baseline of mastery, and to stretch students with special interest in mathematics. The use of ICT helps students to visualise and manipulate mathematical objects with ease, hence promoting interactivity.

We hope you will enjoy the subject as we embark on this exciting journey together to develop important mathematical dispositions that will certainly see you through beyond the examinations, to appreciate mathematics as an important tool in life, and as a discipline of the mind.

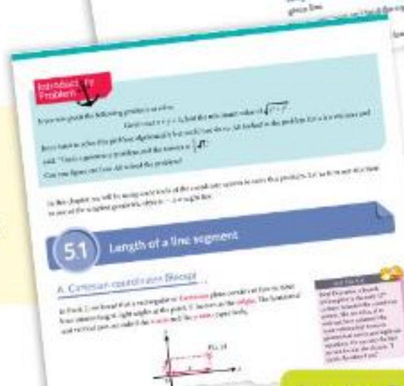
KEY FEATURES


CHAPTER 5

Coordinate Geometry

Chapter Opener
gives students an overview of the topic. It includes *rationales* for learning the chapter to arouse students' *interest* and **big ideas** that *connect* the concepts within the chapter or with other chapters.

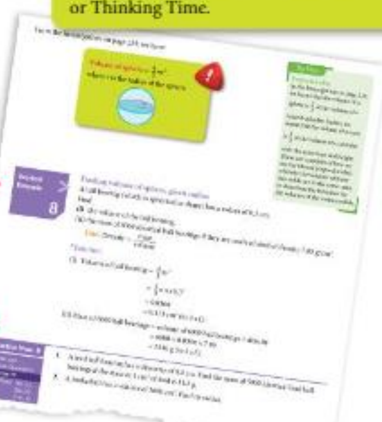
Learning Outcomes
help students to be aware of what they are about to study so as to *monitor* their progress.


5.1 Length of a line segment

Introductory Problem
provides students with a more specific *motivation* to learn the topic, using a problem that helps develop a concept, or an application problem that students will revisit after they have gained necessary knowledge from the chapter.

Recap
revisits relevant prerequisites at the beginning of the chapter or at appropriate junctures so that students are *ready* to learn new knowledge built on their existing schema.

Important Results
summarise important concepts or formulae obtained from Investigation, Class Discussion or Thinking Time.

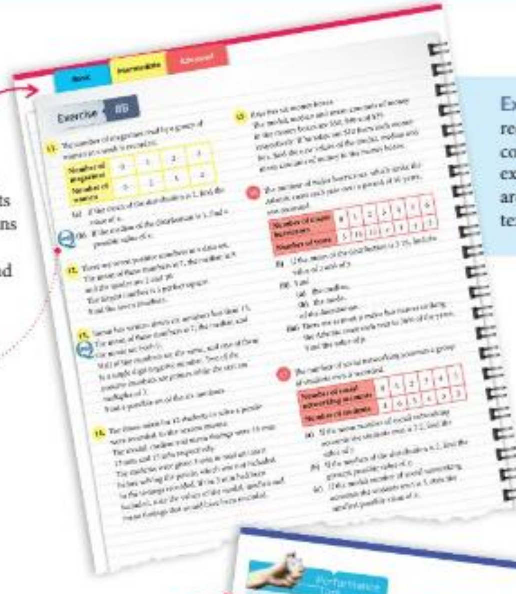


Worked Example
shows students how to present their working clearly when solving related *problems*. In more challenging worked examples, **Pólya's Problem Solving Model** is used to help students learn how to address a problem.

Practise Now
consists of questions that help students achieve *mastery* of procedural *skills*. Puzzles are sometimes used for consolidation to make practice *motivating* and fun.

Similar and Further Questions
follow after Practise Now to help teachers select appropriate questions for students' self-practice.

Exercise
questions are classified into three levels of difficulty – **Basic**, **Intermediate** and **Advanced**. Questions at the Basic level are usually short-answer items to test basic concepts and skills. The Intermediate level contains more structured questions, while the Advanced level involves applications and higher order thinking skills.

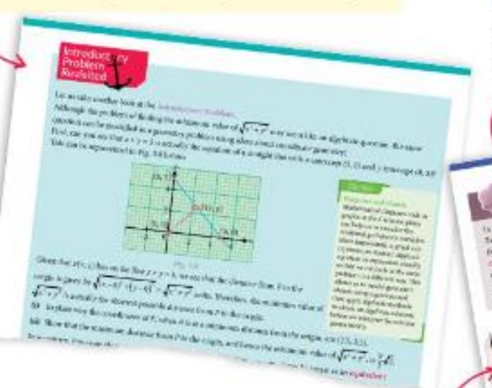


Explanation Questions
require students to communicate their explanations in writing and are spread throughout the textbook.

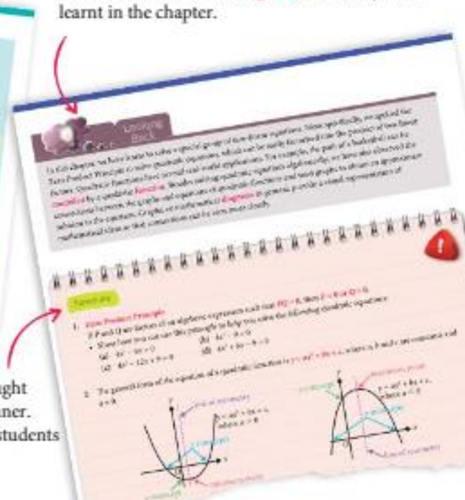
Open-ended Problems
are mathematics problems with more than one correct answer. Solving such problems expose students to real-world problems.

Performance Task
consists of mini-projects designed to develop research and presentation skills of students, through writing a report and/or giving an oral presentation.

Introductory Problem Revisited
revisits an application-based Introductory Problem later in the chapter. This is absent if the Introductory Problem leads directly to the development of a concept.



Looking Back
complements the Chapter Opener and helps students internalise the **big ideas** that they have learnt in the chapter.



Summary
compounds the key concepts taught in the chapter in a succinct manner. Questions are included to help students *reflect* on their learning.

Investigation

Guided investigation provides students the relevant *learning experiences* to explore and discover important mathematical *concepts*. It usually takes the **Concrete-Pictorial-Abstract (C-P-A)** approach to help students construct their knowledge meaningfully. The connections between concrete experiences (manipulative or examples), different pictorial representations and symbolic representations are explicitly made. Some investigations may also involve the use of **Information and Communication Technology (ICT)**.

Class Discussion

Questions are provided to **engage** students in discussion, with the teacher acting as the facilitator. Class discussions provide students the relevant *learning experiences* to think and *reason* mathematically, enhance their oral *communication* skills, and learn new *concepts* and *skills*.

Thinking Time

Key questions are included at appropriate junctures to provide students the relevant *learning experiences* to think critically on their own before sharing their thoughts with their classmates. Mathematical fallacies are sometimes included to check and test students' understanding.

Journal Writing

Journal writing provides opportunities for students to *reflect* on their learning and to *communicate* mathematically in writing. It can also be used as a formative assessment for the teacher to provide feedback for their students.

Reflection

Students are usually required to reflect on what they have learnt at the end of each section so as to *monitor* and *regulate* their own learning. The reflection questions provided can be generic prompts or specific to the topics in the section or chapter, to check if students have understood the key ideas.

MARGINAL NOTES

Big Idea

This provides additional details of the big idea mentioned in the main text.

Recall

Unlike the key feature 'Recap' in the main text, this contains just-in-time recall of prerequisite knowledge that students have already learnt.

Attention

This contains important information that students should know.

Information

This includes information that may be of interest to students.

Reflection

This guides students to think about different methods used to solve a problem.

Problem-solving Tip

This guides students on how to approach a problem in Worked Examples or Practise Now.

Internet Resources

This guides students to search the Internet for valuable information or interesting online games for their independent and self-directed learning.

Just For Fun

This contains puzzles, fascinating facts and interesting stories about mathematics as enrichment for students.

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CHAPTER 1

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CHAPTER 7

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CHAPTER 1

Algebraic Fractions and Formulae

Do you still recall the first mathematical formula you learnt in primary school? It might have been the formula for finding the area of a rectangle:

Area of rectangle = length × breadth

As you have seen over the years, a formula is a concise way of expressing the relationship between two or more variables, often expressing a rule that allows us to compute the value of a missing variable when given the values of the other variables. In other words, we can use a formula to express the equation of a **function** so that we can determine the value of a variable that we are interested in. For example, when we are given the length and breadth of a rectangle, we can find the area of the rectangle. Likewise, given the area of the rectangle and the length of one of its sides, we can find the length of the other side by performing some manipulation.

In this chapter, we are going to learn how we can manipulate, or rewrite, a given formula into a different but equivalent form so that we can use the equivalent formula to predict or determine the value of the missing variables in different situations.

Learning Outcomes

What will we learn in this chapter?

- How to simplify algebraic fractions by addition, subtraction, multiplication and division
- How to express algebraic fractions with linear or quadratic denominators into a single fraction
- How to change the subject of a formula

Introductory Problem

The focal length, f cm, of lenses used in digital cameras to focus an image on the sensing plate is engineered to follow the formula $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, where u cm and v cm are the object distance and the image distance from the centre of the lens respectively.

(i) Given that $f = 20$ and $u = 1000$, find the value of v by substituting the values of f and u into the formula.

(ii)

f	u	v
20	1500	
10	1200	
4	30	
7	160	

Table 1.1

Table 1.1 shows different pairs of values of f and u.
How can you find all the corresponding values of v efficiently?

1.1 Algebraic fractions

In Book 1, we learnt about numerical fractions of the form $\frac{a}{b}$, where a and b are integers, and $b \neq 0$. Examples of numerical fractions are $\frac{3}{5}$, $\frac{12}{7}$ and $-\frac{5}{8}$.

In this section, we will learn about **algebraic fractions** of the form $\frac{A}{B}$, where A and/or B are algebraic expressions, and $B \neq 0$. Examples of algebraic fractions are $\frac{2y}{5}$, $\frac{16}{4h+k}$, $\frac{7x}{(2x+1)(3x-1)}$ and $\frac{3f^2}{f^2-1}$.

The rules for performing operations on algebraic fractions are the same as those for numerical fractions. One important rule is that the value of a fraction remains unchanged if both its numerator and denominator are multiplied or divided by the same number or expression, i.e.

$\frac{A}{B} = \frac{A \times C}{B \times C}$ and $\frac{A}{B} = \frac{A \div C}{B \div C}$, where $B, C \neq 0$.

Reflection

Can you recall why the condition of $B, C \neq 0$ is important?

Note that $\frac{A \times C}{B \times C} = \frac{A}{B} \times \frac{C}{C} = \frac{A}{B} \times 1 = \frac{A}{B}$

and $\frac{A \div C}{B \div C} = \frac{A}{B} \div \frac{C}{C} = \frac{A}{B} \div 1 = \frac{A}{B}$.

Worked Example 1

Simplifying algebraic fractions

Simplify each of the following.

(a) $\dfrac{3xy^3}{9x^4y^2}$ (b) $\dfrac{28x^3(x+y)^2}{63xy^2(x+y)^4}$

Solution

(a) $\dfrac{\overset{1}{\cancel{3}}\,\overset{1}{\cancel{x}}\,y^{\cancel{3}\,1}}{\underset{3}{\cancel{9}}\,x^{\cancel{4}\,3}\,\underset{1}{\cancel{y^2}}} = \dfrac{y}{3x^3}$

Division of y^3 and y^2 by y^2

Division of x and x^4 by x

Division of 3 and 9 by 3

(b) $\dfrac{\overset{4}{\cancel{28}}\,x^{\cancel{3}\,2}\,\overset{1}{\cancel{(x+y)^2}}}{\underset{9}{\cancel{63}}\,\underset{1}{\cancel{x}}\,y^2(x+y)^{\cancel{4}\,2}} = \dfrac{4x^2}{9y^2(x+y)^2}$

Problem-solving Tip

Divide the numerators and the denominators by their common factors. The numerator and the denominator in the final answer should not have any common factors except 1.

Attention

In (a), $\dfrac{x}{x^4} = \dfrac{\overset{1}{\cancel{x}}}{x\times x\times x\times \underset{1}{\cancel{x}}} = \dfrac{1}{x^3}$.

Practise Now 1

Similar and Further Questions
Exercise 1A
Questions 1(a)–(f), 5(a)–(d)

Simplify each of the following.

(a) $\dfrac{8x^5y}{12x^3y^4}$ (b) $\dfrac{9x^4(x-y)^3}{27x^2y^3(x-y)}$

Thinking Time

Bernard did the following:

(a) $\dfrac{\cancel{ab}}{\cancel{ab}} = 0$ (b) $\dfrac{b+4\overset{1}{\cancel{b}}}{a\underset{1}{\cancel{b}}} = \dfrac{b+4}{a}$ (c) $\dfrac{1+\overset{1}{\cancel{4}}}{\underset{1}{\cancel{4}}} = \dfrac{2}{1} = 2$

Explain what is wrong with his working.

Worked Example 2

Simplifying algebraic fractions by factorisation

Simplify each of the following.

(a) $\dfrac{a^2+4ab^2}{3ab}$ (b) $\dfrac{3t}{t^2-2t}$ (c) $\dfrac{x^2-3x}{3x-9}$

Solution

(a) $\dfrac{a^2+4ab^2}{3ab} = \dfrac{\overset{1}{\cancel{a}}(a+4b^2)}{3\underset{1}{\cancel{a}}b}$ extract the common factor a from the numerator and divide the numerator and the denominator by a

$= \dfrac{a+4b^2}{3b}$

(b) $\dfrac{3t}{t^2-2t} = \dfrac{3\overset{1}{\cancel{t}}}{\underset{1}{\cancel{t}}(t-2)}$ extract the common factor t from the denominator and divide the numerator and the denominator by t

$= \dfrac{3}{t-2}$

Attention

$\dfrac{a^2+4ab^2}{3ab} \neq \dfrac{a^2+4\overset{1}{\cancel{a}}b^2}{3\underset{1}{\cancel{a}}b}$

because we have to divide both the numerator (not just $4ab^2$) and the denominator by a: $\dfrac{(a^2+4ab^2)\div a}{3ab\div a}$.

This is why we need to factorise the numerator first: $\dfrac{a(a+4b^2)\div a}{3ab\div a} = \dfrac{\overset{1}{\cancel{a}}(a+4b^2)}{3\underset{1}{\cancel{a}}b}$

(c) $\dfrac{x^2-3x}{3x-9} = \dfrac{x\cancel{(x-3)}^{1}}{3\cancel{(x-3)}_{1}}$

$= \dfrac{x}{3}$

extract the common factor x from the numerator and the common factor 3 from the denominator, and divide the numerator and the denominator by $(x-3)$

Practise Now 2

Similar and Further Questions
Exercise 1A
Questions 2(a)–(f)

Simplify each of the following.

(a) $\dfrac{h^2+7hk}{5hk}$ (b) $\dfrac{15p}{10p^2-5p}$ (c) $\dfrac{z^2-4z}{4z-16}$

Worked Example 3

Simplifying algebraic fractions by factorisation

Simplify each of the following.

(a) $\dfrac{2m^2-4m}{m^2-4}$ (b) $\dfrac{x^2-3xy-4y^2}{3x^2-12xy}$ (c) $\dfrac{a^2-ac+ab-bc}{2ab+ac-2bc-c^2}$

Solution

(a) $\dfrac{2m^2-4m}{m^2-4} = \dfrac{2m\cancel{(m-2)}^{1}}{(m+2)\cancel{(m-2)}_{1}}$

$= \dfrac{2m}{m+2}$

extract the common factor $2m$ from the numerator and factorise the denominator using $a^2-b^2=(a+b)(a-b)$

(b) $\dfrac{x^2-3xy-4y^2}{3x^2-12xy} = \dfrac{\cancel{(x-4y)}^{1}(x+y)}{3x\cancel{(x-4y)}_{1}}$

$= \dfrac{x+y}{3x}$

factorise the numerator using the multiplication frame and extract the common factor $3x$ from the denominator

(c) $\dfrac{a^2-ac+ab-bc}{2ab+ac-2bc-c^2} = \dfrac{a(a-c)+b(a-c)}{a(2b+c)-c(2b+c)}$

$= \dfrac{\cancel{(a-c)}^{1}(a+b)}{(2b+c)\cancel{(a-c)}_{1}}$

$= \dfrac{a+b}{2b+c}$

factorise the numerator and the denominator by grouping

Practise Now 3

Similar and Further Questions
Exercise 1A
Questions 3(a)–(f), 5(e)–(m), 7

1. Simplify each of the following.

(a) $\dfrac{3v^2-9v}{v^2-9}$ (b) $\dfrac{p^2-7pq+12q^2}{5p^2-20pq}$ (c) $\dfrac{x^2-3xy+2xz-6yz}{xy-2xz-3y^2+6yz}$

2. Simplify $\dfrac{n^4-5n^2+6}{n^4-9}$.

Thinking Time

For the following algebraic fractions, are there any values that the variables cannot take on?

- Worked Example 1(**b**): $\dfrac{28x^3(x+y)^2}{63xy^2(x+y)^4}$

- Worked Example 2(c): $\dfrac{x^2-3x}{3x-9}$
- Worked Example 3(a): $\dfrac{2m^2-4m}{m^2-4}$

Hint: What do we mean when we say that the denominator cannot be zero?

1.2 Multiplication and division of algebraic fractions

The procedure for the multiplication and division of algebraic fractions is similar to that of the multiplication and division of numerical fractions, except that we now have to consider the variables.

In Book 1, we learnt that $\frac{1}{2}\times\frac{3}{5} = \frac{1\times3}{2\times5}$.

In general, when we multiply $\frac{a}{b}$ by $\frac{c}{d}$, we have:

$$\frac{a}{b}\times\frac{c}{d} = \frac{a\times c}{b\times d} = \frac{ac}{bd}$$

Likewise, we learnt that $\frac{2}{3}\div\frac{4}{5} = \frac{2}{3}\times\frac{5}{4} = \frac{2\times5}{3\times4}$.

In general, when we divide $\frac{a}{b}$ by $\frac{c}{d}$, we have:

$$\frac{a}{b}\div\frac{c}{d} = \frac{a}{b}\times\frac{d}{c} = \frac{a\times d}{b\times c} = \frac{ad}{bc}$$

Recall

Dividing one fraction by another fraction is the same as multiplying the first fraction by the reciprocal of the second fraction. The reciprocal of a fraction is obtained by interchanging the numerator and the denominator of the fraction, e.g. the reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$.

Worked Example 4

Multiplying and dividing algebraic fractions

Simplify each of the following.

(a) $\dfrac{ab}{c^2}\times\dfrac{4c}{6a^2b}$ (b) $\dfrac{p}{q}\times\dfrac{p^2r}{q^2}\div\dfrac{pr^2}{2q}$ (c) $\dfrac{4x-16}{x+y}\div\dfrac{8}{5x+5y}$

(d) $\dfrac{h}{h^2-2h+1}\times\dfrac{(h-1)^3}{2h+1}$ (e) $\dfrac{(2x-3)^2}{a}\times\dfrac{a^2}{9-6x}$

***Solution**

(a) $\dfrac{\cancel{a}^{1}\cancel{b}^{1}}{c^{\cancel{2}}_{1}}\times\dfrac{{}^{2}\cancel{4}\cancel{c}^{1}}{{}_{3}\cancel{6}a^{\cancel{2}}_{1}\cancel{b}_{1}} = \dfrac{2}{3ac}$

(b) $\dfrac{p}{q}\times\dfrac{p^2r}{q^2}\div\dfrac{pr^2}{2q} = \dfrac{\cancel{p}^{1}}{\cancel{q}_{1}}\times\dfrac{p^2\cancel{r}^{1}}{q^2}\times\dfrac{2\cancel{q}^{1}}{{}_{1}\cancel{p}r^{\cancel{2}}_{1}}$

$= \dfrac{2p^2}{q^2r}$

(c) $\dfrac{4x-16}{x+y} \div \dfrac{8}{5x+5y} = \dfrac{\cancel{4}^{1}(x-4)}{\cancel{x+y}_{1}} \times \dfrac{5(\cancel{x+y}^{1})}{\cancel{8}_{2}}$

$= \dfrac{5(x-4)}{2}$

extract the common factor 4 from $4x - 16$ and the common factor 5 from $5x + 5y$

(d) $\dfrac{h}{h^2-2h+1} \times \dfrac{(h-1)^3}{2h+1} = \dfrac{h}{\cancel{(h-1)^2}_{1}} \times \dfrac{(h-1)^{\cancel{3}\,1}}{2h+1}$

$= \dfrac{h(h-1)}{2h+1}$

factorise $h^2 - 2h + 1$ using $a^2 - 2ab + b^2 = (a - b)^2$

(e) $\dfrac{(2x-3)^2}{a} \times \dfrac{a^2}{9-6x} = \dfrac{(2x-3)^2}{\cancel{a}_{1}} \times \dfrac{a^{\cancel{2}\,1}}{3(3-2x)}$

$= (2x-3)^{\cancel{2}\,1} \times \dfrac{a}{-3\cancel{(2x-3)}_{1}}$

$= -\dfrac{a(2x-3)}{3}$ or $\dfrac{a(3-2x)}{3}$

extract the common factor 3 from $9 - 6x$

Problem-solving Tip

Recall $a - b = -(b - a)$.
$\therefore 3 - 2x = -(2x - 3)$.

Practise Now 4

Similar and Further Questions
Exercise 1A
Questions 4(a)–(d), 6(a)–(l)

1. Simplify each of the following.

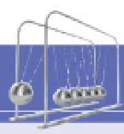
(a) $\dfrac{2a^2}{5c^3} \times \dfrac{15c}{8a^4}$ (b) $\dfrac{p^2}{5q^3} \times \dfrac{35qr}{12p} \div \dfrac{5r^4}{6p^3q^2}$

(c) $\dfrac{2x-6}{5x+5y} \div \dfrac{3}{7y+7x}$ (d) $\dfrac{h^2-6h+9}{h^2-2h} \times \dfrac{h-2}{h-3}$

2. Simplify $\dfrac{3m-n}{n+m} \div \dfrac{2n-6m}{m+n}$.

Reflection

Can we let $(2x - 3)^2 = -(3 - 2x)^2$ instead?



Reflection

1. What do I have to take note of when I simplify an algebraic fraction?
2. For any algebraic fraction to be valid, what condition do I have to check?

Basic Intermediate Advanced

Exercise 1A

1. Simplify each of the following.

(a) $\dfrac{4x^4}{12x^5}$ (b) $\dfrac{16a^3b^4}{24a^5b^2}$

(c) $\dfrac{23q^3r}{69qr^3s}$ (d) $\dfrac{3mn^2p^3}{18m^3np^6}$

(e) $\dfrac{15ac^3}{75a^2b^4c}$ (f) $\dfrac{16xy^3z}{64x^2y^4z^3}$

2. Simplify each of the following.

(a) $\dfrac{xy+3y}{4x+12}$ (b) $\dfrac{8a+4b}{bc+2ac}$

(c) $\dfrac{a^2+2ab}{6a}$ (d) $\dfrac{c^2}{c^2-cd}$

(e) $\dfrac{(m-n)^2}{m^2-mn}$ (f) $\dfrac{5pq}{15p-10pq}$

Exercise 1A

3. Simplify each of the following.

(a) $\dfrac{2a+b}{4a^2-b^2}$ (b) $\dfrac{c^2+2cd-15d^2}{4c^2+20cd}$

(c) $\dfrac{3a-6}{a^2+a-6}$ (d) $\dfrac{x^2+6x-7}{x^2-x}$

(e) $\dfrac{k^2-9}{k^2-7k+12}$ (f) $\dfrac{km+8k}{m^2+4m-32}$

(i) $\dfrac{b^2-a^2}{2a^2+ab-3b^2}$ (j) $\dfrac{y^2-6y-7}{2y^2-17y+21}$

(k) $\dfrac{3x-3y}{ax-ay-x+y}$ (l) $\dfrac{a^2-ab-ac+bc}{a^2+ab-ac-bc}$

(m) $\dfrac{a^2+am-an-mn}{a^2+am+an+mn}$

4. Simplify each of the following.

(a) $\dfrac{15a}{8b^3c}\times\dfrac{4c}{5ab}$ (b) $\dfrac{3(c+d)}{c-d}\times\dfrac{2c-2d}{8c+8d}$

(c) $\dfrac{a-2b}{16}\div\dfrac{4a-8b}{24}$ (d) $\dfrac{8c^3}{6(c+d)}\div\dfrac{2c^2}{3c+3d}$

5. Simplify each of the following.

(a) $\dfrac{9x(a-b)^2}{27x^3(a-b)^3}$ (b) $\dfrac{7a^3(a-3b)^4}{21a^2b(a-3b)^2}$

(c) $\dfrac{8ab^3(2a+3b)^2}{32a^2b(3b+2a)}$ (d) $\dfrac{8an^3(b+c)}{96a^2n(c+b)^2}$

(e) $\dfrac{y^2-2y-15}{y^2-3y-10}$ (f) $\dfrac{8-2m-m^2}{2m^2-3m-2}$

(g) $\dfrac{9x^2-y^2}{y^2-2xy-3x^2}$ (h) $\dfrac{3x^2+5xy-2y^2}{4x^2+7xy-2y^2}$

6. Simplify each of the following.

(a) $\dfrac{5a}{9b}\times\dfrac{ac^2}{2b}\times\dfrac{c^3}{8b^4}$ (b) $\dfrac{6d}{9f^3}\times\dfrac{3f^2}{16d^5}\div\dfrac{8f^2}{27d}$

(c) $\dfrac{2y^3}{x^3}\div\dfrac{4y}{5x}\times\dfrac{64x}{10y^4}$ (d) $\dfrac{3sq}{4p^2r}\div\dfrac{3q^3}{12p^3s^2}\times\dfrac{14ps}{7qr}$

(e) $\dfrac{3w-7}{5w^3}\div\dfrac{21-9w}{27w}$ (f) $\dfrac{6x^2y}{16y-8x}\times\dfrac{12x-24y}{4xy^2}$

(g) $\dfrac{h^2-h-6}{h^2-9}\times\dfrac{h^2}{h^2+2h}$ (h) $\dfrac{c^2-d^2}{c^2-2cd+d^2}\div\dfrac{1}{cd+d^2}$

(i) $\dfrac{m^2-4}{m^2-3m+2}\div\dfrac{m}{m-1}$ (j) $\dfrac{z^2}{z^2-4}\div\dfrac{3z-z^2}{z^2-5z+6}$

(k) $(a^2-4b^2)\div\dfrac{a^2+2ab}{ab}$ (l) $\dfrac{y^2-4y+4}{2-6y}\times\dfrac{2y+4}{3y^2-12}$

7. Simplify $\dfrac{x^2+y^2-z^2+2xy}{x^2-y^2-z^2-2yz}$.

1.3 Addition and subtraction of algebraic fractions

A. Simplification of linear expressions with fractional coefficients (Recap)

In Book 1, we learnt how to simplify linear expressions with fractional coefficients. For example,

$$\begin{aligned}\frac{x}{3}+\frac{2x-5}{7} &= \frac{7x}{21}+\frac{3(2x-5)}{21} && \text{convert to equivalent fractions: LCM of 3 and 7 is 21}\\ &= \frac{7x+3(2x-5)}{21} && \text{combine into a single fraction}\\ &= \frac{7x+6x-15}{21} && \text{Distributive Law}\\ &= \frac{13x-15}{21}.\end{aligned}$$

Big Idea

Equivalence

Equivalent fractions have the same value, e.g. to convert $\frac{x}{3}$ into an equivalent fraction with a denominator of 21, we multiply both the numerator and denominator by 7.

In this section, we will learn how to add and subtract algebraic fractions.

B. Addition and subtraction of algebraic fractions

Worked Example 5

Adding and subtracting algebraic fractions

Express each of the following as a fraction in its simplest form.

(a) $\frac{2}{3a}+\frac{3}{5a}$ (b) $\frac{3}{2b-4c}+\frac{2}{3b-6c}$ (c) $\frac{2h}{h-2k}-\frac{3k}{2k-h}$

*Solution

(a) $\frac{2}{3a}+\frac{3}{5a} = \frac{10}{15a}+\frac{9}{15a}$ convert to equivalent fractions: LCM of $3a$ and $5a$ is $15a$

$= \frac{10+9}{15a}$ combine into a single fraction

$= \frac{19}{15a}$

Attention
We can write $\frac{2a}{3}$ as $\frac{2}{3}a$ but $\frac{2}{3a} \neq \frac{2}{3}a$.

(b) $\frac{3}{2b-4c}+\frac{2}{3b-6c} = \frac{3}{2(b-2c)}+\frac{2}{3(b-2c)}$ extract the common factor 2 from $2b-4c$ and the common factor 3 from $3b-6c$

$= \frac{9}{6(b-2c)}+\frac{4}{6(b-2c)}$ convert to equivalent fractions: LCM of $2(b-2c)$ and $3(b-2c)$ is $6(b-2c)$

$= \frac{9+4}{6(b-2c)}$ combine into a single fraction

$= \frac{13}{6(b-2c)}$

(c) $\frac{2h}{h-2k}-\frac{3k}{2k-h} = \frac{2h}{h-2k}-\frac{3k}{-(h-2k)}$

$= \frac{2h}{h-2k}+\frac{3k}{h-2k}$

$= \frac{2h+3k}{h-2k}$ combine into a single fraction

Attention
$-\frac{3k}{-(h-2k)} = \frac{3k}{h-2k}$

Practise Now 5

Similar and Further Questions
Exercise 1B
Questions 1(a)–(f), 4(a)–(h)

1. Express each of the following as a fraction in its simplest form.

(a) $\frac{6}{5a}+\frac{3}{8a}$ (b) $\frac{4}{2b+3c}-\frac{7}{6b+9c}$ (c) $\frac{h}{2-3k}-\frac{3h}{3k-2}$

2. Express each of the following as a fraction in its simplest form.

(a) $\frac{2m+3n}{3m}-\frac{m-n}{6n}$ (b) $\frac{3p}{4p-4q}-\frac{5p-2q}{3p-3q}$ (c) $\frac{5x}{4x-3y}-\frac{7y}{6y-8x}$

Worked Example 6

Adding and subtracting more complicated algebraic fractions

Express each of the following as a fraction in its simplest form.

(a) $\frac{3}{x+5}+\frac{1}{x-3}$ (b) $\frac{5y}{y^2-4}-\frac{2}{y-2}$

Solution

(a) $\dfrac{3}{x+5}+\dfrac{1}{x-3}=\dfrac{3(x-3)}{(x+5)(x-3)}+\dfrac{x+5}{(x+5)(x-3)}$ convert to equivalent fractions: LCM of $x+5$ and $x-3$ is $(x+5)(x-3)$

$=\dfrac{3(x-3)+x+5}{(x+5)(x-3)}$ combine into a single fraction

$=\dfrac{3x-9+x+5}{(x+5)(x-3)}$ Distributive Law

$=\dfrac{4x-4}{(x+5)(x-3)}$

(b) $\dfrac{5y}{y^2-4}-\dfrac{2}{y-2}=\dfrac{5y}{(y+2)(y-2)}-\dfrac{2}{y-2}$ factorise the denominator y^2-4 using $a^2-b^2=(a+b)(a-b)$

$=\dfrac{5y}{(y+2)(y-2)}-\dfrac{2(y+2)}{(y+2)(y-2)}$ convert to equivalent fractions: LCM of $(y+2)(y-2)$ and $y-2$ is $(y+2)(y-2)$

$=\dfrac{5y-2(y+2)}{(y+2)(y-2)}$ combine into a single fraction

$=\dfrac{5y-2y-4}{(y+2)(y-2)}$ Distributive Law

$=\dfrac{3y-4}{(y+2)(y-2)}$

Practise Now 6

Similar and Further Questions
Exercise 1B
Questions 2(a)–(h), 5(a)–(f), 6(a)–(d), 8

Express each of the following as a fraction in its simplest form.

(a) $\dfrac{2}{x+1}-\dfrac{3}{2x-5}$ (b) $\dfrac{2}{y^2-9}-\dfrac{y}{y-3}$

(c) $\dfrac{1}{z+5}-\dfrac{1}{z-5}+\dfrac{2z}{z^2-25}$ (d) $\dfrac{4}{w+5}-\dfrac{7}{w^2+8w+15}$

Worked Example 7

Adding and subtracting algebraic fractions with common factors in the denominator and numerator

(a) Ali was asked to simplify $\dfrac{3(x+2)}{x^2-4}-\dfrac{1}{x-2}$.

He said that to convert to like fractions, the LCM of the denominators was $x-2$.

Do you agree? Explain why and thus simplify $\dfrac{3(x+2)}{x^2-4}-\dfrac{1}{x-2}$.

(b) Vasi said that the LCM can also be taken as x^2-4. Show that Vasi's method is also correct.

Solution

(a) The fraction $\dfrac{3(x+2)}{x^2-4}$ can be further simplified, before finding the LCM of the denominators to convert to like fractions. Since $\dfrac{3(x+2)}{x^2-4}=\dfrac{3(x+2)}{(x+2)(x-2)}$

$=\dfrac{3}{x-2}$,

I agree with Ali that the LCM is $x-2$.

$$\frac{3(x+2)}{x^2-4}-\frac{1}{x-2}=\frac{3(x+2)}{(x+2)(x-2)}-\frac{1}{x-2}$$

factorise the denominator x^2-4 using $a^2-b^2=(a+b)(a-b)$

$$=\frac{3\cancel{(x+2)}^{1}}{\cancel{(x+2)}_{1}(x-2)}-\frac{1}{x-2}$$

$$=\frac{3}{x-2}-\frac{1}{x-2}$$

$$=\frac{2}{x-2}$$

(b) $$\frac{3(x+2)}{x^2-4}-\frac{1}{x-2}=\frac{3x+6}{x^2-4}-\frac{x+2}{(x+2)(x-2)}$$

$$=\frac{3x+6-(x+2)}{x^2-4}$$

$$=\frac{3x+6-x-2}{x^2-4}$$

$$=\frac{2x+4}{x^2-4}$$

$$=\frac{2(x+2)}{(x+2)(x-2)}$$

extract the common factor 2 from the numerator and factorise the denominator using $a^2-b^2=(a+b)(a-b)$

$$=\frac{2}{x-2}$$

Practise Now 7

Similar and Further Questions
Exercise 1B
Questions 9, 10

Yasir was asked to simplify $\frac{2(x+3)}{x^2-9}+\frac{1}{x+3}$.

He said that to convert to like fractions, the LCM of the denominators was x^2-9.

Do you agree? Explain why and thus simplify $\frac{2(x+3)}{x^2-9}+\frac{1}{x+3}$.

Attention

If each fraction is not first simplified, we have to factorise the expression obtained after addition or subtraction.

Reflection

When adding and subtracting algebraic fractions, what must I do to the expressions in the numerators and/or denominators before finding the LCM of the denominators to combine them into a single fraction?

1.4 Solving equations involving algebraic fractions

After learning how to perform the four operations (addition, subtraction, multiplication and division) on algebraic fractions in Sections 1.2 and 1.3, we shall learn how to solve equations involving algebraic fractions.

Worked Example 8

Solving equations involving algebraic fractions

Solve each of the following equations.

(a) $\frac{a-2}{5}+\frac{a-1}{3}=1$

(b) $\frac{6}{2b-5}-\frac{4}{b-3}=0$

Solution

(a) **Method 1:**

$$\frac{a-2}{5}+\frac{a-1}{3}=1$$

$$\frac{3(a-2)}{15}+\frac{5(a-1)}{15}=1 \quad \text{convert fractions on LHS to equivalent fractions}$$

$$\frac{3(a-2)+5(a-1)}{15}=1 \quad \text{combine fractions on LHS into a single fraction}$$

$$\frac{3a-6+5a-5}{15}=1$$

$$\frac{8a-11}{15}=1$$

$$8a-11=15 \quad \text{multiply both sides by 15}$$

$$8a=26$$

$$a=3\frac{1}{4}$$

Method 2:

$$\frac{a-2}{5}+\frac{a-1}{3}=1$$

$$3(a-2)+5(a-1)=15 \quad \text{multiply both sides by the LCM of 5 and 3, i.e. 15}$$

$$3a-6+5a-5=15$$

$$8a-11=15$$

$$8a=26$$

$$a=3\frac{1}{4}$$

Reflection

(a) Which method do you prefer? Why?

(b) **Method 1:**

$$\frac{6}{2b-5}-\frac{4}{b-3}=0$$

$$6(b-3)-4(2b-5)=0 \quad \text{multiply both sides by the LCM of } 2b-5 \text{ and } b-3, \text{ i.e. } (2b-5)(b-3)$$

$$6b-18-8b+20=0$$

$$-2b+2=0$$

$$-2b=-2$$

$$b=1$$

Method 2:

$$\frac{6}{2b-5}-\frac{4}{b-3}=0$$

$$\frac{6}{2b-5}=\frac{4}{b-3}$$

$6(b-3)=4(2b-5)$ multiply both sides by the LCM of $2b-5$ and $b-3$, i.e. $(2b-5)(b-3)$

$6b-18=8b-20$

$6b-8b=-20+18$

$-2b=-2$

$b=1$

Recall

We learnt how to solve this in Book 1.

Reflection

(b) Which method do you prefer? Why? Is it possible to apply **Method 2** to **(a)**? Why?

Practise Now 8

Similar and Further Questions
Exercise 1B
Questions 3(a)–(g), 7(a)–(e), 11

Solve each of the following equations.

(a) $\frac{a-3}{2}+\frac{2a-1}{7}=4$ **(b)** $\frac{3}{2b+3}-\frac{5}{3b-4}=0$

Basic | Intermediate | Advanced

Exercise 1B

1. Express each of the following as a fraction in its simplest form.

(a) $\frac{7}{6a}+\frac{4}{9a}$ **(b)** $\frac{3}{2b}+\frac{1}{3b}-\frac{5}{6b}$

(c) $\frac{1}{3c}-\frac{1}{3d}$ **(d)** $\frac{f-4h}{3k}-\frac{2f-5h}{8k}$

(e) $\frac{4a}{x-3y}+\frac{3a}{3x-9y}$

(f) $\frac{p+3}{2z}+\frac{p-1}{6z}-\frac{2p+1}{3z}$

2. Express each of the following as a fraction in its simplest form.

(a) $\frac{5}{a}+\frac{3}{a+4}$ **(b)** $\frac{1}{2b}-\frac{3}{b+c}$

(c) $\frac{4}{d-5}+\frac{2}{2d+3}$ **(d)** $\frac{2}{f+5}-\frac{3}{f-1}$

(e) $\frac{11}{3h-7}+\frac{2}{6-5h}$ **(f)** $\frac{3}{k^2-1}+\frac{2}{k-1}$

(g) $\frac{3}{4m^2-1}-\frac{5}{2m+1}$ **(h)** $\frac{2}{n-2}+\frac{3}{(n-2)^2}$

3. Solve each of the following equations.

(a) $\frac{a}{a+2}=\frac{3}{5}$ **(b)** $\frac{1}{b-2}=\frac{2}{b-1}$

(c) $\frac{4}{c+3}-\frac{3}{c+2}=0$ **(d)** $\frac{5}{d+4}-\frac{2}{d-2}=0$

(e) $\frac{6}{f}-\frac{10}{3f}=2$ **(f)** $\frac{5}{6h}+\frac{6}{7h}-\frac{9}{14h}=4$

(g) $\frac{3}{k+1}-\frac{1}{2k+2}=5$

4. Express each of the following as a fraction in its simplest form.

(a) $\frac{5}{2(a-b)}+\frac{4}{3(b-a)}$ **(b)** $\frac{c-1}{3c-7}-\frac{1}{14-6c}$

(c) $\frac{4f}{10f-5d}+\frac{2d}{6f-3d}$ **(d)** $\frac{u+1}{2u-8}-\frac{u+2}{12-3u}$

(e) $\frac{2m-5}{9n-6}-\frac{m+3}{4-6n}$ **(f)** $\frac{h+k}{p-q}+\frac{3h+k}{8q-8p}$

(g) $\frac{5x^2}{6x-6y}-\frac{2x^2}{3y-3x}$

(h) $\frac{3x}{4y-2z}-\frac{2x}{z-2y}+\frac{5}{3z-6y}$

Exercise 1B

5. Express each of the following as a fraction in its simplest form.

(a) $\dfrac{3a}{3a-5}+\dfrac{4a}{4a-1}$

(b) $\dfrac{5}{2b+1}-\dfrac{2b}{(2b+1)^2}$

(c) $\dfrac{h+5}{h^2-6h}-\dfrac{3}{h-6}$

(d) $\dfrac{1}{m}+\dfrac{2}{m-4}+\dfrac{3}{m-3}$

(e) $\dfrac{x+y}{x-y}+\dfrac{x^2-4y^2}{x^2-y^2}-\dfrac{x-3y}{x+y}$

(f) $\dfrac{1}{2z-3}-\dfrac{2}{3-2z}+\dfrac{18}{9-4z^2}$

6. Express each of the following as a fraction in its simplest form.

(a) $\dfrac{2}{a+3}+\dfrac{3}{a^2+4a+3}$

(b) $\dfrac{1}{b^2-5b-6}-\dfrac{b}{b-6}$

(c) $\dfrac{1}{2p^2-8p-10}+\dfrac{2p}{p-5}$

(d) $\dfrac{x}{x+y}+\dfrac{4}{x^2+3xy+2y^2}-\dfrac{3x}{x+2y}$

7. Solve each of the following equations.

(a) $2-\dfrac{5}{x+2}=1\dfrac{3}{5}$

(b) $\dfrac{7x-4}{15}+\dfrac{x-1}{3}=\dfrac{3x-1}{5}-\dfrac{7+x}{10}$

(c) $\dfrac{x+1}{5x-1}+\dfrac{1}{2(1-5x)}=\dfrac{1}{4}$

(d) $\dfrac{x+1}{2x-1}-\dfrac{4}{4x-2}-\dfrac{3}{6x-3}=1$

(e) $\dfrac{3}{2-x}+\dfrac{5}{4-2x}-\dfrac{1}{x-2}=4$

8. Express $\dfrac{\frac{1}{3x}+\frac{2}{y}}{\frac{2}{x}}$ as a fraction in its simplest form.

9. Albert and Joyce were both asked to simplify $\dfrac{b-2}{b^2-5b+6}+\dfrac{b}{b+3}$.

Both students were able to factorise b^2-5b+6 to get $(b-2)(b-3)$.

Albert simplified $\dfrac{b-2}{b^2-5b+6}$ before converting to like fractions, whereas Joyce converted to like fractions directly.

Who do you think will be able to simplify $\dfrac{b-2}{b^2-5b+6}+\dfrac{b}{b+3}$ in a fewer number of steps? Explain why.

10. Given that $\dfrac{2a+7}{a^2+3a+2}$ is the end result of the sum of two algebraic fractions, $\dfrac{P}{Q}$ and $\dfrac{R}{a^2+3a+2}$, write down possible expressions for P, Q and R.

11. Given that $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{2}{x}}=\dfrac{4}{3}$, find the value of $\dfrac{y}{x}$.

1.5 Manipulating algebraic formulae

A. Changing subject of formula

From the **Introductory Problem**, we realised that we can find the value of v by substituting $f = 20$ and $u = 1000$ into the formula $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, and then manipulate the equation to isolate v on the left-hand side in order to find the value of v:

$$\begin{aligned}
\frac{1}{20} &= \frac{1}{1000} + \frac{1}{v} \\
\frac{1}{v} &= \frac{1}{20} - \frac{1}{1000} \\
&= \frac{50-1}{1000} \\
&= \frac{49}{1000} \\
v &= \frac{1000}{49} \\
&= 20.4 \text{ (to 3 s.f.)}
\end{aligned}$$

However, if we want to find the values of v for different pairs of values of f and u, it becomes very troublesome to keep manipulating the equation as such.

Instead, it will be easier to make v the **subject** of the formula:

$$\begin{aligned}
\frac{1}{f} &= \frac{1}{u} + \frac{1}{v} \\
\frac{1}{v} &= \frac{1}{f} - \frac{1}{u} \\
&= \frac{u-f}{fu} \\
v &= \frac{fu}{u-f}
\end{aligned}$$

We can then obtain v directly by substituting the values of f and u into the above formula.

Worked Example 9

Changing the subject of formula

(i) Make l the subject of the formula $P = 2l + 2b$.

(ii) Hence, calculate the value of l when $P = 132$ and $b = 30$.

Solution

(i)

$$P = 2l + 2b$$

$$P - 2b = 2l + 2b - 2b$$ subtract $2b$ from both sides to isolate the term containing l

$$P - 2b = 2l$$

$$\frac{P-2b}{2} = \frac{2l}{2}$$ divide by 2 on both sides

$$l = \frac{P-2b}{2}$$

Attention

Changing the subject of a formula is very similar to using inverses of operations to solve mathematical equations.

Therefore we must remember the following rules:

- adding and subtracting are inverses of one another;
- multiplying and dividing are inverses of one other;
- when using inverses we must perform the same operation on both sides of the formula.

(ii) When $P = 132$, $b = 30$,

$$
\begin{aligned}
l &= \frac{132-2(30)}{2} \\
&= \frac{132-60}{2} \\
&= \frac{72}{2} \\
&= 36
\end{aligned}
$$

Practise Now 9

Similar and Further Questions
Exercise 1C
Questions 1(a)–(d), 6(a)–(d)

1. **(i)** Make a the subject of the formula $v = u + at$.
(ii) Hence, find the value of a when $t = 4$, $u = 10$ and $v = 50$.

2. The simple interest $\$I$ payable on an investment is given by $I = \frac{PRT}{100}$, where $\$P$ is the principal, $R\%$ is the interest rate on the investment per annum and T is the number of years that the investment is held.

(i) Make T the subject of the formula $I = \frac{PRT}{100}$.

(ii) Hence, find the number of years that an initial investment of \$50 000 must be held in a bank that pays simple interest at a rate of 2% per annum to earn an interest of \$4000.

Big Idea

Functions and Models
The simple interest formula is an example of how an algebraic equation, which expresses the relationship between different variables, can model a real-world situation. Through manipulation, the formula can be rewritten in different but equivalent equations with different variables as the subject. This allows for easy predictions of various values in different situations.

Journal Writing

1. Why do we need to change the subject of a formula?
2. What is the difference between a formula and an equation?

Worked Example 10

Changing the subject of formula involving algebraic fraction

(i) Rearrange the formula $y = \frac{2-x}{3+2x}$ to make x the subject.

(ii) Hence, calculate the value of x when $y = -2$.

Solution

(i)

$$
\begin{aligned}
y &= \frac{2-x}{3+2x} \\
(3+2x) \times y &= (3+2x) \times \frac{2-x}{3+2x} && \text{multiply by } 3+2x \text{ on both sides} \\
y(3+2x) &= 2-x \\
3y + 2xy &= 2-x && \text{Distributive Law} \\
3y + 2xy + x &= 2 - x + x && \text{add } x \text{ to both sides} \\
3y + 2xy + x &= 2 \\
3y - 3y + 2xy + x &= 2 - 3y && \text{subtract } 3y \text{ from both sides} \\
2xy + x &= 2 - 3y \\
x(2y+1) &= 2 - 3y && \text{extract the common factor } x \text{ on the LHS}
\end{aligned}
$$

Problem-solving Tip

To change the subject of a formula involving algebraic fractions, e.g. to x, we need to manipulate the equation such that all the terms with x are on the LHS. Then, factorise the expression on the LHS if necessary and divide both sides of the equation such that x is isolated on the LHS.

$$\frac{x(2y+1)}{2y+1} = \frac{2-3y}{2y+1} \quad \text{divide by } 2y+1 \text{ on both sides}$$

$$x = \frac{2-3y}{2y+1}$$

(ii) When $y = -2$,

$$x = \frac{2-3(-2)}{2(-2)+1}$$
$$= \frac{2+6}{-4+1}$$
$$= -\frac{8}{3}$$

Practise Now 10

Similar and Further Questions
Exercise 1C
Questions 2(a)–(d), 7(a)–(d), 8

1. **(i)** Rearrange the formula $y = \dfrac{2x+5}{3x-7}$ to make x the subject.
   (ii) Hence, find the value of x when $y = -3$.
2. **(i)** Rearrange the formula $p = a + \dfrac{bx^2}{3k}$ to make k the subject.
   (ii) Hence, find the value of k when $a = 1$, $b = -2$, $p = 3$ and $x = 9$.

Worked Example 11

Changing the subject of formula involving cube root

It is given that $\sqrt[3]{ax+b} = k$.

(i) Express x in terms of a, b and k.
(ii) Hence, calculate the value of x when $a = 4$, $b = 3$ and $k = -1$.
(iii) For x to be defined, $a \neq t$. State the value of t.

Solution

(i)

$$\sqrt[3]{ax+b} = k$$
$$ax + b = k^3 \quad \text{take the cube on both sides}$$
$$ax + b - b = k^3 - b \quad \text{subtract } b \text{ from both sides}$$
$$ax = k^3 - b$$
$$\frac{ax}{a} = \frac{k^3-b}{a} \quad \text{divide by } a \text{ on both sides}$$
$$x = \frac{k^3-b}{a}$$

(ii) When $a = 4$, $b = 3$, $k = -1$,

$$x = \frac{(-1)^3-3}{4}$$
$$= \frac{-1-3}{4}$$
$$= \frac{-4}{4}$$
$$= -1$$

(iii) For x to be defined, $a \neq 0$.
$\therefore t = 0$

Practise Now 11

Similar and Further Questions
Exercise 1C
Questions 3(a)–(d), 9(a)–(d), 10–12

1. It is given that $3y = \sqrt{b^2 - 4ax}$.
   (i) Express x in terms of a, b and y.
   (ii) Hence, find the value of x when $a = -5$, $b = 4$ and $y = 2$.
   (iii) For x to be defined, $a \neq t$. State the value of t.

2. (i) Rearrange $p = a + \frac{bx^2}{3k}$ to make x the subject.
   (ii) Hence, find the values of x when $a = -1$, $b = 2$, $k = 1$ and $p = 5$.
   OPEN (iii) Given that $3k(p - a) < 0$, suggest a possible value of b such that x is defined.

B. Finding the value of an unknown in a formula

Worked Example 12

Finding the value of an unknown in a formula without changing the subject of the formula

Given that $y = \sqrt{\frac{64}{3x+1}}$,

(a) calculate
   (i) the value of y when $x = 1$,
   (ii) the value of x when $y = 2$,

(b) explain if it is possible to have a value of y if $x \leq -\frac{1}{3}$.

Solution

(a) (i) When $x = 1$,

$$\begin{aligned} y &= \sqrt{\frac{64}{3(1)+1}} \\ &= \sqrt{\frac{64}{3+1}} \\ &= \sqrt{\frac{64}{4}} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

(ii) When $y = 2$,

$$\begin{aligned} 2 &= \sqrt{\frac{64}{3x+1}} \\ 4 &= \frac{64}{3x+1} \quad \text{take the square on both sides} \\ 4(3x+1) &= 64 \\ 3x + 1 &= 16 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

(b) If $x = -\frac{1}{3}$, $3x + 1 = 0$ and $\frac{64}{3x+1}$ will be undefined. If $x < -\frac{1}{3}$, $3x + 1 < 0$ and $\frac{64}{3x+1} < 0$.
Hence, no real solutions can be obtained for y.
$\therefore$ it is not possible to find a value of y.

Practise Now 12

Similar and Further Questions
Exercise 1C
Questions 4, 5(a), (b), 13(a)–(d), 14(a), (b), 15

1. Given that $y = \sqrt{\frac{x+7}{x-2}}$, where $x \neq k$, find
   (a) the value of y when $x = 5$,
   (b) the value of x when $y = 4$,
   (c) the value of k.

2. Given that $a = \sqrt[3]{\dfrac{5b+16}{2b-23}}$, find the value of b when $a = 3$.

3. Given that $\sqrt[3]{\dfrac{x+y}{x-y}} = z$,

(a) find the value of x when $y = 4$ and $z = 2$,

(b) explain if it is possible to have a value of z if $x = y$.

Introductory Problem Revisited

Now that you have gained some understanding of how to change the subject of a formula, can you solve the **Introductory Problem**? Discuss with your classmates.

Reflection

1. What should I do to change the subject of a formula?
2. What have I learnt in this section or chapter that I am still unclear of?

Basic | Intermediate | Advanced

Exercise 1C

1. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $ax + by = k$ [y]

(b) $PV = nRT$ [n]

(c) $5b - 2d = 3c$ [d]

(d) $R = m(a + g)$ [a]

2. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $\dfrac{a}{m} = b + c$ [a]

(b) $5q - r = \dfrac{2p}{3}$ [p]

(c) $\dfrac{k+a}{5} = 3k$ [k]

(d) $A = \dfrac{1}{2}(a+b)h$ [b]

3. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $\sqrt[3]{h-k} = m$ [h]

(b) $b = \sqrt{D+4ac}$ [D]

(c) $P = \dfrac{V^2}{R}$ [V]

(d) $A = \dfrac{\theta}{360} \times \pi r^2$ [θ]

4. Given that $\sqrt{ax^2 - b} = c$, find the values of x when $a = 2$, $b = 7$ and $c = 5$.

Advanced Intermediate Basic

Exercise 1C

5. Given that $a^2(b + c) = 2b - c$, find
(a) the values of a when $b = 7$ and $c = 2$,
(b) the value of c when $a = 4$ and $b = -1$.

6. Rearrange each of the following formulae to make the letter in the brackets the subject.
(a) $F = \frac{9}{5}C + 32$ [C]
(b) $A = 2\pi r^2 + \pi rl$ [l]
(c) $s = ut + \frac{1}{2}at^2$ [u]
(d) $S = \frac{n}{2}[2a + (n-1)d]$ [d]

7. Rearrange each of the following formulae to make the letter in the brackets the subject.
(a) $\frac{1}{h+1} + 2 = k$ [h]
(b) $z = \frac{y(z-y)}{x}$ [z]
(c) $\frac{px}{q} = p + q$ [p]
(d) $\frac{1}{a} + \frac{1}{b} = 1$ [b]

8. Given that $V = \pi r^2 h + \frac{2}{3}\pi r^3$,
(i) make h the subject of the formula,
(ii) find the value of h when $V = 1000$ and $r = 7$.

9. Rearrange each of the following formulae to make the letter in the brackets the subject.
(a) $V = \frac{4}{3}\pi r^3$ [r]
(b) $v^2 = u^2 + 2as$ [u]
(c) $y = (x - p)^2 + q$ [x]
(d) $t = \sqrt{\frac{4z}{m-3}}$ [z]

10. Given that $a = \sqrt{\frac{3b+c}{b-c}}$, and $3b + c > 0$,
(i) express b in terms of a and c,
(ii) find the value of b when $a = 2$ and $c = 5$.
(iii) For b to be defined, $a^2 \neq t$. State the value of t.

11. The time taken, T seconds, for a pendulum to complete one oscillation is given by the formula $T = 2\pi\sqrt{\frac{l}{g}}$, where l m is the length of the pendulum and g is taken to be 10 m s^{-2}.
(i) Express l in terms of T and g.
(ii) Hence, find the length of the pendulum if it takes 12 seconds to complete 20 oscillations.

12. The amount of energy, E joules (J), stored in an object with a mass of m kg is given by the formula $E = mgh + \frac{1}{2}mv^2$, where h m is the height of the object above the ground, v m s^{-1} is the velocity of the object and g is taken to be 10 m s^{-2}.
(i) Given that $v \geqslant 0$, rearrange the formula $E = mgh + \frac{1}{2}mv^2$ to make v the subject.
(ii) Hence, find the velocity of an object with a mass of 0.5 kg, if it is 2 m above the ground and has 100 J of energy.

13. Given that $\frac{m(nx - y^2)}{p} = 3n$, where $p \neq k$, find
(a) the value of p when $m = 5$, $n = 7$, $x = 4$ and $y = -2$,
(b) the value of n when $m = 14$, $p = 9$, $x = 2$ and $y = 3$,
(c) the values of y when $m = 5$, $n = 4$, $p = 15$ and $x = 42$,
(d) the value of k.

14. Given that $A = \frac{1}{3}\pi r^2 h + \frac{4}{3}\pi r^3$, find
(a) the value of A when $\pi = 3.142$, $h = 15$ and $r = 7$,
(b) the value of h when $\pi = 3.142$, $A = 15\,400$ and $r = 14$.

>

Basic Intermediate Advanced

Exercise 1C

15. Given that $y = 3x + \sqrt[3]{a+b^2}$,

(i) find

(a) the value of y when $a = 13$, $b = 15$ and $x = 3.8$,

(b) the value of a when $b = 13$, $x = 8.5$ and $y = 35$,

(c) the values of b when $a = 23$, $x = 15.6$ and $y = 56$.

(ii) explain if it is possible to have a value of y if $a + b^2 < 0$.

Looking Back

By the time you complete this chapter, we will have covered all the foundational skills of algebraic manipulation: from writing algebraic expressions to solving algebraic equations; from working with simple linear algebraic expressions to working with expressions containing more complex algebraic fractions. An important idea that lies at the heart of many of the techniques — **equivalence**: the idea that there is some form of "sameness" even though the expressions may look different.

In Book 1, we learnt that two expressions are equivalent if the value of both expressions is the same for any value we substitute into the variables, e.g.

$$a(b + c) = ab + ac, \text{ for any values of } a, b, \text{ and } c.$$

In Book 2, we encountered special algebraic identities such as

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Besides equivalent expressions, we also learnt about equivalent equations where the solution set is the same. For instance, in Book 1, we learnt how to solve $5x + 4 = 2x - 8$ and we see that this equation has the same solution as $3x = -12$, which is an equivalent equation obtained by performing a series of operations that preserved the equality on both sides of the equation.

In Book 2, we explored how we can solve two linear simultaneous equations in two variables. In particular, we are able to solve simultaneous linear equations by means of elimination and substitution, which gives us the same solution set even though we have rewritten the equations in a different form. Furthermore, we learnt how we can work with inequalities by writing equivalent inequalities that have the same solution set.

Finally, in this chapter, we have learnt how we can work with algebraic fractions by writing equivalent algebraic fractions obtained by multiplying or dividing both the numerator and denominator by the same non-zero number. This technique is based on the idea of equivalent fractions, something that you have already learnt and are familiar with since primary school! So, if there is one golden rule in algebraic manipulation, it will be this: check that the expressions or equations are equivalent, that is, they have the same value when evaluated for the variables, or that they have the same solution set.

Summary

1. Equivalent fractions

The value of a fraction remains unchanged if both its numerator and denominator are multiplied or divided by the same non-zero number or expression,

i.e. $\frac{a}{b} = \frac{a \times c}{b \times c}$ and $\frac{a}{b} = \frac{a \div c}{b \div c}$, where $b, c \neq 0$.

An example is $\frac{3}{10} = \frac{15}{50}$.

- Give two other examples of equivalent fractions.

2. Simplifying algebraic fractions

To simplify an algebraic fraction, we divide its numerator and its denominator by the highest common factor.

E.g. $\frac{\cancel{3}^{1}x^{\cancel{4}^{1}}}{\cancel{9}_{3}x^{\cancel{3}^{1}}} = \frac{x}{3}$

$$\begin{aligned} \frac{a^2 + 4ab}{3ab} &= \frac{\cancel{a}^{1}(a+4b)}{3\cancel{a}_{1}b} \\ &= \frac{a+4b}{3b} \end{aligned}$$

- Simplify (a) $\frac{2x^2}{8x^4}$, (b) $\frac{a+3a^2b^3}{a^3b^2}$.

3. Multiplying and dividing algebraic fractions

(a) When we multiply $\frac{a}{b}$ by $\frac{c}{d}$, we have:

$$\begin{aligned} \frac{a}{b} \times \frac{c}{d} &= \frac{a \times c}{b \times d} \\ &= \frac{ac}{bd}, \end{aligned}$$

where $b, d \neq 0$.

- Simplify (a) $\frac{a^2b}{c} \times \frac{ac}{bd^2}$, (b) $\frac{c}{bd^3} \times \frac{ad^2}{b^2c}$.

(b) When we divide $\frac{a}{b}$ by $\frac{c}{d}$, we have:

$$\begin{aligned} \frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \times \frac{d}{c} \\ &= \frac{a \times d}{b \times c} \\ &= \frac{ad}{bc}, \end{aligned}$$

where $b, c, d \neq 0$.

- Simplify (a) $\frac{5b^2}{cd} \div \frac{25a}{c^2d^3}$, (b) $\frac{(a+b)^2}{(c-d)} \div \frac{(c-d)^2}{(a+b)^3}$.

Summary

4. Adding and subtracting algebraic fractions

To combine two fractions, both fractions must have the same denominator. We find the LCM of the denominators and express each fraction as an equivalent fraction with the LCM as a denominator.

For example:

$$\begin{aligned}\frac{2y}{x+1}-\frac{y}{x-2} &= \frac{2y(x-2)-y(x+1)}{(x+1)(x-2)}\\ &= \frac{2xy-4y-xy-y}{(x+1)(x-2)}\\ &= \frac{xy-5y}{(x+1)(x-2)}\end{aligned}$$

5. Changing the subject of a formula

To make a variable the subject of a formula, we need to ensure that all terms involving the variable are isolated on one side of the equation. In the final expression, the subject is the only term with that variable and has a coefficient of 1.

For example:

To express a in terms of u, v and s given that $v^2 = u^2 + 2as$,

$$\begin{aligned}v^2 &= u^2 + 2as\\ 2as &= v^2 - u^2\\ a &= \frac{v^2-u^2}{2s}\end{aligned}$$

CHAPTER 2

Quadratic Equations and Graphs

Predicting the trajectory of an object has always been a fascinating challenge for mankind in the development of machinery. To develop 'flying machines' such as rockets, it is important to understand how objects move through the air under the pull of gravity. What is the path of a falling object, a cannonball or a water-propelled rocket?

The motion of an object under the pull of gravity can be **modelled** by quadratic functions. In this chapter, we will explore how the ideas of **equivalence** and **diagrams** can be used to solve quadratic equations.

Learning Outcomes

What will we learn in this chapter?

- How to solve quadratic equations by factorisation
- What quadratic functions and the properties of their graphs are
- How to draw graphs of quadratic functions
- How to sketch the graphs of quadratic equations of the form $y = (x - h)(x - k)$, $y = -(x - h)(x - k)$, $y = (x - p)^2 + q$ and $y = -(x - p)^2 + q$
- Why quadratic equations and quadratic functions have useful applications in mathematics and in real-world contexts

Introductory Problem

1. If a and b are real numbers such that $ab = 0$, what can we say about a or b?
2. If x and y are real numbers such that $xy = 6$, can we say that $x = 6$ or $y = 6$?

In this chapter, we will learn how to apply the solution of Question 1 in the **Introductory Problem** to solve quadratic equations. In addition, we will learn about quadratic functions and their graphs, and how to apply them in mathematics and in real-world contexts.

2.1 Solving quadratic equations by factorisation

A. Solving linear equations (Recap)

In Book 1, we learnt how to solve linear equations in one variable by *isolating* the unknown on one side of the equation. For example,

$$
\begin{aligned}
2x + 1 &= 4 \\
2x &= 4 - 1 && \text{subtract 1 from both sides of the equation} \\
&= 3 \\
x &= \frac{3}{2} && \text{divide both sides of the equation by 2}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{5}{3}y - 7 &= 3 \\
-\frac{5}{3}y &= 3 + 7 && \text{add 7 to both sides of the equation} \\
&= 10 \\
5y &= 10 \times (-3) && \text{multiply both sides of the equation by } -3 \\
&= -30 \\
y &= -\frac{30}{5} && \text{divide both sides of the equation by 5} \\
&= -6
\end{aligned}
$$

Attention

For algebra, we can leave numerical answers as improper fractions or mixed numbers. But for non-numerical terms such as $-\frac{5}{3}y$, we usually write the fractional coefficient as an improper fraction.

Class Discussion — Linear equations

Solve the following linear equations. If your classmate does not obtain the correct answer, explain to him or her what he or she has done wrong.

(a) $3x + 2 = 5$ **(b)** $4y - 3 = 7$ **(c)** $\frac{7}{2}x - 8 = 6$ **(d)** $-\frac{3}{4}y - 9 = -3$

B. Solving quadratic equations

How can we solve a quadratic equation such as $x^2 - 3x = 0$ or $x(x - 3) = 0$? How can we *isolate* the unknown x on one side of the equation?

To do this, we need to learn about the Zero Product Principle.

In arithmetic, we have learnt that the product of any number and zero is equal to zero. For example,

$$\begin{aligned} 2 \times 0 &= 0; \\ 0 \times 8 &= 0; \\ -3 \times 0 &= 0; \\ 0 \times (-7) &= 0; \\ 0 \times 0 &= 0. \end{aligned}$$

What about the converse, shown in **Introductory Problem** Question 1? If a and b are real numbers such that $ab = 0$, what can we say about a or b?

If two numbers, a and b, are non-zero, their product, ab, can never be zero. So, if the product of two real numbers is 0, one (or both) of the two numbers must be 0 because only $a \times 0 = 0$, $0 \times b = 0$ and $0 \times 0 = 0$.

Therefore, if $ab = 0$, then $a = 0$ or $b = 0$ (or both equal to 0).

Similarly, in algebra, if two factors of an algebraic expression, P and Q, are such that $PQ = 0$, then $P = 0$ or $Q = 0$ (or both equal to 0).

Big Idea

Notations

In mathematics, the convention for the operator 'or' is inclusive. For example, $a = 0$ or $b = 0$ includes the case where both a and b are equal to 0.

Zero Product Principle

If a and b are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$.

If P and Q are factors of an algebraic expression such that $PQ = 0$, then $P = 0$ or $Q = 0$.

We shall use this principle to solve quadratic equations of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.

Worked Example 1

Solving quadratic equations of the form $ax^2 + bx = 0$ (where $b \neq 0$)

Solve the following quadratic equations.

(a) $x(x - 3) = 0$ **(b)** $4x^2 + 6x = 0$

*Solution

(a) $\underbrace{x}_{\text{first factor}}\underbrace{(x - 3)}_{\text{second factor}} = 0$

$x = 0$ or $x - 3 = 0$ if $ab = 0$, then $a = 0$ or $b = 0$

$x = 3$

(b) **Method 1:**

$4x^2 + 6x = 0$

$2x(2x + 3) = 0$ factorise by extracting common factor $2x$

$2x = 0$ or $2x + 3 = 0$ if $ab = 0$, then $a = 0$ or $b = 0$

$x = 0$ or $x = -\dfrac{3}{2}$

Problem-solving Tip

It is a good practice to *check* your answers by substituting the values of the unknown found into the original equation, e.g. in **(a)**,

if $x = 0$, LHS $= 0(0 - 3)$
$= 0$
$=$ RHS;

if $x = 3$, LHS $= 3(3 - 3)$
$= 0$
$=$ RHS.

Note that the answer is $x = 0$ **or** 3 (*not* $x = 0$ and 3) because x cannot be 0 and 3 at the same time.

Method 2:

$$
\begin{aligned}
4x^2 + 6x &= 0 \\
2x^2 + 3x &= 0 && \text{divide both sides of equation by common factor 2} \\
x(2x + 3) &= 0 && \text{factorise by extracting common factor } x \\
x = 0 \quad \text{or} \quad 2x + 3 &= 0 && \text{if } ab = 0 \text{, then } a = 0 \text{ or } b = 0 \\
x &= -\frac{3}{2}
\end{aligned}
$$

Attention

For **Method 2**, we cannot divide both sides of the equation by the common factor x because x can be 0 and we cannot divide by 0. See Class Discussion below.

Practise Now 1

Similar and Further Questions
Exercise 2A
Questions 1(a)–(h), 11

Solve the following quadratic equations.

(a) $x(x - 2) = 0$ **(b)** $4x(x + 1) = 0$

(c) $3x^2 + 18x = 0$ **(d)** $-\frac{2}{3}x^2 + 5x = 0$

Problem-solving Tip

For **(d)**, it may be easier to remove the fractional coefficient first, by multiplying both sides of the equation by 3.

Class Discussion

Mathematical fallacy

The following shows a *wrong* solution for Worked Example 1**(b)**.

$$
\begin{aligned}
4x^2 + 6x &= 0 && \text{divide both sides of equation by common factor } 2x \\
2x + 3 &= 0 \\
x &= -\frac{3}{2}
\end{aligned}
$$

Discuss why the solution is wrong.

Worked Example 2

Solving quadratic equations of the form $ax^2 + bx + c = 0$ (where $b \neq 0$ and $c \neq 0$)

Solve the following quadratic equations.

(a) $(3x + 7)(x - 4) = 0$

(b) $2y^2 + 7y - 15 = 0$

(c) $9n^2 - 12n + 4 = 0$

Solution

(a) $(3x + 7)(x - 4) = 0$

$$
\begin{aligned}
3x + 7 = 0 \quad &\text{or} \quad x - 4 = 0 && \text{if } ab = 0 \text{, then } a = 0 \text{ or } b = 0 \\
3x = -7 \quad &\text{or} \quad x = 4 \\
x = -\frac{7}{3}
\end{aligned}
$$

Big Idea

Equivalence

We learnt in Book 1 that to solve a linear equation, we convert the equation to another equivalent equation, until the unknown is isolated. Recall that equivalent equations have the same solution. This also applies to a quadratic equation, e.g. in **(b)**. With each manipulation, the equations are *equivalent* and will lead to the same solution. We can show this by substituting in the solution of y into any of the intermediate equations.

(b)

$$
\begin{aligned}
2y^2 + 7y - 15 &= 0 \\
(2y - 3)(y + 5) &= 0 && \text{factorise using multiplication frame} \\
2y - 3 = 0 \quad \text{or} \quad y + 5 &= 0 && \text{if } ab = 0 \text{, then } a = 0 \text{ or } b = 0 \\
y = \frac{3}{2} \quad \text{or} \quad y &= -5
\end{aligned}
$$

$\times$	y	$+5$
$2y$	$2y^2$	$+10y$
-3	$-3y$	-15

(c)
$$9n^2 - 12n + 4 = 0$$
$$(3n)^2 - 2(3n)(2) + 2^2 = 0 \quad \text{express LHS as } a^2 - 2ab + b^2, \text{ where } a = 3n \text{ and } b = 2$$
$$(3n - 2)^2 = 0 \quad \text{apply } a^2 - 2ab + b^2 = (a - b)^2$$
$$3n - 2 = 0 \quad \text{if } k^2 = 0, \text{ then } k = 0$$
$$n = \frac{2}{3}$$

Reflection

All the quadratic equations in Worked Examples 1 and 2(**a**) and 2(**b**) have two solutions. Why does the quadratic equation in Worked Example 2(**c**) only have one solution? Section 2.2B provides an explanation to this.

Practise Now 2

Similar and Further Questions
Exercise 2A
Questions 2(a)–(f), 3(a)–(j), 12(a)–(d), 13, 22, 23

1. Solve the following quadratic equations.

(a) $(x + 5)(x - 7) = 0$ **(b)** $(3y + 2)(4y - 5) = 0$

(c) $w^2 + 8w + 16 = 0$ **(d)** $\frac{3}{2}z^2 - 5z + 4 = 0$

(e) $4h^2 - 20h + 25 = 0$ **(f)** $3k^2 + 30k + 48 = 0$

2. **(i)** Solve the equation $3x^2 - 10x + 8 = 0$.

(ii) Hence, solve the equation $3(y + 1)^2 - 10(y + 1) + 8 = 0$.

3. **(i)** If $x = -2$ is a solution of the equation $x^2 + px + 8 = 0$, find the value of p.

(ii) Hence, find the other solution of the equation.

Problem-solving Tip

For Question 1(**d**), it may be easier to remove the fractional coefficient first, by multiplying both sides of the equation by 2.

Worked Example 3

Solving quadratic equations of the form $ax^2 + c = 0$ (where $c \neq 0$)

(a) Solve $25x^2 - 9 = 0$.

(b) Explain why $25x^2 + 9 = 0$ has no real solutions.

Solution

(a) **Method 1:**
$$25x^2 - 9 = 0$$
$$(5x)^2 - 3^2 = 0$$
$$(5x + 3)(5x - 3) = 0 \quad \text{apply } a^2 - b^2 = (a + b)(a - b)$$
$$5x + 3 = 0 \quad \text{or} \quad 5x - 3 = 0 \quad \text{if } ab = 0, \text{ then } a = 0 \text{ or } b = 0$$
$$x = -\frac{3}{5} \quad \text{or} \quad x = \frac{3}{5}$$
$$\therefore x = \pm\frac{3}{5}$$

Method 2:
$$25x^2 - 9 = 0$$
$$25x^2 = 9$$
$$x^2 = \frac{9}{25}$$
$$x = \pm\sqrt{\frac{9}{25}}$$
$$= \pm\frac{3}{5}$$

(b) $25x^2 + 9 = 0$

$25x^2 = -9$

Since $x^2 \geqslant 0$ for all real values of x, then $25x^2 \geqslant 0$.

Thus $25x^2$ can never be equal to -9.

$\therefore 25x^2 + 9 = 0$ has no real solutions.

Big Idea

Notations

The notation $\pm$ is used to denote + or – in a concise and precise manner, e.g. $\pm\frac{3}{5}$ means $+\frac{3}{5}$ or $-\frac{3}{5}$.

Problem-solving Tip

(a) For **Method 2**, since $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$ and $\left(-\frac{3}{5}\right)^2 = \frac{9}{25}$, then $x^2 = \frac{9}{25}$ implies $x = \pm\frac{3}{5}$.

Reflection

(a) Which method do you prefer? Why?

Practise Now 3

Similar and Further Questions
Exercise 2A
Questions 4(a)–(f), 14(a), (b)

1. Solve each of the following equations.

 (a) $9x^2 - 4 = 0$ (b) $100y^2 - 25 = 0$

 (c) $36m^2 = 1$ (d) $\frac{1}{4}n^2 - 4 = 0$

2. Explain why $16x^2 + 49 = 0$ has no real solutions.

Worked Example 4

Solving quadratic equations of the form $ax^2 + bx + c = d$

Solve each of the following equations.

(a) $x(x + 1) = 6$ (b) $(2y - 1)(y - 4) = 9$

***Solution**

(a)
$$\begin{aligned}
x(x + 1) &= 6 \\
x^2 + x &= 6 \\
x^2 + x - 6 &= 0 && \text{subtract 6 from both sides of the equation} \\
(x + 3)(x - 2) &= 0 && \text{factorise using multiplication frame} \\
x + 3 = 0 \quad &\text{or} \quad x - 2 = 0 && \text{if } ab = 0 \text{, then } a = 0 \text{ or } b = 0 \\
x = -3 \quad &\text{or} \quad x = 2
\end{aligned}$$

(b)
$$\begin{aligned}
(2y - 1)(y - 4) &= 9 \\
2y^2 - 9y + 4 &= 9 \\
2y^2 - 9y - 5 &= 0 && \text{subtract 9 from both sides of the equation} \\
(2y + 1)(y - 5) &= 0 && \text{factorise using multiplication frame} \\
2y + 1 = 0 \quad &\text{or} \quad y - 5 = 0 \\
y = -\frac{1}{2} \quad &\text{or} \quad y = 5
\end{aligned}$$

Attention

From the **Introductory Problem** Question 2, we realise that if $ab = 6$, it does *not* mean $a = 6$ or $b = 6$ because we can have $a = 2$ and $b = 3$, or $a = 12$ and $b = \frac{1}{2}$. Are there other possible values for a and b if $ab = 6$?
Therefore, $x(x + 1) = 6$ does *not* mean $x = 6$ or $x + 1 = 6$.

Practise Now 4

Similar and Further Questions
Exercise 2A
Questions 5(a)–(f), 15(a), (b), 24

Solve each of the following equations.

(a) $x(x + 6) = -5$ (b) $9y(1 - y) = 2$

(c) $(3t + 5)(t - 2) = -6$ (d) $(2v + 1)^2 = \frac{1}{5}(v + 2)$

Class Discussion

Mathematical fallacy: 2 = 1?

$$\begin{aligned}
&\text{Suppose} & x &= y \\
&\text{Then} & x^2 &= xy && \text{multiply by } x \\
& & x^2 - y^2 &= xy - y^2 && \text{subtract } y^2 \text{ from both sides of the equation} \\
& & (x + y)(x - y) &= y(x - y) && \text{factorise} \\
& & x + y &= y && \text{divide both sides of the equation by } x - y \\
&\text{Since } x = y \text{, then} & 2y &= y \\
& & 2 &= 1 && \text{divide both sides of the equation by } y
\end{aligned}$$

What is wrong in the above steps?

C. Applications of quadratic equations in mathematical problems and in real-world contexts

Worked Example 5

Finding two numbers given their sum

Two consecutive positive odd numbers are such that the sum of their squares is 130. Find the two numbers.

Solution

Let the smaller number be x.

Then the next consecutive odd number is $x + 2$.

$$\begin{aligned} \therefore \quad x^2 + (x+2)^2 &= 130 \\ x^2 + x^2 + 4x + 4 &= 130 \\ 2x^2 + 4x - 126 &= 0 \\ x^2 + 2x - 63 &= 0 \\ (x-7)(x+9) &= 0 \end{aligned}$$

$$x - 7 = 0 \quad \text{or} \quad x + 9 = 0$$

$$x = 7 \quad \text{or} \quad x = -9 \text{ (rejected since } x > 0)$$

When $x = 7$, $x + 2 = 7 + 2$
$= 9$

$\therefore$ the two consecutive positive odd numbers are 7 and 9.

Practise Now 5

Similar and Further Questions
Exercise 2A
Questions 6–10, 25

1. Two consecutive positive even numbers are such that the sum of their squares is 164. Find the two numbers.
2. The difference between two positive numbers is 5 and the square of their sum is 169. Find the two numbers.

Problem-solving Tip

2. There is no need to use two variables. Try to formulate the equation using one variable.

Worked Example 6

Finding dimensions of rectangular garden

The perimeter of a rectangular garden is 50 m and its area is 150 m^2. Calculate the length and the breadth of the garden.

Solution

Let the length of the rectangular garden be x m.

Then the breadth of the rectangular garden is $\left(\frac{50-2x}{2}\right)$ m = $(25 - x)$ m.

$$\begin{aligned} x(25 - x) &= 150 \\ 25x - x^2 &= 150 \\ x^2 - 25x + 150 &= 0 \\ (x - 10)(x - 15) &= 0 \end{aligned}$$

$$x - 10 = 0 \quad \text{or} \quad x - 15 = 0$$

$$x = 10 \quad \text{or} \quad x = 15$$

When $x = 10$, breadth of garden $= 25 - 10$
$= 15$ m

When $x = 15$, breadth of garden $= 25 - 15$
$= 10$ m

$\therefore$ the length of garden is 15 m and the breadth of garden is 10 m.

Problem-solving Tip

The length of a rectangle usually refers to the longer side.

Practise Now 6

Similar and Further Questions
Exercise 2A
Questions 16–21

The perimeter of a rectangle is 20 cm and its area is 24 cm^2. Find the length and the breadth of the rectangle.

Reflection

1. What do I already know about the factorisation of quadratic expressions that could help solve quadratic equations?
2. What have I learnt in this section that I am still unclear of?

Basic | Intermediate | Advanced

Exercise 2A

1. Solve each of the following equations.
(a) $a(a - 9) = 0$
(b) $b(b + 7) = 0$
(c) $5c^2 + 25c = 0$
(d) $11d^2 - d = 0$
(e) $-4h^2 - 16h = 0$
(f) $3k - 81k^2 = 0$
(g) $-\frac{1}{2}x(2x+3) = 0$
(h) $\frac{4}{5}y^2 - 4y = 0$

2. Solve each of the following equations.
(a) $(m - 4)(m - 9) = 0$
(b) $(n - 3)(n + 5) = 0$
(c) $(p + 1)(p + 2) = 0$
(d) $(7q - 6)(4q - 5) = 0$
(e) $(5s + 3)(2 - s) = 0$
(f) $(-2t - 5)(8t - 5) = 0$

3. Solve each of the following equations.
(a) $s^2 + 10s + 21 = 0$
(b) $t^2 - 16t + 63 = 0$
(c) $3u^2 + 49u + 60 = 0$
(d) $6w^2 - 29w + 20 = 0$
(e) $\frac{1}{3}x^2 + 2x - 9 = 0$
(f) $9y^2 + 21y - 18 = 0$
(g) $m^2 - 16m + 64 = 0$
(h) $k^2 + 12k + 36 = 0$
(i) $25p^2 + 70p + 49 = 0$
(j) $\frac{4}{9}q^2 - \frac{4}{3}q + 1 = 0$

4. Solve each of the following equations.
(a) $a^2 - 16 = 0$
(b) $4b^2 - 100 = 0$
(c) $121 - c^2 = 0$
(d) $25d^2 - \frac{1}{4} = 0$
(e) $9x^2 - 64 = 0$
(f) $\frac{1}{18} - \frac{1}{2}y^2 = 0$

5. Solve each of the following equations.
(a) $k(2k + 5) = 3$
(b) $2m(m - 5) = 5m - 18$
(c) $(n - 2)(n + 4) = 27$
(d) $(p - 1)(p - 6) = 126$
(e) $3r^2 - 5(r + 1) = 7r + 58$
(f) $(3s + 1)(s - 4) = -5(s - 1)$

6. The sum of a whole number and twice the square of the number is 10. Find the number.

7. If four times a whole number is subtracted from three times the square of the number, the result 15 is obtained. Find the number.

8. Two consecutive positive numbers are such that the sum of their squares is 113. Find the two numbers.

9. The difference between two positive numbers is 7 and the square of their sum is 289. Find the two numbers.

10. The difference between two numbers is 9 and the product of the numbers is 162. Find the two numbers.

Exercise 2A

11. Solve the equation $7x^3 + 21x^2 = 0$.

12. Solve each of the following equations.

(a) $7f + f^2 = 60$ (b) $15 = 8h^2 - 2h$

(c) $\frac{1}{2}x^2 - \frac{11}{4}x + \frac{5}{4} = 0$ (d) $2 - 3.5y - 9.75y^2 = 0$

13. (i) Solve the equation $6x^2 - x - 15 = 0$.

(ii) Hence, solve the equation $6(y-3)^2 - (y-3) - 15 = 0$.

14. (a) Solve $\frac{4}{9} - \frac{d^2}{25} = 0$.

(b) Explain why $\frac{4}{9} + \frac{d^2}{25} = 0$ has no real solutions.

15. Solve each of the following equations.

(a) $\frac{1}{3}(2q-3)(q-4) = 6$

(b) $\frac{1}{4}(t+2)(2t-6) = \frac{t}{2} + 1$

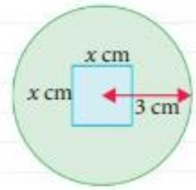
16. The perimeter of a rectangular campsite is 64 m and its area is 207 m^2. Find the length and the breadth of the campsite.

17. A rectangular field, 70 m long and 50 m wide, is surrounded by a concrete path of uniform width. Given that the area of the path is 1024 m^2, find the width of the path.

18. The length of a side and the corresponding height of a triangle are $(x + 3)$ cm and $(2x - 5)$ cm respectively. Given that the area of the triangle is 20 cm^2, find the value of x.

19. A piece of wire 44 cm long is cut into two parts. Each part is bent to form a square. Given that the total area of the two squares is 65 cm^2, find the perimeter of each square.

20. The figure shows an ancient coin which was once used in China. The coin is in the shape of a circle of radius 3 cm with a square of sides x cm removed from its centre. The area of each face of the coin is 7π cm^2.



(i) Form an equation in x and show that it reduces to $2\pi - x^2 = 0$.

(ii) Solve the equation $2\pi - x^2 = 0$.

(iii) Find the perimeter of the square.

21. Nadia walks at an average speed of $(x + 2)$ km/h for $(x - 3)$ hours and cycles at an average speed of $(3x + 5)$ km/h for x hours. She covers a total distance of 74 km.

(i) Form an equation in x and show that it reduces to $4x^2 + 4x - 80 = 0$.

(ii) Solve the equation $4x^2 + 4x - 80 = 0$.

(iii) Find the time taken for her entire journey.

22. Solve the equation $9x^2y^2 - 12xy + 4 = 0$, expressing y in terms of x.

23. (i) If $x = 5$ is a solution of the equation $x^2 - qx + 10 = 0$, find the value of q.

(ii) Hence, find the other solution of the equation.

24. Solve the equation $x - (2x - 3)^2 = -6(x^2 + x - 2)$.

25. When $(x + 1)^2$ is divided by $x - 2$, the quotient is 16 and the remainder is $x - 3$. Find the possible values of x.

2.2 Quadratic functions and graphs

In Books 1 and 2, we learnt about

- linear expressions (of the form $ax + b$), e.g. $2x + 1$;
- linear equations (of the form $ax + b = 0$), e.g. $2x + 1 = 0$;
- linear functions (of the form $y = ax + b$), e.g. $y = 2x + 1$.

In Chapter 1 and Section 2.1 of Book 3, we have learnt about

- quadratic expressions (of the form $ax^2 + bx + c$), e.g. $2x^2 - x - 6$;
- quadratic equations (of the form $ax^2 + bx + c = 0$), e.g. $2x^2 - x - 6 = 0$.

In this section, we will learn about **quadratic functions** (of the form $y = ax^2 + bx + c$).

A. Quadratic functions

Investigation — Relationship between area of square and its length

Go to www.sl-education.com/tmsoupp3/pg32 or scan the QR code on the right and open the geometry template 'Area of Square'.

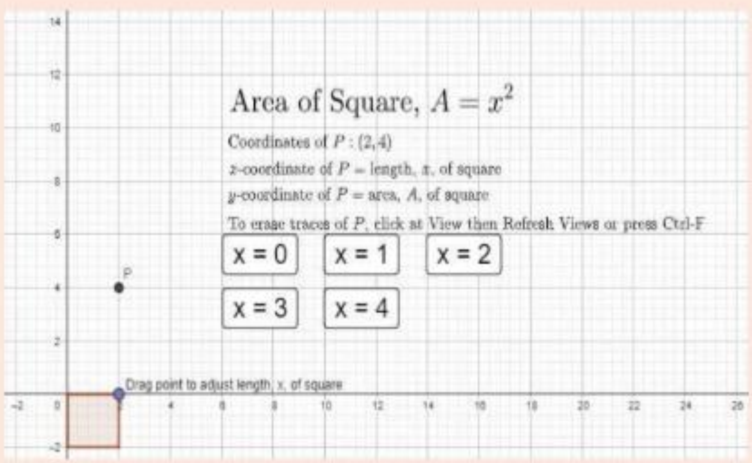


Fig. 2.1

In the template, the x-coordinate of the point P represents the length, x units, of a square while the y-coordinate of P represents its area, A units^2. Hence, P will trace out the graph of $A = x^2$.

Click on each of the buttons 'Set $x = 0$', 'Set $x = 1$', etc. to obtain the graph of $A = x^2$ and answer the following questions.

1. For each value of x, how many corresponding values of A are there? Is $A = x^2$ the equation of a function?
2. What do you notice about the shape of the graph? Is it linear or non-linear? Explain your answer.

In Book 2, we learnt that a function is a relationship between two variables x and y such that every specified input x produces *exactly one output* y.

From the Investigation on page 32, we have observed that for every value of x, there is exactly one corresponding value of A, so $A = x^2$ is the equation of a function. Since x^2 is a quadratic expression, $A = x^2$ is the equation of a **quadratic function**.

In Book 2, we also learnt that graphs of linear functions are straight lines. From the Investigation on page 32, we have observed that the graphs of quadratic functions are *non-linear*. In fact, the graphs of quadratic functions belong to a family of curves called **parabolas**.

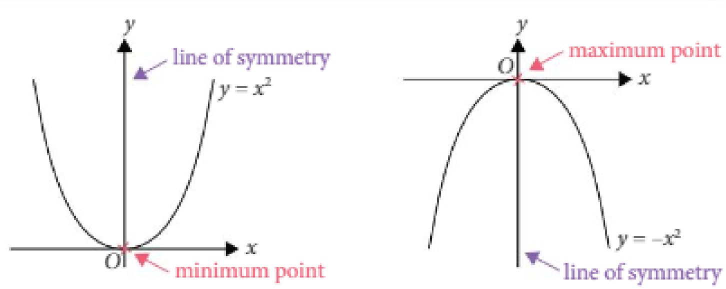
B. Graphs of quadratic functions

Investigation — Graphs of $y = x^2$ and $y = -x^2$

1. Using a graphing software, draw each of the following graphs.
   (a) $y = x^2$ (b) $y = -x^2$
2. Study the graphs and answer each of the following questions.
   (a) Both graphs pass through a particular point on the coordinate axes. What are the coordinates of this point?
   (b) State the lowest or highest point of each graph.
   (c) Both graphs are symmetrical about one of the axes. Name the axis. Hence, state the equation of the line of symmetry of the graphs.
   (d) Which point on the graph does the line of symmetry pass through?

From the above Investigation, we observe that:

- The graph of $y = x^2$ opens *upwards* indefinitely and has a **minimum point**.
- The graph of $y = -x^2$ opens *downwards* indefinitely and has a **maximum point**.
- The minimum point and the maximum point are called **turning points**.
- The graphs of $y = x^2$ and $y = -x^2$ are symmetrical about the y-axis, i.e. the equation of the **line of symmetry** of each graph is $x = 0$.
- The line of symmetry of $y = x^2$ and $y = -x^2$ is a vertical line that passes through its turning point.



(a) (b)

Fig. 2.2

Investigation — Graphs of quadratic functions $y = ax^2 + bx + c$

Go to www.sl-education.com/tmsoupp3/pg34 or scan the QR code on the right and open the geometry template 'Graphs of Quadratic Functions'.

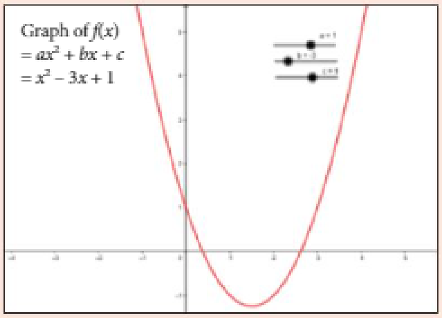


Fig. 2.3

Part 1: Effect of value of *a*

Adjust the value of b and of c to 0.

1. Increase the value of a. What do you notice about the shape of the graph?
2. Decrease the value of a but ensure that it is positive. What do you notice about the shape of the graph?
3. Decrease the value of a until it becomes negative. What do you notice about the shape of the graph?
4. How does the value of a affect the shape of the graph? What happens when a is positive and when a is negative?

Part 2: Effect of value of *c*

Adjust the value of a and of b to 1.

5. Increase the value of c. What do you notice about the position of the graph?
6. Decrease the value of c. What do you notice about the position of the graph?
7. How does the value of c affect the position of the graph?

Part 3: Quadratic graphs

8. Change the values of a, b and c to obtain each of the quadratic graphs in Table 2.1. Complete the table. The first one has been done for you.

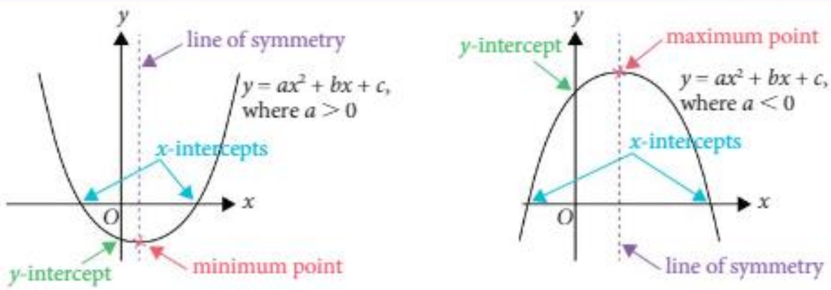
   Note: An ***x*-intercept** refers to the x-coordinate of a point of intersection of a graph with the x-axis.

Quadratic Graph	Coefficient of x^2	Opens upwards / downwards	Coordinates of minimum / maximum point	Equation of line of symmetry	x-intercept(s)	y-intercept
$y = x^2 - 4x + 3$	1	Opens upwards	$(2, -1)$	$x = 2$	1, 3	3
$y = -x^2 - 2x + 3$						
$y = x^2 - 4x + 4$						
$y = -4x^2 + 12x - 9$						
$y = 2x^2 + 2x + 1$						
$y = -3x^2 + x - 4$						

Table 2.1

From the Investigation on page 34, we observe that:

- For $a > 0$, the graph of $y = ax^2 + bx + c$ opens *upwards* indefinitely and has a **minimum point**.
- For $a < 0$, the graph of $y = ax^2 + bx + c$ opens *downwards* indefinitely and has a **maximum point**.
- The *smaller* the absolute value of a, the *wider* the graph of $y = ax^2 + bx + c$ opens.
- The **line of symmetry** of the graph of $y = ax^2 + bx + c$ is a vertical line that passes through its turning point.
- The graph of $y = ax^2 + bx + c$ may have 0, 1 or 2 x-intercept(s) but it has only 1 y-intercept.



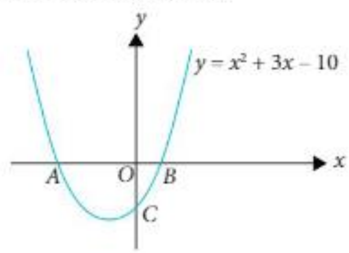
(a) (b)

Fig. 2.4

Worked Example 7

Finding coordinates of points given sketch of curve

In the figure, the curve $y = x^2 + 3x - 10$ cuts the x-axis at two points A and B, and the y-axis at the point C. Calculate the coordinates of A, B and C.



Solution

When $y = 0$, $x^2 + 3x - 10 = 0$

$$(x - 2)(x + 5) = 0$$
$$x - 2 = 0 \quad \text{or} \quad x + 5 = 0$$
$$x = 2 \quad \text{or} \quad x = -5$$

$\therefore$ the coordinates of A and B are $(-5, 0)$ and $(2, 0)$ respectively.

When $x = 0$, $y = 0^2 + 3(0) - 10$

$$= -10$$

$\therefore$ the coordinates of C are $(0, -10)$.

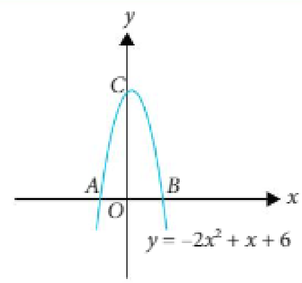
Problem-solving Tip

The y-coordinate of A and of B is 0. So we let $y = 0$ in the equation $y = x^2 + 3x - 10$ to find the x-coordinate of A and of B. Also, from the given graph, we observe that the x-coordinate of B is greater than the x-coordinate of A.

Practise Now 7

Similar and Further Questions
Exercise 2B
Questions 1, 4

In the figure, the curve $y = -2x^2 + x + 6$ cuts the x-axis at two points A and B, and the y-axis at the point C. Find the coordinates of A, B and C.



Worked Example 8

Drawing the graph of $y = ax^2 + bx + c$, where $a > 0$

The variables x and y are connected by the equation $y = x^2 + 2x + 2$. Some values of x and the corresponding values of y are given in the table.

x	-3	-2	-1	0	1	2	3
y	p	2	1	2	5	10	17

(a) Calculate the value of p.

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = x^2 + 2x + 2$ for $-3 \leqslant x \leqslant 3$.

(c) Use your graph in part **(b)** to find

(i) the value of y when $x = 2.3$,

(ii) the values of x when $y = 3$,

(iii) the minimum value of y and the value of x at which this occurs.

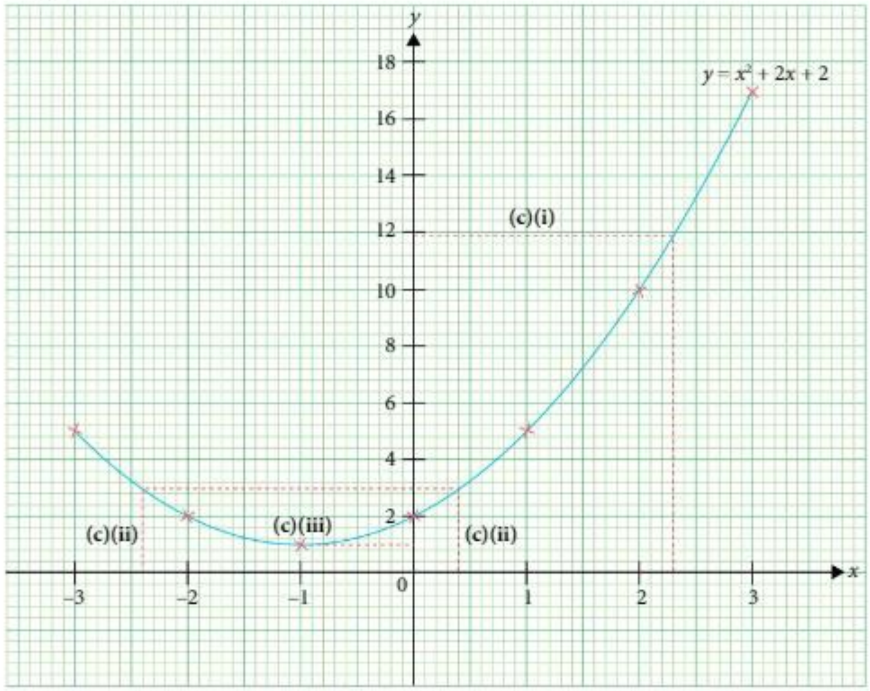
(d) State the equation of the line of symmetry of the graph.

Solution

(a) When $x = -3$, $y = (-3)^2 + 2(-3) + 2$
$= 5$

$\therefore p = 5$

(b)



(c) (i) When $x = 2.3$, $y = 11.8$
(ii) When $y = 3$, $x = -2.4$ or 0.4
(iii) Minimum value of $y = 1$
Minimum value of y occurs when $x = -1$
(d) The equation of the line of symmetry of the graph is $x = -1$.

Recall

(d) The line of symmetry of the graph of a quadratic function is a vertical line that passes through the turning point.

Practise Now 8

Similar and Further Questions
Exercise 2B
Questions 2, 9

The variables x and y are connected by the equation $y = 2x^2 - 8x + 11$. Some values of x and the corresponding values of y are given in the table.

x	–1	0	1	2	3	4	5
y	21	11	5	3	q	11	21

(a) Find the value of q.
(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 2x^2 - 8x + 11$ for $-1 \leqslant x \leqslant 5$.
(c) Use your graph in part **(b)** to find
(i) the value of y when $x = 1.5$,
(ii) the values of x when $y = 8$,
(iii) the minimum value of y and the value of x at which this occurs.
(d) State the equation of the line of symmetry of the graph.

Worked Example 9

Drawing the graph of $y = ax^2 + bx + c$, where $a < 0$

The variables x and y are connected by the equation $y = 7 + 5x - 2x^2$. Some values of x and the corresponding values of y are given in the table.

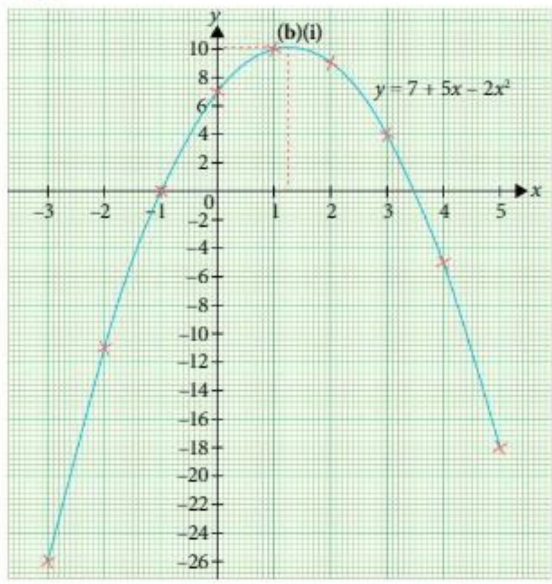
x	–3	–2	–1	0	1	2	3	4	5
y	–26	–11	0	7	10	9	4	–5	–18

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = 7 + 5x - 2x^2$ for $-3 \leqslant x \leqslant 5$.

(b) Use your graph in part **(a)** to find
  (i) the coordinates of the turning point,
  (ii) the equation of the line of symmetry of the graph.

*Solution

(a)



(b) (i) The coordinates of the turning point are (1.25, 10.2).
  (ii) The equation of the line of symmetry of the graph is $x = 1.25$.

Problem-solving Tip

- After plotting the points and *before* drawing the curve, we realise that (1, 10) is not the turning point because the plotted points are not symmetrical about the line $x = 1$, e.g. the points (0, 7) and (2, 9) are not level (i.e. do not lie on the same horizontal line).
- To locate the turning point, we can find two points on the graph that are level, e.g. the x-intercepts (one of which is plotted). From the graph, the other x-intercept lies between $x = 3$ and $x = 4$. Let's try $x = 3.5$. Substituting $x = 3.5$ into the equation, $y = 0$. So 3.5 is the other x-intercept. (If $x = 3.5$ does not work, try other values of x.)
- Therefore, the line of symmetry will pass through the midpoint of (–1, 0) and (3.5, 0). So x-coordinate of turning point
  = x-coordinate of midpoint of (–1, 0) and (3.5, 0)
  $= \dfrac{-1 + 3.5}{2}$
  $= 1.25$
  and y-coordinate of turning point
  $= 7 + 5(1.25) - 2(1.25)^2$
  $= 10.125$
  Based on the scale used, we can only plot and read the y-coordinate of the turning point as 10.2.

Practise Now 9

Similar and Further Questions
Exercise 2B
Questions 3, 5

The variables x and y are connected by the equation $y = 5 - 3x - 2x^2$. Some values of x and the corresponding values of y are given in the table.

x	–3	–2	–1	0	1	2
y	–4	3	6	5	0	–9

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 5 - 3x - 2x^2$ for $-3 \leqslant x \leqslant 2$.

(b) Use your graph in part **(a)** to find
  (i) the coordinates of the turning point,
  (ii) the equation of the line of symmetry of the graph.

C. Applications of quadratic functions in real-world contexts

Many real-world objects and phenomena can be **modelled** using quadratic functions. For example, do you know that the path of a basketball as it travels through the air follows the shape of a quadratic curve? There are many ways to model the path of the basketball by superimposing a coordinate system. In Fig. 2.5, we can see that the path of the basketball can be modelled by superimposing a coordinate system, with x being the time in seconds after the ball was released and y being the height of ball in metres above the point of release.

x	0	0.5	1	1.5	2	2.5	3
y	0.012	0.357	0.552	0.597	0.492	0.237	−0.168

Big Idea

Functions and Models
Functions express the relationship between variables. Hence, we can use them to create models to depict or approximate real-world situations, which can allow us to predict trends or simulate possible scenarios.

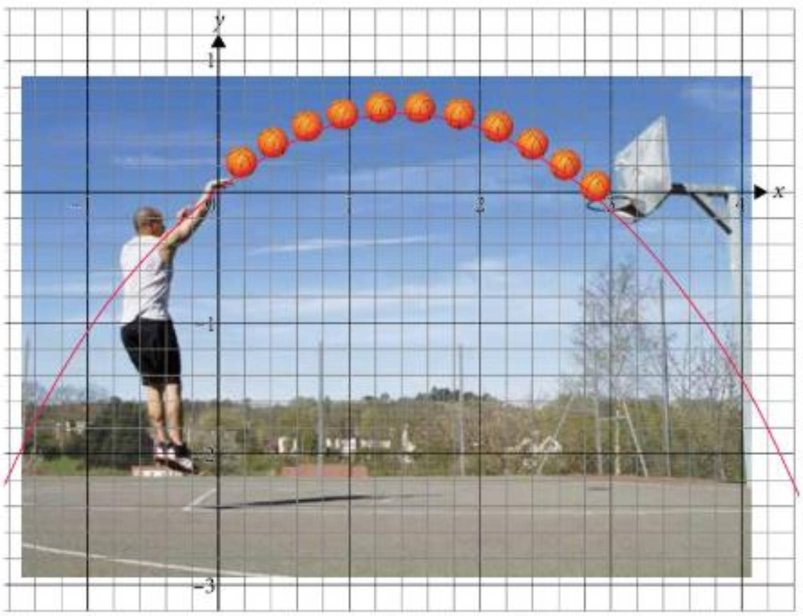


Fig. 2.5

The best-fit curve that describes the position of the ball at any time x is the graph of a quadratic function as shown in Fig. 2.6. Displaying the coordinates and the best-fit curve on a graphing software as shown in Fig. 2.6, we model the equation of the path of the basketball as $y = -0.3x^2 + 0.84x + 0.012$.

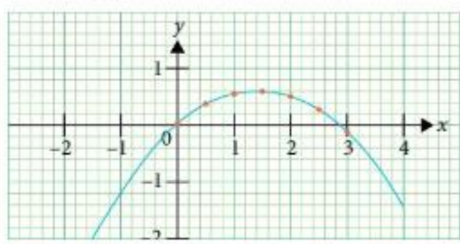


Fig. 2.6

Similarly, for other applications, we can collect data about the variables of interest and use a function to model the behaviour of these variables.

Worked Example 10

Modelling of real-world scenario using quadratic function and its graph

Ken kicks a soccer ball into the air such that the height, h metres, of the ball from the ground can be modelled by $h = 27x - 6x^2$, where x is the horizontal distance travelled by the ball in metres.

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 m on the x-axis and 1 cm to represent 5 m on the h-axis, draw the graph of $h = 27x - 6x^2$ for $0 \leqslant x \leqslant 5$.

(b) Use your graph in part **(a)** to find

(i) the maximum height of the ball above the ground,

(ii) the horizontal distance travelled by the ball before it hits the ground.

Big Idea

Functions and Models

In Worked Example 10, a quadratic function is used to model the movement of the ball.

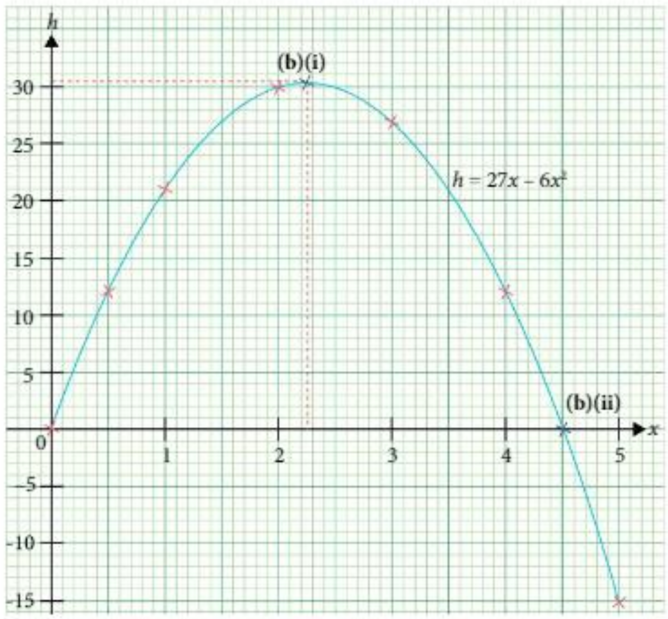
(i) What are the characteristics of the phenomenon here?

(ii) Why do you think a quadratic function is used to model this phenomenon?

*Solution

(a)

x	0	0.5	1	2	3	4	5
h	0	12	21	30	27	12	−15



Problem-solving Tip

Just like Worked Example 9, we can check our reading of the turning point by first finding two points on the graph that are level. We observe from the graph that when $x = 4.5$, $h = 0$.

So, the x-coordinate of turning point

$= x$-coordinate of midpoint of $(0, 0)$ and $(4.5, 0)$

$= \frac{0 + 4.5}{2}$

$= 2.25$

and y-coordinate of turning point $= 27(2.25) - 6(2.25)^2$

$= 30.375$.

Based on the scale used, we can only plot and read the y-coordinate of the turning point as 30.5.

(b) **(i)** Maximum height of ball = 30.5 m

(ii) Horizontal distance travelled by the ball before it hits the ground = 4.5 m

Practise Now 10

Similar and Further Questions
Exercise 2B
Questions 6, 7, 10, 11

1. A stone is thrown vertically upwards from the top of a cliff. Its height, h metres, above ground level, can be modelled by $h = 28 + 42x - 12x^2$, where x metres is the horizontal distance travelled by the stone.

(a) Assuming that the height of the thrower is negligible, find the height of the cliff.

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 m on the x-axis and 1 cm to represent 5 m on the h-axis, draw the graph of $h = 28 + 42x - 12x^2$ for $0 \leqslant x \leqslant 4.5$.

(c) Use your graph in part (b) to find

(i) the maximum height of the stone above ground level and the corresponding horizontal distance travelled,

(ii) the horizontal distance travelled before the stone hits the ground.

2. The cross section of a farmland can be modelled by the equation $y = 1 + 0.45x - 0.025x^2$, where x and y are the horizontal distance from the farmhouse and the height of the farmland above sea level respectively, measured in metres.

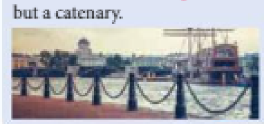
(i) On a sheet of graph paper, using a scale of 1 cm to represent 2 m on the x-axis and 4 cm to represent 1 m on the y-axis, draw the graph of $y = 1 + 0.45x - 0.025x^2$ for $0 \leqslant x \leqslant 20$.

(ii) Explain the meaning of the constant term 1 in the equation.

(iii) Find the greatest height of the farmland above sea level and the corresponding distance from the farmhouse.

Information

The shape of some real-world objects may look like a parabola, but they are not. For example, a chain hanging freely under its own weight, or the Gateway Arch, in USA, is *not a parabola*, but a catenary.



Worked Example 11

Modelling of real-world scenario using quadratic function

The height, y metres, of an object projected directly upwards from ground level can be modelled by $y = 45t - 5t^2$, where t is the time in seconds after it leaves the ground.

(i) Calculate the height of the object 5 seconds after it leaves the ground.

(ii) After how many seconds will the object strike the ground again?

Solution

(i) When $t = 5$, $y = 45(5) - 5(5^2)$

$= 100$

$\therefore$ height of object 5 seconds after it leaves the ground $= 100$ m

(ii) When $y = 0$, $45t - 5t^2 = 0$

$5t(9 - t) = 0$

$5t = 0$ or $9 - t = 0$

$t = 0$ or $t = 9$

$\therefore$ the object will strike the ground again after 9 seconds.

Attention

The height, y metres, of the object can be modelled by $y = 45t - 5t^2$ only when $y \geqslant 0$ because the ground is level. What would a negative y-value mean?

Practise Now 11

Similar and Further Questions
Exercise 2B
Questions 8, 12, 13

1. The height, y metres, of a model rocket launched directly upwards from level ground can be modelled by $y = 96t - 4t^2$, where t is the time in seconds after it leaves the ground.

(i) Find the height of the rocket 12 seconds after it leaves the ground.

(ii) After how many seconds will the rocket strike the ground again?

2. An athlete throws a javelin across a field. The height, h metres, of the javelin above ground level, can be modelled by $h = -\frac{1}{85}(x^2 - 65x - 204)$, where x is the horizontal distance from the point it was thrown in metres.

(i) What does the value of h at $x = 0$ mean?

(ii) Did the athlete achieve a new personal best, given that his record is a horizontal distance of 70 m?

(iii) Find the maximum height that the javelin reached while in the air.

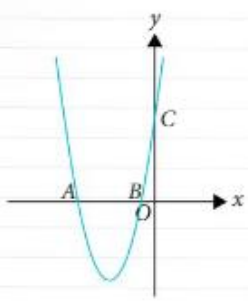
Reflection

1. What do I already know about linear functions from Book 2 that could help me learn quadratic functions in this section?
2. What are the similarities and differences between real-world applications of quadratic *equations* in Section 2.1C and quadratic *functions* in Section 2.2C?
3. What have I learnt in this section or chapter that I am still unclear of?

Basic **Intermediate** **Advanced**

Exercise 2B

1. In the figure, the curve $y = x^2 + 7x + 6$ cuts the x-axis at two points A and B, and the y-axis at the point C. Calculate the coordinates of A, B and C.



2. The variables x and y are connected by the equation $y = x^2 + 2x - 8$. Some values of x and the corresponding values of y are given in the table.

x	−5	−4	−3	−2	−1	1	2	3
y	7	a	−5	−8	−9	b	0	7

(a) Find the value of a and of b.

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 + 2x - 8$ for $-5 \leqslant x \leqslant 3$.

(c) Use your graph in part **(b)** to find

(i) the values of x when $y = 3$,

(ii) the minimum value of y.

(d) State the equation of the line of symmetry of the graph.

3. The variables x and y are connected by the equation $y = 2 - 3x - 2x^2$. Some values of x and the corresponding values of y are given in the table.

x	−4	−3	−2	−1	0	1	2
y	−18	p	0	3	2	q	−12

(a) Find the value of p and of q.

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 2 - 3x - 2x^2$ for $-4 \leqslant x \leqslant 2$.

(c) Use your graph in part **(b)** to find

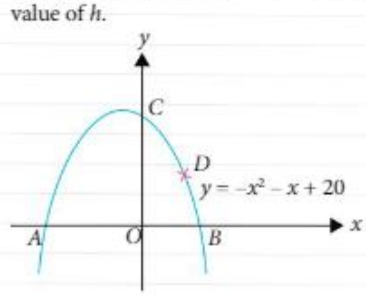
(i) the coordinates of the turning point,

(ii) the equation of the line of symmetry of the graph.

4. The diagram shows the curve $y = -x^2 - x + 20$.

(i) The curve cuts the x-axis at two points A and B, and the y-axis at the point C. Find the coordinates of A, B and C.

(ii) The point $D(3, h)$ lies on the curve. Find the value of h.



Exercise 2B

5. The variables x and y are connected by the equation $y = 10 - x - x^2$. Some values of x and the corresponding values of y are given in the table.

x	–4	–3	–2	–1	0	1	2	3
y	–2	4	8	10	10	8	4	–2

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 10 - x - x^2$ for $-4 \leqslant x \leqslant 3$.

(b) Use your graph in part **(a)** to find

(i) the coordinates of the turning point,

(ii) the equation of the line of symmetry of the graph.

(c) **(i)** Draw the line $y = 1.6$ on your graph in part **(a)**.

(ii) Hence, solve the equation $10 - x - x^2 = 1.6$.

6. If a store prices each book at $\$x$, $(64 - 8x)$ books will be sold.

(i) The total amount of money earned from the sale of the books is $\$y$. Express y in terms of x.

(ii) On a sheet of graph paper, using a scale of 2 cm to represent \$1 on the x-axis and 1 cm to represent \$10 on the y-axis, draw the graph of y for $0 \leqslant x \leqslant 8$.

(iii) Use your graph in part **(ii)** to find the amount at which the store should price each book such that the total amount earned is at a maximum.

7. The length of a side and the corresponding height of a triangle are $(x + 2)$ cm and $(7 - x)$ cm respectively.

(i) Write down a formula for the area, A, of the triangle in terms of x, and show that

$$A = 7+\frac{5}{2}x-\frac{1}{2}x^2.$$

(ii) On a sheet of graph paper, using a scale of 1 cm to represent 1 cm on the x-axis and 1 cm to represent 1 cm² on the A-axis, draw the graph of $A = 7+\frac{5}{2}x-\frac{1}{2}x^2$ for $0 \leqslant x \leqslant 8$.

(iii) Use your graph in part **(ii)** to find the length of the side of the triangle and its corresponding height that will result in its maximum area.

8. A hawk drops its prey from a certain height above the ground. The height, h metres, of the prey can be modelled by $h = 20 - 4t - 3t^2$, where t is the time in seconds after it is dropped by the hawk.

(i) At what height above the ground does the hawk drop its prey?

(ii) After how many seconds will the prey fall onto the ground?

9. The variables x and y are connected by the equation $y = x^2 - 2x$. Some values of x and the corresponding values of y are given in the table.

x	–2	–1	0	1	2	3	4
y	8	3	0	–1	0	3	8

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on each axis, draw the graph of $y = x^2 - 2x$ for $-2 \leqslant x \leqslant 4$.

(b) Use your graph in part **(a)** to find

(i) the values of x when $y = 1$,

(ii) the minimum value of y.

(c) State the equation of the line of symmetry of the graph.

(d) By drawing an appropriate line in your graph in part **(a)**, solve the equation $x^2 - 2x = x$.

10. The value of an asset can be modelled by $y = 2x^2 - 4x + 7$, where y is the value of the asset in thousands of dollars and x is the time in years.

(i) On a sheet of graph paper, using a scale of 5 cm to represent 1 year on the x-axis and 1 cm to represent \$1000 on the y-axis, draw the graph of $y = 2x^2 - 4x + 7$ for $0 \leqslant x \leqslant 3$.

(ii) Use your graph in part **(i)** to find the minimum value of the asset and the time at which it occurs.

Basic | Intermediate | Advanced

Exercise 2B

11. The path of a pirate ship adventure ride at a theme park follows the shape of a parabola. The ship swings back and forth, accelerating to the base and then upwards. The height of a rider above ground level, h m, can be modelled by the equation $h = 1.2x^2 - 12x + 30$, where x is the horizontal distance of the rider from where the ride begins.

(i) On a sheet of graph paper, using a scale of 2 cm to represent 1 m on the x-axis and 4 cm to represent 10 m on the h-axis, draw the graph of $h = 1.2x^2 - 12x + 30$ for $0 \leqslant x \leqslant 10$.

(ii) Explain the meaning of the constant term 30 in the equation.

(iii) When a rider is 19.2 m above ground level, water is splashed on him. Find the horizontal distance a rider travels between the two splashes.

12. The height, y metres, of an object projected directly upwards from the ground can be modelled by $y = 17t - 5t^2$, where t is the time in seconds after it leaves the ground.

(i) Find the height of the object 2.5 seconds after it leaves the ground.

(ii) After how many seconds will the object strike the ground again?

(iii) 1.5 seconds after the object has been projected, a second object is also projected directly upwards from the ground. Given that the equations of motion of the two objects are the same, what is the distance between the two objects one second after the second object has been projected?

13. The height, h metres, of a ball projected directly upwards from the ground can be modelled by $h = 56t - 7t^2$, where t is the time in seconds after it leaves the ground.

(i) Find the height of the ball 3.5 seconds after it leaves the ground.

(ii) After how many seconds will the ball strike the ground again?

(iii) When will the ball be 49 m above the ground? Briefly explain why there are two possible answers.

2.3 Sketching graphs of quadratic functions

A. Graphs of quadratic equation expressed in the general form $y = ax^2 + bx + c$ (Recap)

In Section 2.2, we have learnt how to **draw** (i.e. **plot**) the graphs of quadratic functions of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$. Fig. 2.7 summarises some of their characteristics.

We have also learnt that the minimum point and the maximum point are called *turning points*.

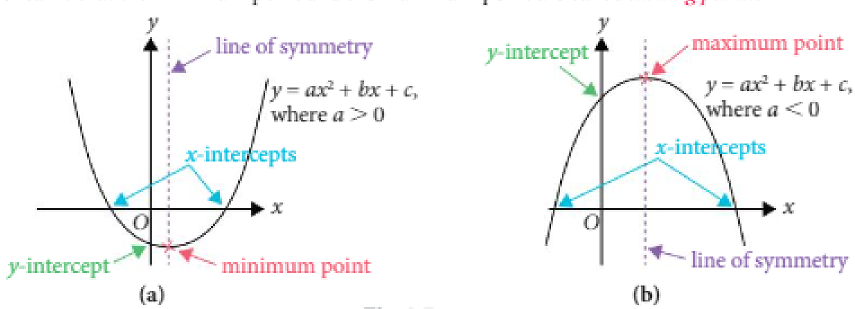


Fig. 2.7

Let us learn how to **sketch** graphs of quadratic functions which are expressed in the *factorised form* $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$, and in the *completed-square form* $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$, where the coefficient of x^2 is either 1 or -1.

Table 2.2 shows some similarities and differences between drawing (i.e. plotting) and sketching a graph.

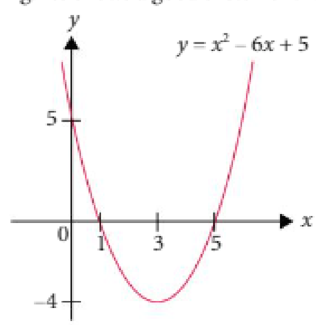
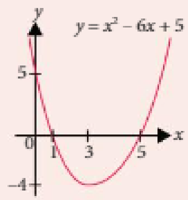
Drawing (i.e. plotting) a graph	Sketching a graph
Draw on graph paper	Draw on plain paper (or foolscap paper)
Use a pencil to draw	Use a pencil to draw
Use a scale for the axes	A scale for the axes prevents distortion of the curve. Fig. 2.8 shows a good sketch of the curve.  Fig. 2.8
Need to plot about 7 or more points at equally-spaced intervals in order to draw the curve passing through these points	Only need to plot the **critical points** which the curve passes through: • the point where the curve cuts the y-axis • the points where the curve cuts the x-axis (if any) • the turning point.
Can read the coordinates of any un-plotted points on the drawn curve to a certain degree of accuracy	Cannot read the coordinates of any un-plotted points on the curve

Table 2.2

Attention

If we do not use a scale for the axes, we may end up with a distorted curve caused by the non-uniform scale from 1 to 3, and from 3 to 5, on the x-axis.



Before we can sketch the graphs of quadratic functions of the form $y = \pm(x - h)(x - k)$ and $y = \pm(x - p)^2 + q$, we need to study some of their characteristics, such as the relationships between the x-intercepts or the coordinates of the turning point, and the constants h and k, or p and q.

B. Graphs of quadratic equations expressed in factorised form $y = \pm(x - h)(x - k)$

Investigation Characteristics of graphs of $y = \pm(x - h)(x - k)$

1. Use a graphing software to plot the graphs of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$ shown in Table 2.3. Copy and complete the table. The first row has been done for you. Note that the values for h and k are interchangeable, i.e. for the first row, you can also write $h = 5$ and $k = -1$.

Function	h	k	x-intercepts	Average of x-intercepts	Equation of line of symmetry	Turning point (state maximum or minimum)
$y = (x + 1)(x - 5)$	-1	5	$-1, 5$	$\frac{-1+5}{2} = 2$	$x = 2$	Minimum, $(2, -9)$
$y = (x - 2)(x - 6)$						
$y = (x + 7)(x + 3)$						
$y = -(x - 4)(x + 10)$						
$y = x(8 - x)$						
$y = -\left(x + \frac{7}{2}\right)(x - 9)$						

Table 2.3

2. (a) What is the relationship between the x-intercepts and the values of h and k?
   (b) What is the relationship between the equation of the line of symmetry and the x-intercepts?
   (c) What is the relationship between the equation of the line of symmetry and the x- or y-coordinate of the turning point?
   (d) Given the x-coordinate of the turning point, how do you find its y-coordinate?

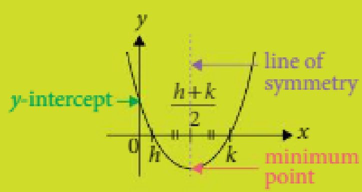
3. **Without using any graphing software**, copy and complete Table 2.4 below, using what you have learnt in Question 2.

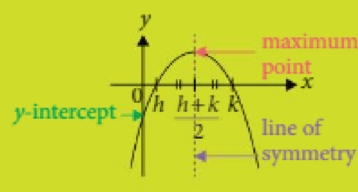
Function	h	k	x-intercepts	Equation of line of symmetry	Turning point (state maximum or minimum)
$y = (x - 1)(x - 9)$					
$y = (x + 2)(x - 7)$					
$y = (3 - x)(x + 4)$					
$y = -x\left(x + \frac{5}{3}\right)$					

Table 2.4

From the Investigation on page 46, we observe that:

For the graph of $y = \pm(x - h)(x - k)$,
- the x-intercepts are h and k;
- the equation of the line of symmetry is $x = \frac{h+k}{2}$;
- the x-coordinate of the turning point is $\frac{h+k}{2}$.





Big Idea

Equivalence
Expressing $y = ax^2 + bx + c$ in its **equivalent form** $y = \pm(x - h)(x - k)$ (when it is possible to do so) can help us find the x-intercepts and the coordinates of the turning point of its graph easily.

Worked Example 12

Sketching the graph of $y = (x - h)(x - k)$

Sketch the graph of $y = (x - 1)(x - 5)$.

Solution

Find y-intercept:

When $x = 0$, $y = (x - 1)(x - 5)$
$= (0 - 1)(0 - 5)$
$= 5$ y-intercept

Find x-intercept(s):

Method 1:

Comparing $y = (x - 1)(x - 5)$ and
$y = (x - h)(x - k)$, $h = 1$ and $k = 5$.
$\therefore$ x-intercepts are 1 and 5.

Method 2:

When $y = 0$, $(x - 1)(x - 5) = 0$
$x - 1 = 0$ or $x - 5 = 0$
$x = 1$ or $x = 5$ x-intercepts

Find turning point:

x-coordinate of turning point $= \frac{1+5}{2}$ average of x-intercepts
$= 3$

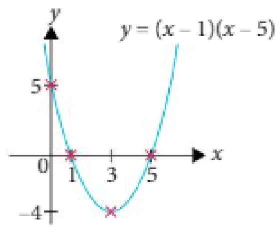
y-coordinate of turning point $= (3 - 1)(3 - 5)$
$= (2)(-2)$
$= -4$

Problem-solving Tip

Since the coefficient of x^2 is positive, we expect the curve to be U-shaped.

Step 1: Find the y-intercept.
Step 2: Find the x-intercepts using one of the two methods. Which one do you prefer? Explain.
Step 3: Find the coordinates of the turning point.
Step 4: Sketch the graph:
(i) Use a scale to draw the axes.
(ii) Label only the **critical values** (i.e. x-intercepts, y-intercept and the coordinates of the turning point) on the axes.
(iii) Mark out the **critical points** using crosses.
(iv) Draw the curve passing through the critical points. Try to draw the curve for $x > 5$ such that it looks *symmetrical* about the line of symmetry $x = 3$.
(v) Label the equation on the curve. There is no need to draw the line of symmetry, but you could draw it using a dotted line if it helps you to draw the curve for $x > 5$.

Sketch:



Practise Now 12

Similar and Further Questions
Exercise 2C
Questions 1(a)–(f), 3–6

Sketch the graph of each of the following functions.

(a) $y = (x - 2)(x - 6)$

(b) $y = (x + 4)(x + 7)$

(c) $y = -x(x - 5)$

(d) $y = (3 - x)(x + 8)$

C. Graphs of quadratic equations expressed in completed-square form $y = \pm(x - p)^2 + q$

Investigation Characteristics of graphs of $y = \pm(x - p)^2 + q$

1. Use a graphing software to plot the graphs of the form $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$ in Table 2.5. Then copy and complete the table. The first row has been done for you. Unlike the previous Investigation where the values for h and k are interchangeable, the values of p and q are not interchangeable here. Why?

Function	p	q	Turning point (state maximum or minimum)	Equation of line of symmetry	Number of x-intercepts
$y = (x + 1)^2 + 5$	–1	5	Minimum, (–1, 5)	$x = -1$	0
$y = (x - 4)^2$					
$y = (x + 8)^2 - 9$					
$y = -(x - 7)^2 + \frac{3}{2}$					
$y = -\left(x + \frac{5}{3}\right)^2$					
$y = -x^2 - 6$					

Table 2.5

2. (a) What is the relationship between the coordinates of the turning point and the values of p and q?
   (b) What is the relationship between the equation of the line of symmetry and the x- or y-coordinate of the turning point?
   (c) When will the curve just touch the x-axis (i.e. there is only one x-intercept)? Explain.
   Hint: Look at the value of p or q; and the coordinates of the turning point.
   (d) If the coefficient of x^2 is positive, when will the curve cut the x-axis at two points, or not at all?
   Hint: Look at the value of p or q; and the coordinates of the turning point.
   (e) If the coefficient of x^2 is negative, when will the curve cut the x-axis at two points, or not at all?
   Hint: Look at the value of p or q; and the coordinates of the turning point.
   (f) Given the equation of the curve, how do you find the x-intercepts?
   (g) Given the x-coordinate of the turning point, how do you find its y-coordinate?

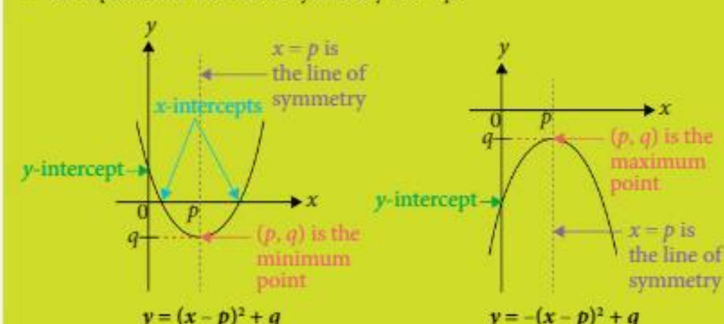
3. **Without using any graphing software**, copy and complete Table 2.6 using what you have deduced in Question 2.

Function	p	q	Turning point (state maximum or minimum)	Equation of line of symmetry	Number of x-intercepts
$y = (x - 7)^2 + 2$					
$y = x^2 - \frac{8}{3}$					
$y = -(x + 14)^2 + 16$					
$y = -\left(x - \frac{5}{2}\right)^2$					

Table 2.6

From the above Investigation, we observe that:

For the graph of $y = \pm(x - p)^2 + q$,
- the coordinates of the turning point are (p, q);
- the equation of the line of symmetry is $x = p$.



Big Idea

Equivalence
Expressing $y = ax^2 + bx + c$ in its **equivalent form** $y = \pm(x - p)^2 + q$ can help us find the coordinates of the turning point of its graph easily.

Worked Example 13

Sketching the graph of $y = (x - p)^2 + q$

Sketch the graph of $y = (x - 1)^2 - 4$.
Indicate clearly the coordinates of the points where the graph crosses the axes and the minimum point on the curve.

Solution

Find y-intercept:

When $x = 0$, $y = (x - 1)^2 - 4$
$= (0 - 1)^2 - 4$
$= -3$ $\quad$ y-intercept

Find x-intercept(s):

When $y = 0$, $(x - 1)^2 - 4 = 0$
$(x - 1)^2 = 4$
$x - 1 = \pm 2$
$x = 1 \pm 2$
$= -1$ or 3 $\quad$ x-intercepts

Find turning point:

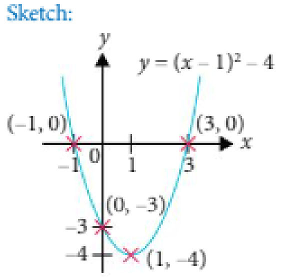
Method 1:

Comparing $y = (x - 1)^2 - 4$ and $y = (x - p)^2 + q$, $p = 1$ and $q = -4$.
$\therefore$ coordinates of minimum point $= (1, -4)$

Method 2:

Let $(x - 1)^2 = 0$. Then $x - 1 = 0$, i.e. $x = 1$.
When $(x - 1)^2 = 0$, $y = (x - 1)^2 - 4 = -4$.
$\therefore$ coordinates of minimum point $= (1, -4)$

Sketch:



Problem-solving Tip

Step 1: Find the y-intercept.
Step 2: Find the x-intercepts.
Step 3: Find the coordinates of the turning point using one of the two methods. For **Method 2**, since the *perfect square* $(x - 1)^2 \geq 0$ for all real values of x, then $y = (x - 1)^2 - 4 \geq -4$. So, if we let $(x - 1)^2 = 0$, we will find the coordinates of the minimum point. Which method do you prefer? Explain.
Step 4: Sketch the graph:
(i) Use a scale to draw the axes.
(ii) Label only the **critical values** (i.e. x-intercepts, y-intercept and the coordinates of the turning point) on the axes.
(iii) Mark out the **critical points** using crosses.
(iv) Draw the curve passing through the critical points. The curve should look *symmetrical* about the line of symmetry $x = 1$.
(v) Label the equation of the graph.
(vi) Label the coordinates of the critical points in the sketch, as stated in the question.

Practise Now 13

Similar and Further Questions
Exercise 2C
Questions 2(a)-(f), 10

1. Sketch the graph of $y = (x - 2)^2 - 9$.
   Indicate clearly the coordinates of the points where the graph crosses the axes and the minimum point on the curve.
2. Sketch the graph of $y = -(x + 5)^2 + 1$.
   Indicate clearly the coordinates of the points where the graph crosses the axes and the maximum point on the curve.

Problem-solving Tip

2. How do you use **Method 2** in Worked Example 12 to find the coordinates of the maximum point?
   Hint: Since $(x + 1)^2 \geq 0$, then $-(x + 1)^2 \leq 0$.

Worked Example 14

Sketching the graph of $y = -(x + p)^2 + q$

(i) Express $y = -x^2 - 6x - 11$ in the form $y = -(x + p)^2 + q$.

(ii) Hence, sketch the graph of $y = -x^2 - 6x - 11$.

Solution

(i)
$$\begin{aligned} y &= -x^2 - 6x - 11 \\ &= -(x^2 + 6x) - 11 \\ &= -(x^2 + 6x + 3^2 - 3^2) - 11 \\ &= -[(x + 3)^2 - 9] - 11 \\ &= -(x + 3)^2 + 9 - 11 \\ &= -(x + 3)^2 - 2 \end{aligned}$$

(ii) Find y-intercept:

Method 1:
When $x = 0$,
$$\begin{aligned} y &= -x^2 - 6x - 11 \\ &= -11 \end{aligned}$$

Method 2:
When $x = 0$,
$$\begin{aligned} y &= -(x + 3)^2 - 2 \\ &= -(0 + 3)^2 - 2 \\ &= -11 \end{aligned}$$

Find x-intercept(s):

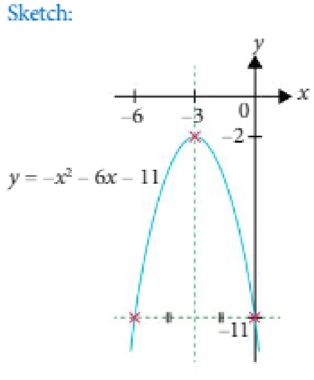
When $y = 0$, $-(x + 3)^2 - 2 = 0$
$$(x + 3)^2 = -2$$
Since $(x + 3)^2 \geqslant 0$, there are no real solutions.

Find turning point:

Method 1:
Comparing $y = -(x + 3)^2 - 2$ and $y = -(x - p)^2 + q$,
$p = -3$ and $q = -2$.
$\therefore$ coordinates of maximum point $= (-3, -2)$

Method 2:
Let $(x + 3)^2 = 0$. Then $x + 3 = 0$, i.e. $x = -3$.
When $(x + 3)^2 = 0$, $y = -(x + 3)^2 - 2 = 0 - 2 = -2$.
$\therefore$ coordinates of maximum point $= (-3, -2)$

Sketch:



Attention

$-(x + 3)^2 - 2 = 0$ having no real solutions means that the curve $y = -x^2 - 6x - 11$ does not cut the x-axis.

Problem-solving Tip

A quadratic curve that does not cut the x-axis has only two critical points: the y-intercept and the turning point. To sketch such a curve, first draw the line of symmetry passing through the turning point. Then reflect the point $(0, -11)$ on the y-axis in the line of symmetry to get the third point $(-6, -11)$. Sketch the curve passing through these three points. There is no need to write down the coordinates of the critical points since the question did not ask for them.

Practise Now 14

Similar and Further Questions
Exercise 2C
Questions 7(a), (b), 8(a), (b), 9(a), (b)

1. **(i)** Express $y = -x^2 + 4x - 6$ in the form $y = -(x - p)^2 + q$.
   (ii) Hence, sketch the graph of $y = -x^2 + 4x - 6$.
2. **(i)** Express $y = x^2 + x + 1$ in the form $y = (x + p)^2 + q$.
   (ii) Hence, sketch the graph of $y = x^2 + x + 1$.
3. **(i)** Express $y = -x^2 - 6x - 9$ in the form $y = -(x + p)^2 + q$.
   (ii) Hence, sketch the graph of $y = -x^2 - 6x - 9$.

Thinking Time

We have learnt how to sketch the graphs of quadratic functions in the following forms:
(a) $y = \pm(x - h)(x - k)$, **(b)** $y = \pm(x - p)^2 + q$.

1. Consider the graph of $y = (x - 1)^2 - 4$ in Worked Example 13.
   (i) How many times does this graph cut the x-axis?
   (ii) Can we express $y = (x - 1)^2 - 4$ in the form of $y = \pm(x - h)(x - k)$? If yes, show how it is done.
2. Consider the graph of the quadratic function $y = (x - 3)(x - 3)$.
   (i) Is $y = (x - 3)(x - 3)$ in the form of $y = \pm(x - h)(x - k)$ or $y = \pm(x - p)^2 + q$ or both? Explain.
   (ii) How many times does the graph of $y = (x - 3)(x - 3)$ cut the x-axis?
   (iii) Sketch the graph of $y = (x - 3)(x - 3)$.
3. Consider the graph of $y = -x^2 - 6x - 11$ in Worked Example 14.
   (i) How many times does this graph cut the x-axis?
   (ii) Can we express $y = -x^2 - 6x - 11$ in the form of $y = \pm(x - h)(x - k)$? Explain.
4. To sketch a quadratic function with the equation $y = x^2 + bx + c$, how do we decide whether we should express it in the factorised form, $y = (x - h)(x - k)$, or in the completed-square form, $y = (x - p)^2 + q$?

From the above Thinking Time, we observe that some quadratic functions cannot be expressed in the *factorised form*, $y = \pm(x - h)(x - k)$, because they do not cut the x-axis at all.
In general, all quadratic functions can be expressed in the *completed-square form*, $y = \pm(x - p)^2 + q$.

Class Discussion

Matching quadratic graphs with the corresponding functions

Match the graphs with their respective functions and justify your answers. If your classmate does not obtain the correct answer, explain to him or her what he or she has done wrongly.

A: $y = -(x + 1)(x + 6)$	**B:** $y = (x + 1)(x + 6)$	**C:** $y = (x - 1)(x + 6)$	**D:** $y = -(x - 1)(x - 6)$
E: $y = x^2 - 7x + 6$	**F:** $y = -x^2 - 5x + 6$	**G:** $y = x^2 - 5x - 6$	**H:** $y = -x^2 + 5x + 6$
I: $y = x^2 + 4x + 6$	**J:** $y = -x^2 + 4x - 6$	**K:** $y = x^2 + 2x + 1$	**L:** $y = -(x - 1)^2$

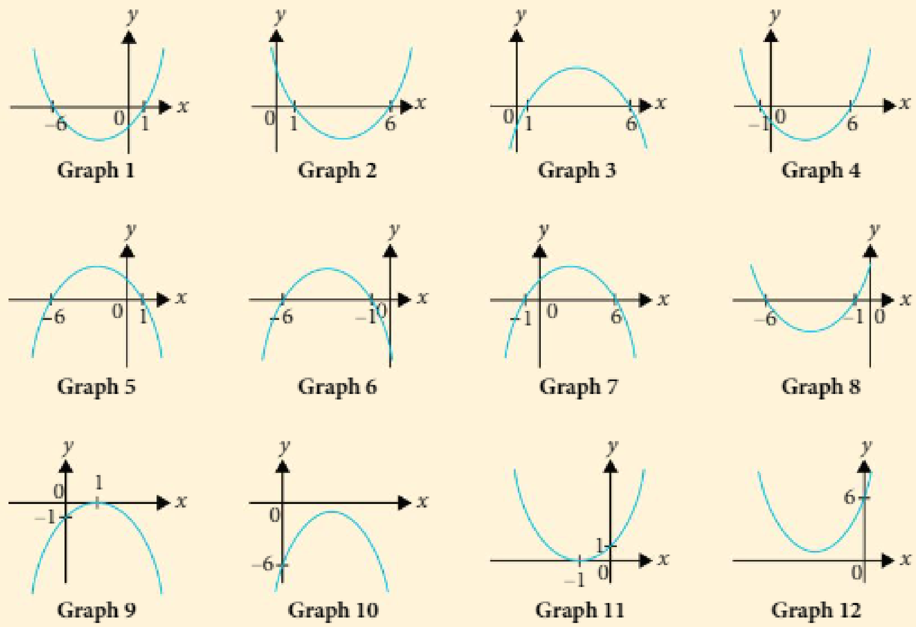


Fig. 2.9

Reflection

1. What have I previously learnt about the characteristics of the graphs of quadratic functions of the form $y = x^2 + bx + c$? How can this help me sketch the graphs of quadratic functions of the form $y = \pm(x - h)(x - k)$ and $y = \pm(x - p)^2 + q$?
2. What have I learnt in this section that I am still unclear of?

Basic Intermediate Advanced

Exercise 2D

1. Sketch the graph of each of the following functions.
   (a) $y = (x + 1)(x + 3)$ (b) $y = (x - 2)(x + 4)$
   (c) $y = -(x + 1)(x - 5)$ (d) $y = -(x - 1)(x + 6)$
   (e) $y = (3 - x)(x + 2)$ (f) $y = (2 - x)(4 - x)$

2. Sketch the graph of each of the following functions. Indicate clearly the coordinates of the maximum or the minimum point and the equation of the line of symmetry.
   (a) $y = x^2 - 4$ (b) $y = -x^2 + 6$
   (c) $y = (x - 3)^2 - 2$ (d) $y = (x + 1)^2 - 3$
   (e) $y = -(x + 2)^2 + 3$ (f) $y = -(x - 4)^2 + 5$

3. (i) Factorise $x^2 + \frac{3}{4}x$.
   (ii) Hence, sketch the graph of $y = x^2 + \frac{3}{4}x$.

4. Sketch the graph of $y = -(x^2 - x)$.

5. (i) Factorise $x^2 + x - 6$ completely.
   (ii) Hence, sketch the graph of $y = x^2 + x - 6$.

6. Sketch the graph of $y = x^2 - 4x + 3$.

7. (a) (i) Express $y = -x^2 - 7x - 15$ in the form $y = -(x + p)^2 + q$.
   (ii) Hence, sketch the graph of $y = -x^2 - 7x - 15$.
   (b) (i) Express $y = x^2 - 3x + 4$ in the form $y = (x - p)^2 + q$.
   (ii) Hence, sketch the graph of $y = x^2 - 3x + 4$.

8. (a) (i) Express $y = -x^2 - 10x - 25$ in the form $y = -(x + p)^2 + q$.
   (ii) Hence, sketch the graph of $y = -x^2 - 10x - 25$.
   (b) (i) Express $y = x^2 - 8x + 16$ in the form $y = (x - p)^2 + q$.
   (ii) Hence, sketch the graph of $y = x^2 - 8x + 16$.

9. (a) (i) Express $y = -x^2 + 6x - 6$ in the form $y = -(x - p)^2 + q$.
   (ii) Hence, sketch the graph of $y = -x^2 + 6x - 6$.
   (b) (i) Express $y = x^2 - 8x + 5$ in the form $y = (x - p)^2 + q$.
   (ii) Hence, sketch the graph of $y = x^2 - 8x + 5$.

10. The graph of $y = (x - h)^2 + k$ has a minimum point at $\left(-\frac{1}{2}, \frac{3}{4}\right)$.
   (i) State the value of h and of k.
   (ii) Hence, sketch the graph of $y = (x - h)^2 + k$, indicating the coordinates of the point of intersection of the graph with the y-axis.

Looking Back

In this chapter, we have learnt to solve a special group of non-linear equations. More specifically, we applied the Zero Product Principle to solve quadratic equations which can be easily factorised into the product of two linear factors. Quadratic functions have several real-world applications. For example, the path of a basketball can be **modelled** by a quadratic **function**. Besides solving quadratic equations algebraically, we have also observed the connections between the graphs and equations of quadratic functions and used graphs to obtain an approximate solution to the equation. Graphs, or mathematical **diagrams** in general, provide a visual representation of mathematical ideas so that connections can be seen more clearly.

Summary

1. **Zero Product Principle**

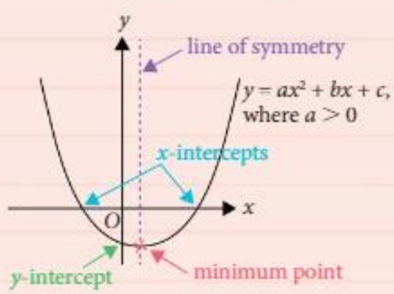
   If P and Q are factors of an algebraic expression such that $PQ = 0$, then $P = 0$ or $Q = 0$.

   - Show how you can use this principle to help you solve the following quadratic equations:

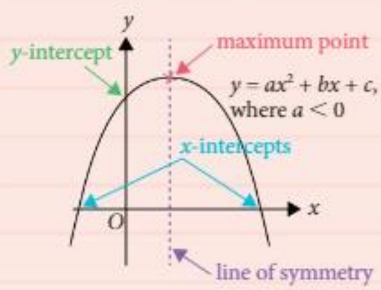
     (a) $4x^2 - 9x = 0$ **(b)** $4x^2 - 9 = 0$

     (c) $4x^2 - 12x + 9 = 0$ **(d)** $4x^2 + 5x - 9 = 0$

2. The general form of the equation of a quadratic function is $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.



$a > 0$



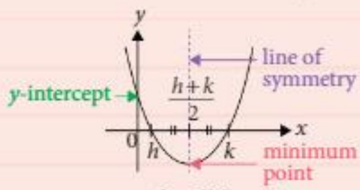
$a < 0$

- For $a > 0$, the graph of $y = ax^2 + bx + c$ opens *upwards* indefinitely and has a **minimum point**.
- For $a < 0$, the graph of $y = ax^2 + bx + c$ opens *downwards* indefinitely and has a **maximum point**.
- The *smaller* the absolute value of a, the *wider* the graph of $y = ax^2 + bx + c$ opens.
- The **line of symmetry** of the graph of $y = ax^2 + bx + c$ is a vertical line that passes through its turning point.
- The graph of $y = ax^2 + bx + c$ may have 0, 1 or 2 x-intercept(s) but it has only 1 y-intercept.

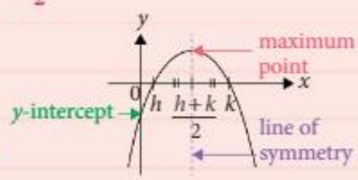
Summary

3. Graphs of quadratic functions of the form $y = \pm(x - h)(x - k)$

- The x-intercepts are h and k.
- The equation of the line of symmetry is $x = \frac{h+k}{2}$.
- The x-coordinate of the turning point is $\frac{h+k}{2}$.



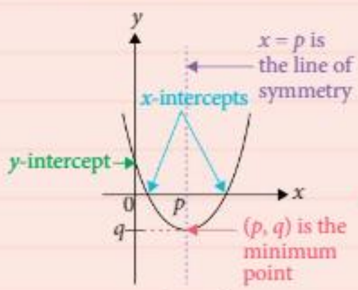
$y = (x - h)(x - k)$



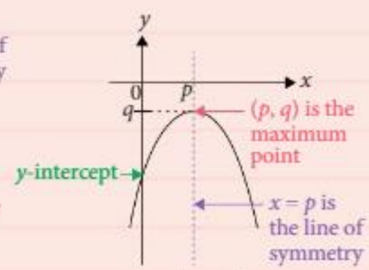
$y = -(x - h)(x - k)$

4. Graphs of quadratic functions of the form $y = \pm(x - p)^2 + q$

- The coordinates of the turning point are (p, q).
- The equation of the line of symmetry is $x = p$.



$y = (x - p)^2 + q$



$y = -(x - p)^2 + q$

CHAPTER 3

Quadratic and Fractional Equations

Ancient babylonian tablets, similar to the one in the picture, provide us with fascinating insights into the lives and knowledge of people who lived thousands of years ago. Thousands of mathematics problems found on tablets dated around 1600 BC denote a range of numeric and advanced mathematical practices in the Babylonian period.

On one of these tablets is an interesting problem, translated as "I take away the side from the area of a square to get 870".

Written in algebraic notation, this would be expressed as a quadratic equation $x^2 - x = 870$. Can you solve this?

The general method to solve any quadratic equation was finally developed after 3000 years, in the 16th century. Everyone thought that the problem of solving quadratic equation was thus laid to rest. Then, in September 2019, Loh Po-Shen, a mathematics professor from Carnegie Mellon University, rediscovered a technique used by the ancient Babylonians that made the solving of quadratic equations much easier!

In this chapter, we will learn about the idea behind the standard general solution to any quadratic equation.

Learning Outcomes

What will we learn in this chapter?

- How to solve quadratic equations in one variable by
  - completing the square for equations of the form $x^2 + bx + c = 0$
  - use of formula
  - graphical method
- How to solve fractional equations that can be reduced to quadratic equations
- How to formulate a quadratic equation in one variable to solve problems
- Why quadratic equations and quadratic functions have useful applications in mathematics and real-world contexts

Introductory Problem

In Chapter 2, we have learnt how to solve a quadratic equation by factorising the corresponding quadratic expression and then using the Zero Product Principle to solve the equation:

If P and Q are factors of an algebraic expression such that $PQ = 0$, then $P = 0$ or $Q = 0$.

Solve the following quadratic equations:

(a) $x^2 + 4x + 3 = 0$

(b) $x^2 + 4x - 3 = 0$

In the **Introductory Problem**, you could use factorisation to solve part **(a)**, but not part **(b)**. Let us now learn how to solve quadratic equations that cannot be solved by the factorisation method.

3.1 Solving quadratic equations by completing the square

A. Solving quadratic equations by factorisation (Recap)

Recall that a quadratic equation is of the form

$$ax^2 + bx + c = 0, \text{ where } a, b \text{ and } c \text{ are real numbers and } a \neq 0.$$

We have also learnt how to solve a quadratic equation by factorisation, which we will recap below.

Worked Example 1

Solving quadratic equation by factorisation

Solve the equation $x^2 - 5x - 6 = 0$.

Solution

$$
\begin{aligned}
x^2 - 5x - 6 &= 0 \\
(x - 6)(x + 1) &= 0 \quad \text{factorise using multiplication frame} \\
x - 6 = 0 \quad &\text{or} \quad x + 1 = 0 \quad \text{Zero Product Principle} \\
x = 6 \quad &\text{or} \quad x = -1
\end{aligned}
$$

Big Idea

Equivalence

Recall that two equations are **equivalent** if they have the same solution(s), e.g. $x^2 - 5x - 6 = 0$ and $(x - 6)(x + 1) = 0$ are equivalent equations. Converting an equation into a sequence of appropriate equivalent equations is the basis of solving an equation.

Practise Now 1

Similar and Further Questions
Exercise 3A
Questions 1(a)–(d)

Solve each of the following equations.

(a) $x^2 - 7x - 8 = 0$

(b) $2y^2 + 3y - 20 = 0$

B. Solving quadratic equations of the form $(x + r)^2 = s$

The quadratic equation in Worked Example 1 can be solved using factorisation. But we have seen from the **Introductory Problem** that some quadratic equations, e.g. $x^2 + 4x - 3 = 0$, cannot be solved this way. Such quadratic equations can be expressed in the form $(x + r)^2 = s$, where r and s are real numbers (which will be covered later in this chapter).

Let us first see how an equation of the form $(x + r)^2 = s$ can be solved easily by taking the square roots on both sides of the equation to obtain the solutions, as shown in Worked Example 2.

Worked Example 2

Solving quadratic equation of the form $(x + r)^2 = s$

Solve the equation $(x + 3)^2 = 14$.

Solution

$(x + 3)^2 = 14$

$x + 3 = \pm\sqrt{14}$ take square roots on both sides

$x = -3\pm\sqrt{14}$

$= 0.742$ or -6.74 (to 3 s.f.)

Big Idea

Notations

The notation $\pm$ is used to represent '+' or '–' in a *concise manner*. For example, $\pm\sqrt{14}$ means $+\sqrt{14}$ or $-\sqrt{14}$; and $-3\pm\sqrt{14}$ means $-3+\sqrt{14}$ or $-3-\sqrt{14}$.

Practise Now 2

Similar and Further Questions
Exercise 3A
Questions 2(a)–(h)

Solve each of the following equations.

(a) $(x + 7)^2 = 100$

(b) $(2y - 5)^2 = 11$

C. Perfect squares

We encountered the algebraic expression $(x + 3)^2$ in Worked Example 2. What do you call such an expression?
We learnt about **perfect squares** in Book 2.
For example, the numbers 9 and 16 are perfect squares because they can be expressed as the square of an integer: $9 = 3^2$ and $16 = 4^2$.
Algebraic expressions like a^2, b^2, $(a + b)^2$ and $(a - b)^2$ are also perfect squares because they are the squares of the expressions a, b, $(a + b)$ and $(a - b)$ respectively.
They can also be represented as squares pictorially. For example, in Fig. 3.1**(a)**, a^2, b^2, and $(a + b)^2$ can be represented as squares.

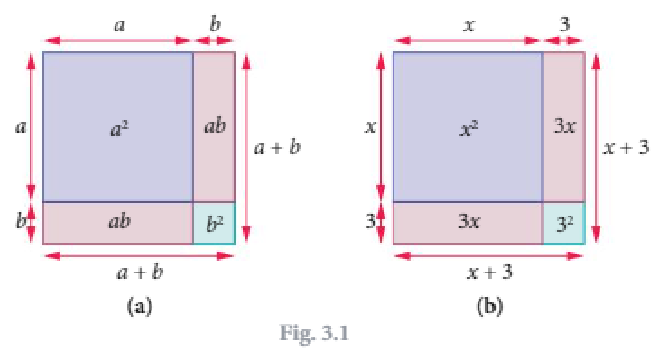
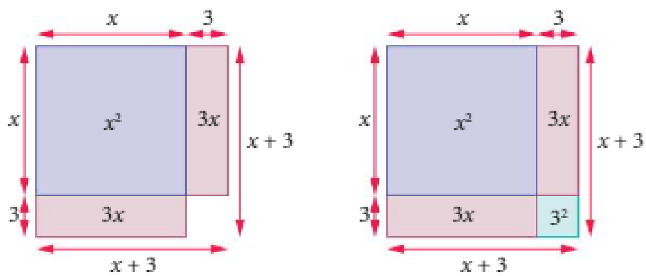


Fig. 3.1

Similarly, the expression $(x + 3)^2$ in Worked Example 2 is a perfect square and it can be represented pictorially as shown in Fig. 3.1**(b)**.

D. Completing the square

We learnt in the previous section that $(x + 3)^2 = x^2 + 6x + 9$ is a perfect square.
But is $x^2 + 6x$ also a perfect square?
By viewing it pictorially in Fig. 3.2(**a**), we realise that $x^2 + 6x$ is not a perfect square.
However, we can make $x^2 + 6x$ into a perfect square by adding $3^2 = 9$ to the expression $x^2 + 6x$.
This makes it $x^2 + 6x + 9 = (x + 3)^2$ as shown in Fig. 3.2(**b**).



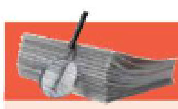
(a) **(b)**

Fig. 3.2

This process is called **completing the square** because we add the number $3^2 = 9$ to complete $x^2 + 6x$ into a square. It is very important to note that $x^2 + 6x \neq (x + 3)^2$.
What number do we add to a quadratic expression of the form $x^2 + bx$ to make it into a perfect square? Let us investigate.

Big Idea

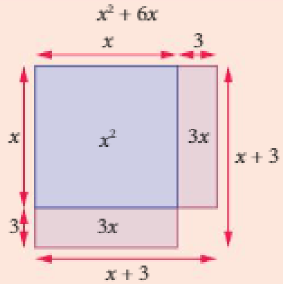
Equivalence
Since $(x + 3)^2 = x^2 + 6x + 9$ for all values of x, $(x + 3)^2 \neq x^2 + 6x$. Therefore, $x^2 + 6x$ is **not equivalent** to $(x + 3)^2$ and we cannot write the two expressions as being equal.



Investigation

Completing the square for quadratic expressions of the form $x^2 + bx$

To make a quadratic expression of the form $x^2 + bx$ into a perfect square $(x + r)^2$, we have to add a number, t, to $x^2 + bx$ to complete the square.
In this Investigation, we will find a relationship between t and b, and between r and b.
Complete Table 3.1. The first one has been done for you.

	Quadratic expression $x^2 + bx$	Number, t, that must be added to complete the square	Half the coefficient of x, i.e. $\frac{b}{2}$	Completed square $(x + r)^2$	r
(a)	$x^2 + 6x$ 	3^2	$\frac{6}{2} = 3$	$(x + 3)^2$	3

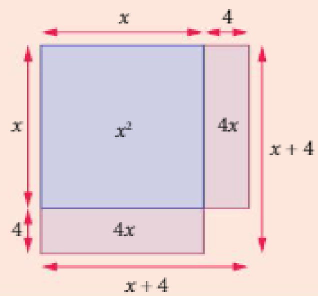
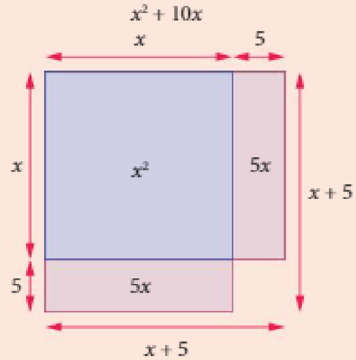
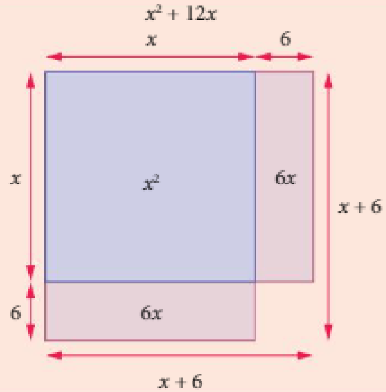
	Quadratic expression $x^2 + bx$	Number, t, that must be added to complete the square	Half the coefficient of x, i.e. $\frac{b}{2}$	Completed square $(x + r)^2$	r
(b)	$x^2 + 8x$ 				
(c)	$x^2 + 10x$ 				
(d)	$x^2 + 12x$ 				

Table 3.1

1. What is the relationship between t and b?
2. What is the relationship between r and b?
3. Based on your answer to Question 1, what number must you add to $x^2 + 18x$ to complete the square?
4. Based on your answer to Question 2, what is the completed square $(x + r)^2$ after completing the square for $x^2 + 18x$?

From the Investigation on pages 60 and 61, we observe the following:

To complete the square for $x^2 + bx$, we add the number $\left(\frac{b}{2}\right)^2$, so that the completed square is $\left(x+\frac{b}{2}\right)^2$.

Since $x^2 + bx \neq \left(x+\frac{b}{2}\right)^2$, how do we express $x^2 + bx$ in terms of a perfect square?

Worked Example 3 will show you how this is done.

Worked Example 3

Completing the square for quadratic expressions of the form $x^2 + bx$

Express each of the following expressions in the form $(x + r)^2 + u$.

(a) $x^2 + 10x$ **(b)** $x^2 - 5x$

Solution

(a) $x^2 + 10x = \left[x^2+10x+\left(\frac{10}{2}\right)^2\right]-\left(\frac{10}{2}\right)^2$

$= (x^2 + 10x + 5^2) - 5^2$

$= (x + 5)^2 - 25$ $\quad a^2 + 2ab + b^2 = (a + b)^2$

(b) $x^2 - 5x = \left[x^2-5x+\left(-\frac{5}{2}\right)^2\right]-\left(-\frac{5}{2}\right)^2$

$= \left(x-\frac{5}{2}\right)^2-\frac{25}{4}$ $\quad a^2 - 2ab + (-b)^2 = (a - b)^2$, where $b = \frac{5}{2}$

Problem-solving Tip

(a) The coefficient of x is 10. We divide this by 2 and get 5. So we add 5^2 to $x^2 + 10x$. But $x^2 + 10x \neq x^2 + 10x + 5^2$. To make them equal, we have to subtract 5^2 like this: $x^2 + 10x = x^2 + 10x + 5^2 - 5^2$. How do you explain that the two expressions above are equal?

(b) The coefficient of x is -5.

Practise Now 3

Similar and Further Questions
Exercise 3A
Questions 3(a)–(h)

Express each of the following expressions in the form $(x + r)^2 + u$.

(a) $x^2 + 12x$ **(b)** $x^2 - 7x$

(c) $x^2 + 1.6x$ **(d)** $x^2 - \frac{3}{4}x$

Worked Example 4

Completing the square for quadratic expressions of the form $x^2 + bx + c$

Express

(a) $x^2 + 2x + 3$ in the form $(x + r)^2 + u$;

(b) $-x^2 + 7x - 5$ in the form $-(x + r)^2 + u$.

Solution

(a) $x^2 + 2x + 3 = (x^2 + 2x + 1^2) - 1^2 + 3$

$= (x + 1)^2 + 2$

(b) $-x^2 + 7x - 5 = -(x^2 - 7x) - 5$

$= -\left[x^2-7x+\left(-\frac{7}{2}\right)^2-\left(-\frac{7}{2}\right)^2\right]-5$

$= -\left(x-\frac{7}{2}\right)^2+\frac{49}{4}-\frac{20}{4}$

$= -\left(x-\frac{7}{2}\right)^2+\frac{29}{4}$

Problem-solving Tip

(a) Leave the constant term 3 alone and complete the square for $x^2 + 2x$ just like in Worked Example 3.

(b) Group the first two terms and factorise out -1 so that the coefficient of x^2 in the brackets is now positive. Then complete the square for $x^2 - 7x$ just like in Worked Example 3.

Practise Now 4

Similar and Further Questions
Exercise 3A
Questions 4(a)–(h)

1. Express each of the following expressions in the form $(x + r)^2 + u$.
   (a) $x^2 + 14x + 5$
   (b) $x^2 + 7x - 1.2$
   (c) $x^2 - 9x + 3$
   (d) $x^2 - \frac{6}{5}x - 4$
2. Express each of the following expressions in the form $-(x + r)^2 + u$.
   (a) $-x^2 + 6x - 2$
   (b) $-x^2 + 9x - 3.5$
   (c) $-x^2 - 7x + 5$
   (d) $-x^2 - \frac{4}{9}x - 1$

E. Solving quadratic equations by completing the square

After learning how to complete the square for a quadratic expression, we can now solve quadratic equations by completing the square.

Worked Example 5

Solving quadratic equation by completing the square

Solve the equation $x^2 + 4x - 3 = 0$, giving your answer correct to 2 decimal places.

*Solution

Method 1:

$$\begin{aligned} x^2 + 4x - 3 &= 0 \\ (x^2 + 4x + 2^2) - 2^2 - 3 &= 0 && \text{add } 2^2 - 2^2 = 0 \text{ to LHS} \\ (x + 2)^2 - 7 &= 0 && \text{complete the square} \\ (x + 2)^2 &= 7 \\ x + 2 &= \pm\sqrt{7} && \text{take square roots on both sides} \\ x &= -2 \pm \sqrt{7} \\ &= 0.65 \text{ or } -4.65 \text{ (to 2 d.p.)} \end{aligned}$$

Method 2:

$$\begin{aligned} x^2 + 4x - 3 &= 0 \\ x^2 + 4x &= 3 && \text{add 3 to both sides} \\ x^2 + 4x + 2^2 &= 3 + 2^2 && \text{add } 2^2 \text{ to both sides} \\ (x + 2)^2 &= 7 && \text{complete the square} \\ x + 2 &= \pm\sqrt{7} && \text{take square roots on both sides} \\ x &= -2 \pm \sqrt{7} \\ &= 0.65 \text{ or } -4.65 \text{ (to 2 d.p.)} \end{aligned}$$

Big Idea

Equivalence

In **Method 1**, why can we add $2^2 - 2^2$ to $x^2 + 4x - 3$? Are the two expressions $x^2 + 4x - 3$ and $(x^2 + 4x + 2^2) - 2^2 - 3$ equivalent? Explain.

In **Method 2**, why can we add 2^2 to both sides of the equation? Are the two equations, $x^2 + 4x = 3$ and $x^2 + 4x + 2^2 = 3 + 2^2$, equivalent? Explain.

Reflection

Which method do you prefer? Why?

Information

From the solutions of $x^2 + 4x - 3 = 0$, we can infer that

$$x^2 + 4x - 3 = \left[x - \left(-2 + \sqrt{7}\right)\right]\left[x - \left(-2 - \sqrt{7}\right)\right] = \left(x + 2 - \sqrt{7}\right)\left(x + 2 + \sqrt{7}\right).$$

Hence although we cannot factorise equations such as $x^2 + 4x - 3$ using a multiplication frame, *all quadratic expressions can be expressed as a product of two linear factors as shown in the above example.*

Practise Now 5

Similar and Further Questions
Exercise 3A
Questions 5(a)–(f), 6, 7(a)–(d), 8

1. Solve each of the following equations, giving your answers correct to 2 decimal places.
   (a) $x^2 + 6x - 4 = 0$
   (b) $y^2 + 7y + 5 = 0$
   (c) $z^2 - z - 1 = 0$
2. Solve the equation $(x + 4)(x - 3) = 15$.

Reflection

The first quadratic equation in the **Introductory Problem**, $x^2 + 4x + 3 = 0$, can be solved by factorisation but not the second one, $x^2 + 4x - 3 = 0$.

We solved the second equation by completing the square in Worked Example 5.

1. Do you think the first equation can also be solved by completing the square? If yes, show how this can be done.
2. Do you prefer to solve the first equation by factorisation or by completing the square? Why?
3. How do you solve the quadratic equation $2x^2 + 8x - 6 = 0$, where the coefficient of x^2 is not equal to 1?

Basic | Intermediate | Advanced

Exercise 3A

1. Solve each of the following equations.
(a) $x^2 + 7x - 18 = 0$ (b) $2x^2 + 5x - 7 = 0$
(c) $5y^2 - 28y + 15 = 0$ (d) $4z^2 - 49 = 0$

2. Solve each of the following equations, giving your answers correct to 2 decimal places where necessary.
(a) $(x + 1)^2 = 9$ (b) $(2y + 1)^2 = 16$
(c) $(5h - 4)^2 = 81$ (d) $(7 - 3k)^2 = \frac{9}{16}$
(e) $(m + 3)^2 = 11$ (f) $(2n - 3)^2 = 23$
(g) $(5 - w)^2 = 7$ (h) $\left(\frac{1}{2} - t\right)^2 = 10$

3. Express each of the following expressions in the form $(x + r)^2 + u$.
(a) $x^2 + 20x$ (b) $x^2 - 15x$
(c) $x^2 + \frac{1}{2}x$ (d) $x^2 - \frac{2}{9}x$
(e) $x^2 + 0.2x$ (f) $x^2 - 1.4x$
(g) $-x^2 - 10x$ (h) $-x^2 + 11x$

4. Express each of the following expressions in the form $(x + r)^2 + u$.
(a) $x^2 - 6x + 1$ (b) $x^2 + 3x - 2$
(c) $x^2 + 9x - 1.8$ (d) $x^2 - \frac{2}{7}x + 7$
(e) $-x^2 + 10x - 2$ (f) $-x^2 + 13x - \frac{13}{2}$
(g) $-x^2 - 9x - 20.25$ (h) $-x^2 - \frac{3}{4}x + 3$

5. Solve each of the following equations by completing the square, giving your answers correct to 2 decimal places.
(a) $x^2 + 2x - 5 = 0$ (b) $y^2 - 12y + 9 = 0$
(c) $z^2 - 5z - 5 = 0$ (d) $p^2 + \frac{1}{4}p - 3 = 0$
(e) $q^2 - \frac{6}{7}q + \frac{2}{49} = 0$ (f) $r^2 + 0.6r - 1 = 0$

6. (i) Express $x^2 + 17x - 30$ in the form $(x + a)^2 + b$.
(ii) Hence solve the equation $x^2 + 17x - 30 = 0$, giving your answers correct to one decimal place.

7. Solve each of the following equations.
(a) $a(a + 4) = 3a + 1$ (b) $(b + 1)^2 = 7b$
(c) $(c - 2)(c + 5) = c$ (d) $d(d - 4) = 2(d + 7)$

8. Given the equation $y^2 - ay - 6 = 0$, where a is a constant, find the solutions for y in terms of a.

3.2 Solving quadratic equations using formula

From Section 3.1, you may have realised that solving a quadratic equation by completing the square can be quite tedious.

In this section, we will learn how to generalise this method into a general formula that can be used to solve all quadratic equations.

Completing the square for specific equation		Completing the square in general	
$x^2 + 4x - 3 = 0$		$ax^2 + bx + c = 0$	
		$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	divide throughout by a
$x^2 + 4x = 3$	add 3 to both sides	$x^2 + \frac{b}{a}x = -\frac{c}{a}$	subtract $\frac{c}{a}$ from both sides
$x^2 + 4x + 2^2 = 3 + 2^2$	add 2^2 to both sides	$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	add $\left(\frac{b}{2a}\right)^2$ to both sides
$(x + 2)^2 = 7$	complete the square	$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$ $= \frac{b^2 - 4ac}{4a^2}$	complete the square
$x + 2 = \pm\sqrt{7}$	take square roots on both sides	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ $= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	take square roots on both sides
$x = -2 \pm \sqrt{7}$		$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	
$= 0.65$ or -4.65 (to 2 d.p.)		$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	

In general:

The solutions of a quadratic equation $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Worked Example 6

Solving quadratic equation using formula

Solve the equation $3x^2 - 4x - 5 = 0$.

Solution

$3x^2 - 4x - 5 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-5)}}{2(3)} \qquad a = 3,\ b = -4,\ c = -5$$

$$= \frac{4 \pm \sqrt{76}}{6}$$

$= 2.12$ or -0.786 (to 3 s.f.)

Practise Now 6

Similar and Further Questions
Exercise 3B
Questions 1(a)–(f), 2(a)–(f), 4(a)–(f)

Solve each of the following equations.

(a) $2x^2 + 3x - 7 = 0$
(b) $-5x^2 + 8x + 1 = 0$
(c) $3x^2 - 5 - x = 0$
(d) $1 - x^2 - 7x = 0$
(e) $(x - 1)^2 = 4x - 5$
(f) $(x + 3)(x - 1) = 8x - 7$

Class Discussion

Number of real solutions to quadratic equations

Solve the following equations using the quadratic formula.

(a) $3x^2 + 5x - 4 = 0$
(b) $4x^2 - 12x + 9 = 0$
(c) $2x^2 + 5x + 8 = 0$

1. Can you solve the equation in part **(c)**? Why not?
   What do you notice about the sign of the number which you have to take the square root of?
2. We say that the equation in part **(c)** has no real solutions.
   State the number of real solutions each of the three equations above have.
3. Can you solve the equation in part **(b)**, i.e. $4x^2 - 12x + 9 = 0$, by factorisation?
   Which method do you prefer? Why?

From the above Class Discussion, we observe that a quadratic equation can have *two, one or no real solutions*. We also learnt that the quadratic formula can be used to solve all quadratic equations even if they have no real solutions. In other words, we can still solve such equations but their solutions are not real.

Reflection

We have learnt 3 methods to solve a quadratic equation thus far:

(a) factorisation;
(b) completing the square;
(c) quadratic formula.

1. Which method cannot be used to solve all quadratic equations?
2. If a quadratic equation can be solved by all 3 methods, which method do you prefer? Why?
3. If a quadratic equation cannot be solved using the method in Question 1, which of the two remaining methods do you prefer? Why?

3.3 Solving fractional equations reducible to quadratic equations

In Chapter 1, we learnt about *algebraic fractions* of the form $\frac{A}{B}$, where A and/or B are algebraic expressions, and $B \neq 0$.

Examples of algebraic fractions are $\frac{2}{x+2}$ and $\frac{7h}{h(h-3)}$.

Equations that contain one or more algebraic fractions are known as **fractional equations**. We have learnt how to solve fractional equations such as $\frac{6}{2b-5} - \frac{4}{b-3} = 0$.

We will now learn how to solve fractional equations that can be reduced to quadratic equations, such as $\frac{2}{x+2} = 5x - 1$ and $\frac{3}{x+2} + \frac{x-1}{5-x} = 2$.

Worked Example 7

Solving fractional equation reducible to quadratic equation

Solve the equation $\frac{2}{x+2} = 5x - 1$.

Solution

$$\frac{2}{x+2} = 5x - 1$$

$$\frac{2}{x+2} \times (x+2) = (5x-1) \times (x+2) \quad \text{multiply both sides by } (x+2)$$

$$2 = (5x-1)(x+2)$$
$$= 5x^2 + 9x - 2$$
$$0 = 5x^2 + 9x - 4$$

$$5x^2 + 9x - 4 = 0 \quad \text{if } h = k, \text{ then } k = h$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-9 \pm \sqrt{9^2 - 4(5)(-4)}}{2(5)} \quad a = 5,\ b = 9,\ c = -4$$

$$= \frac{-9 \pm \sqrt{161}}{10}$$

$$= 0.369 \text{ or } -2.17 \text{ (to 3 s.f.)}$$

Big Idea

Equivalence
There is no known procedure to solve a fractional equation directly, so expressing it as a quadratic equation (when it is possible to do so) can help us solve it. Thus we see the value of expressing an equation in a suitable **equivalent form** to aid us in finding the solutions.

Practise Now 7

Similar and Further Questions
Exercise 3B
Questions 3(a)–(f), 5(a), (b), 6

Solve each of the following equations.

(a) $\frac{4}{p} = 2p - 3$

(b) $\frac{2q}{10-3q} = 10 - 3q$

(c) $\frac{6}{x+4} = x + 3$

(d) $\frac{3}{12-y} = 3y - 1$

Worked Example 8

Solving more complicated fractional equation reducible to quadratic equation

Solve the equation $\frac{3}{x+2} + \frac{x-1}{5-x} = 2$.

*Solution

$$\frac{3}{x+2} + \frac{x-1}{5-x} = 2$$

$$\left(\frac{3}{x+2} + \frac{x-1}{5-x}\right) \times (x+2)(5-x) = 2 \times (x+2)(5-x)$$

multiply both sides by LCM of the two denominators: $(x + 2)(5 - x)$

$$3(5 - x) + (x - 1)(x + 2) = 2(-x^2 + 3x + 10)$$

$$15 - 3x + (x^2 + x - 2) = -2x^2 + 6x + 20$$

$$x^2 - 2x + 13 = -2x^2 + 6x + 20$$

$$3x^2 - 8x - 7 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-7)}}{2(3)}$$

$a = 3$, $b = -8$, $c = -7$

$$= \frac{8 \pm \sqrt{148}}{6}$$

$$= 3.36 \text{ or } -0.694 \text{ (to 3 s.f.)}$$

Practise Now 8

Similar and Further Questions
Exercise 3B
Questions 7(a)–(h), 8(a)–(d)

Solve each of the following equations.

(a) $\frac{1}{x+6} + \frac{2}{3-x} = 5$

(b) $\frac{5}{y-3} + \frac{y-1}{y-2} = 7$

(c) $\frac{3}{n-2} - \frac{1}{(n-2)^2} = -2$

(d) $\frac{6}{1-2t} + \frac{3t}{1-4t^2} = 1$

Exercise 3B

1. Solve each of the following equations.

(a) $x^2 + 4x + 1 = 0$ (b) $3x^2 + 6x - 1 = 0$
(c) $3x^2 - 5x - 17 = 0$ (d) $-3x^2 - 7x + 9 = 0$
(e) $2 + 2x^2 - 7x = 0$ (f) $10x - 5x^2 - 2 = 0$

2. Solve each of the following equations.

(a) $x^2 + 5x = 21$ (b) $10x^2 - 12x = 15$
(c) $8x^2 = 3x + 6$ (d) $4x^2 + 1 = -4x$
(e) $9 - 5x^2 = -3x$ (f) $16x - 61 = x^2$

3. Solve each of the following equations.

(a) $\frac{8}{x} = 2x + 1$ (b) $x + \frac{7}{x} = 9$
(c) $\frac{x+1}{5-x} = x$ (d) $3x - 1 = -\frac{9}{3x+5}$
(e) $2x + 1 = \frac{x+1}{x-5}$ (f) $\frac{5x}{x+4} = 4x + 1$

4. Solve each of the following equations.

(a) $x(x + 1) = 1$
(b) $3(x + 1)(x - 1) = 7x$
(c) $(x - 1)^2 - 2x = 0$
(d) $x(x - 5) = 7 - 2x$
(e) $(5x - 9)(x - 1) - x(x - 2) = 0$
(f) $(4x - 3)^2 + (4x + 3)^2 = 25$

5. Solve each of the following equations.

(a) $\frac{x-1}{x+1} = \frac{8x}{1-x}$
(b) $\frac{(x-2)(x-3)}{(x-1)(x+2)} = \frac{2}{3}$

6. Find the value(s) of x that satisfy the equation $\frac{x(x-3)}{(x+1)^2} = \frac{3}{5}$.

7. Solve each of the following equations.

(a) $\frac{x}{2} = \frac{4}{x} - 1$ (b) $\frac{2}{x+5} = 1 - \frac{x+1}{5}$
(c) $\frac{x-2}{5} + \frac{1}{2x-3} = 2$ (d) $\frac{4}{x} - \frac{1}{x-1} = 9$
(e) $\frac{1}{x+2} + \frac{1}{x-2} = \frac{3}{11}$ (f) $\frac{7}{x-1} - \frac{x+1}{x+3} = \frac{1}{2}$
(g) $\frac{5}{x-2} = 2 - \frac{4}{(x-2)^2}$ (h) $\frac{5}{x-1} + \frac{x}{(x-1)^2} = 1$

8. Solve each of the following equations.

(a) $\frac{1}{x} + \frac{2}{x-1} + \frac{3}{x+1} = 0$
(b) $\frac{1}{x^2-9} - \frac{2}{3-x} = 1$
(c) $\frac{3}{x-3} + \frac{x+1}{x^2-5x+6} = 1$
(d) $\frac{4}{x-1} + 2 = \frac{x+2}{2x^2+3x-5}$

3.4 Solving quadratic equations by graphical method

In Sections 3.1 and 3.2, we have learnt how to solve quadratic equations by completing the square and using the quadratic formula. We can also solve them using a graphical method.

Consider the quadratic equation $2x^2 - 5x - 6 = 0$.

To solve it, we can plot the graph of the corresponding quadratic function $y = 2x^2 - 5x - 6$ on graph paper (we will do this in Worked Example 9 later; the sketch in Fig. 3.3 will help you visualise the process).

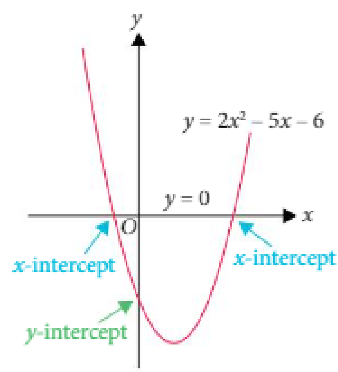


Fig. 3.3

To solve $2x^2 - 5x - 6 = 0$ means we want the y-values of $y = 2x^2 - 5x - 6$ to be equal to zero.

Since the equation of the x-axis is $y = 0$, this means that the y-values of $y = 2x^2 - 5x - 6$ are equal to zero when the graph of $y = 2x^2 - 5x - 6$ cuts the x-axis.

Therefore, the solutions of $2x^2 - 5x - 6 = 0$ will be the x-coordinates of the points of intersection of the graph of $y = 2x^2 - 5x - 6$ and the x-axis. We have learnt in Chapter 2 that these x-coordinates are called the *x-intercepts* (similar to what the y-intercept is).

Worked Example 9 shows you how to do this.

Attention

The solutions of an equation are also called its **roots**.

Worked Example 9

Solving quadratic equation by graphical method

The variables x and y are connected by the equation $y = 2x^2 - 5x - 6$.

(i) Complete the table for $y = 2x^2 - 5x - 6$.

x	-2	-1	0	1	2	3	4
y							

(ii) Draw the graph of $y = 2x^2 - 5x - 6$ for $-2 \leqslant x \leqslant 4$.

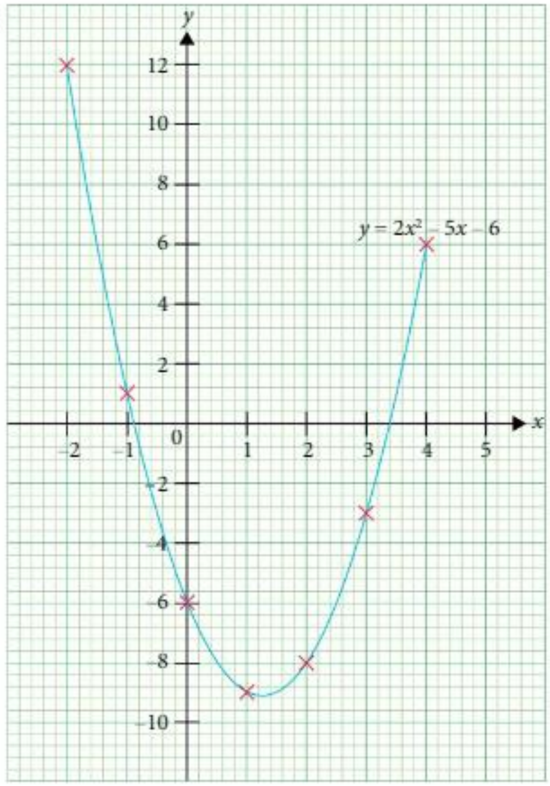
(iii) Hence, solve the equation $2x^2 - 5x - 6 = 0$.

Solution

(i)

x	-2	-1	0	1	2	3	4
y	12	1	-6	-9	-8	-3	6

(ii)



(iii) From the graph, $x = -0.9$ or 3.4.

Problem-solving Tip

(iii) The solutions of $2x^2 - 5x - 6 = 0$ are the x-intercepts. Since we can only read up to half of the smallest square of the graph grid, the solutions are accurate to the nearest 0.1 in this worked example.

Practise Now 9

Similar and Further Questions
Exercise 3C
Questions 1, 2, 4, 5

1. The variables x and y are connected by the equation $y = 2x^2 - 4x - 1$.
   (i) Complete the table for $y = 2x^2 - 4x - 1$.

x	-2	-1	0	1	2	3	4
y							

   (ii) Draw the graph of $y = 2x^2 - 4x - 1$ for $-2 \leqslant x \leqslant 4$.
   (iii) Hence, solve the equation $2x^2 - 4x - 1 = 0$.

2. By drawing the graph of $y = 7 - 4x - 3x^2$ for $-3 \leqslant x \leqslant 2$, solve the equation $7 - 4x - 3x^2 = 0$ graphically.

Worked Example 10

Solving quadratic equation by graphical method

The variables x and y are connected by the equation $y = x^2 - 4x + 4$.

(i) Complete the table for $y = x^2 - 4x + 4$.

x	-1	0	1	2	3	4	5
y							

(ii) Draw the graph of $y = x^2 - 4x + 4$ for $-1 \leqslant x \leqslant 5$.

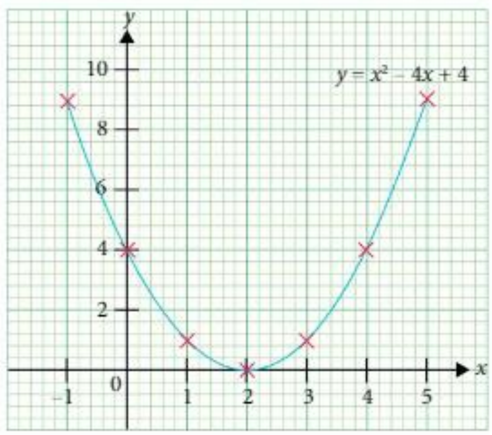
(iii) Hence, solve the equation $x^2 - 4x + 4 = 0$.

*Solution

(i)

x	-1	0	1	2	3	4	5
y	9	4	1	0	1	4	9

(ii)



(iii) From the graph, $x = 2$.

Reflection

If we solve $x^2 - 4x + 4 = 0$ by factorising the left-hand side, will we obtain the same solution?

Practise Now 10

Similar and Further Questions
Exercise 3C
Questions 3, 6, 7

1. The variables x and y are connected by the equation $y = x^2 - 6x + 9$.

   (i) Complete the table for $y = x^2 - 6x + 9$.

x	-1	0	1	2	3	4	5	6
y								

   (ii) Draw the graph of $y = x^2 - 6x + 9$ for $-1 \leqslant x \leqslant 6$.

   (iii) Hence, solve the equation $x^2 - 6x + 9 = 0$.

2. By drawing the graph of $y = 8x - x^2 - 16$ for $0 \leqslant x \leqslant 8$, solve the equation $8x - x^2 - 16 = 0$ graphically.

Thinking Time

Draw the graph of $y = 2x^2 + 4x + 3$ for $-2 \leqslant x \leqslant 4$.

1. **(a)** State the number of points of intersection between the graph and the x-axis.

   (b) How many real solutions are there to the equation $2x^2 + 4x + 3 = 0$? Explain.

2. We have learnt that a quadratic equation can have *two, one or no real solutions* (see Class Discussion on page 66). How do you interpret the number of real solutions of a quadratic equation graphically?

Journal Writing

We have learnt four methods to solve a quadratic equation:

(a) factorisation, (b) completing the square,
(c) quadratic formula, (d) graphical method.

Write down the advantages and disadvantages of using each method.
When solving a quadratic equation, how would you choose which method to use?

Advanced | Intermediate | Basic

Exercise 3C

1. The variables x and y are connected by the equation $y = 2x^2 - 5x + 1$.

(i) Complete the table for $y = 2x^2 - 5x + 1$.

x	-1	0	1	2	3	4
y						

(ii) Draw the graph of $y = 2x^2 - 5x + 1$ for $-1 \leqslant x \leqslant 4$.

(iii) Hence, solve the equation $2x^2 - 5x + 1 = 0$.

2. The variables x and y are connected by the equation $y = 7 - 5x - 3x^2$.

(i) Complete the table for $y = 7 - 5x - 3x^2$.

x	-3	-2	-1	0	1	2
y						

(ii) Draw the graph of $y = 7 - 5x - 3x^2$ for $-3 \leqslant x \leqslant 2$.

(iii) Hence, solve the equation $7 - 5x - 3x^2 = 0$.

3. The variables x and y are connected by the equation $y = x^2 + 6x + 9$.

(i) Complete the table for $y = x^2 + 6x + 9$.

x	-5	-4	-3	-2	-1	0
y						

(ii) Draw the graph of $y = x^2 + 6x + 9$ for $-5 \leqslant x \leqslant 0$.

(iii) Hence, solve the equation $x^2 + 6x + 9 = 0$.

4. **(i)** Draw the graph of $y = 3x^2 + 4x - 5$ for $-3 \leqslant x \leqslant 2$.

(ii) Hence, solve the equation $3x^2 + 4x - 5 = 0$ graphically.

5. By drawing the graph of $y = 5 - 2x - x^2$ for $-4 \leqslant x \leqslant 2$, solve the equation $5 - 2x - x^2 = 0$ graphically.

6. **(i)** Draw the graph of $y = 4x^2 + 12x + 9$ for $-4 \leqslant x \leqslant 2$.

(ii) Hence, solve the equation $4x^2 + 12x + 9 = 0$ graphically.

7. By drawing the graph of $y = 10x - 25 - x^2$ for $0 \leqslant x \leqslant 10$, solve the equation $10x - 25 - x^2 = 0$ graphically.

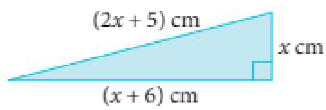
3.5 Applications of quadratic equations and functions in real-world contexts

Some problems in the real world can be solved by formulating a quadratic equation and solving it, as shown in Worked Examples 11 and 12 below.
Others can be **modelled** using a quadratic function, as shown in Worked Example 13.

Worked Example 11

Solving problems in real-world contexts using quadratic equation

The owner of a piece of land wishes to build a fence along its perimeter. On a map with a scale of 1 : 2000, the piece of land is in the shape of a right-angled triangle with sides of length x cm, $(x + 6)$ cm and $(2x + 5)$ cm.



(i) Write down an equation to represent this information and show that it simplifies to $2x^2 + 8x - 11 = 0$.
(ii) Solve the equation $2x^2 + 8x - 11 = 0$.
(iii) Explain why one of the solutions in part **(ii)** must be rejected as the length of one side of the triangle.
(iv) Hence, find the actual length, in metres, of the fence needed.

Solution

(i)

$$(2x + 5)^2 = x^2 + (x + 6)^2 \quad \text{Pythagoras' Theorem}$$
$$4x^2 + 20x + 25 = x^2 + x^2 + 12x + 36$$
$$2x^2 + 8x - 11 = 0 \text{ (shown)}$$

(ii)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-8 \pm \sqrt{8^2 - 4(2)(-11)}}{2(2)} \qquad a = 2,\ b = 8,\ c = -11$$
$$= \frac{-8 \pm \sqrt{152}}{4}$$
$$= 1.08 \text{ or } -5.08 \text{ (to 3 s.f.)}$$

(iii) The sides of the triangle must be more than 0 cm, hence $x > 0$ and -5.08 must be rejected.

(iv)

$$\text{Perimeter of triangle} = x + (x + 6) + (2x + 5)$$
$$= 4x + 11$$
$$= 4\left(\frac{-8 + \sqrt{152}}{4}\right) + 11$$
$$= 15.329 \text{ cm (to 5 s.f.)}$$

$$\text{Actual length of fence needed} = 15.329 \text{ cm} \times 2000$$
$$= 30\,658 \text{ cm}$$
$$= 307 \text{ m (to 3 s.f.)}$$

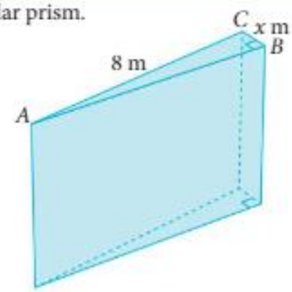
Problem-solving Tip

To find the perimeter, substitute the exact value of x or a value with a degree of accuracy higher than 3 s.f.

Practise Now 11

Similar and Further Questions
Exercise 3D
Questions 1, 5

The diagram shows a structure in the shape of a triangular prism.
The base, triangle ABC, has dimensions as shown.
It is given that AB is longer than BC.



(i) The perimeter of triangle ABC is 17 m. Write down an expression, in terms of x, for the length of AB.

(ii) Hence, formulate an equation in x and show that it simplifies to $2x^2 - 18x + 17 = 0$.

(iii) Solve the equation $2x^2 - 18x + 17 = 0$, giving both answers correct to 3 decimal places.

(iv) Why do we need to reject one answer in part **(iii)**?

(v) Given that the height of the structure is 7 m, find the volume of the structure.

Worked Example 12

Solving problems in real-world contexts using fractional equation reducible to quadratic equation

A family travels 315 km from Bandhi to Karachi. The average speed of a car is 18 km/h more than the speed of a coach. Let the average speed of the coach be x km/h.

(i) Write down an expression, in terms of x, for the number of hours the family would take to reach Karachi if they chose to travel by car.

If they chose to travel by car instead of by coach, they would reach Karachi 1 hour and 15 minutes earlier.

(ii) Write down an equation to represent this information and show that it simplifies to $x^2 + 18x - 4536 = 0$.

(iii) Solve the equation $x^2 + 18x - 4536 = 0$, giving both your answers correct to 3 decimal places.

(iv) Find the time taken by the family to travel by car, giving your answer correct to the nearest minute.

*Solution

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

Stage 1: Understand the problem

What information is given and what information is not given?

- What is given:
  - Distance between Bandhi and Karachi
  - Difference in average speeds of car and coach
  - Difference in time taken to travel by car and by coach
- What is not given:
  - Average speed of car and of coach
  - Time taken to travel by car and by coach

What are we supposed to find?

- Time taken to travel by car

What assumptions do we have to make?

For such questions, we have to assume:

- Distance travelled by car = distance travelled by coach = distance between Bandhi and Karachi

Stage 2: Think of a plan

What do we know about the relationship between speed, distance and time that could help us formulate an equation?

Stage 3: Carry out the plan

(i) Speed of car = $(x + 18)$ km/h

Time taken if travelling by car $= \dfrac{\text{distance}}{\text{speed}}$

$$= \frac{315}{x+18} \text{ h}$$

(ii) Time taken if travelling by coach $= \dfrac{315}{x}$ h

$$1 \text{ h } 15 \text{ min} = \frac{1\times60+15}{60} \text{ h} = \frac{5}{4} \text{ h}$$

$$\therefore \frac{315}{x} - \frac{315}{x+18} = \frac{5}{4}$$

$$\left(\frac{315}{x} - \frac{315}{x+18}\right) \times 4x(x+18) = \frac{5}{4} \times 4x(x+18)$$

multiply both sides by LCM of the three denominators: $4x(x + 18)$

$$1260(x + 18) - 1260x = 5x(x + 18)$$

$$22\,680 = 5x^2 + 90x$$

$$5x^2 + 90x - 22\,680 = 0$$

$$x^2 + 18x - 4536 = 0 \text{ (shown)}$$

(iii) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-18 \pm \sqrt{18^2 - 4(1)(-4536)}}{2(1)}$$

$a = 1$, $b = 18$, $c = -4536$

$$= \frac{-18 \pm \sqrt{18\,468}}{2}$$

$= 58.949$ or -76.949 (to 3 d.p.)

(iv) Time taken if travelling by car $= \dfrac{315}{x+18}$

$= \dfrac{315}{58.949+18}$ (since speed $x > 0$)

$= 4.0936$ h (to 5 s.f.)

$= 4.0936 \times 60$ min

$= 246$ min (to nearest minute)

Problem-solving Tip

Since the times taken are in hours, there is a need to convert 1 h 15 min to hours.

Stage 4: Look back

How can we check that the answer is correct besides verifying the calculations in the above working step by step?

Can we substitute $x = 58.949$ into $\dfrac{315}{x}$ to find the travel time by coach in hours, and then calculate the difference in time taken to travel by car and by coach, in hours, to see if we obtain 1.25 hours (i.e. 1 h 15 min)?

Practise Now 12

Similar and Further Questions
Exercise 3D
Questions 2, 3, 6–10

Bernard drove 600 km from City P to City Q. The average speed of his return journey was 7 km/h faster and the time taken was 15 minutes less.

(i) If he drove at an average speed of x km/h on his journey from City P to City Q, write down an equation to represent this information and show that it simplifies to $x^2 + 7x - 16\,800 = 0$.

(ii) Solve the equation $x^2 + 7x - 16\,800 = 0$, giving both your answers correct to 2 decimal places.

(iii) Find the time taken for the return journey, in hours and minutes, giving your answer correct to the nearest minute.

Worked Example 13

Modelling problems in real-world contexts by quadratic function

Imran threw a ball into the air. The path of the ball could be modelled by the equation

$$h = -t^2 + 4t + 1,$$

where t, in seconds, is the time from the moment the ball was thrown, and h, in metres, is the height of the ball above the ground. Find the difference in height between the ball at its highest point and the point from which it was thrown.

Solution

When $t = 0$, $h = -t^2 + 4t + 1$
$= 1$ h-intercept

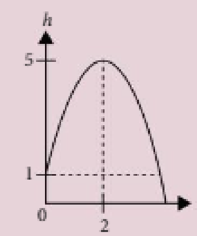
$$\begin{aligned} h &= -t^2 + 4t + 1 \\ &= -(t^2 - 4t) + 1 \\ &= -(t^2 - 4t + 2^2 - 2^2) + 1 \\ &= -(t - 2)^2 + 4 + 1 \\ &= -(t - 2)^2 + 5 \end{aligned}$$

$\therefore$ h-coordinate of maximum point = 5

$\therefore$ difference in height $= 5 - 1$
$= 4$ m

Problem-solving Tip

A sketch is useful to visualise that the difference in height between the ball at its highest point and the point from which it was thrown, is equal to the difference between the h-coordinate of the maximum point and the h-intercept, i.e. $5 - 1 = 4$.



Practise Now 13

Similar and Further Questions
Exercise 3D
Questions 4, 11

1. Vasi participates in an obstacle course. He has to jump off a platform to ring a bell that is 3 metres above the ground. The path of his jump can be modelled by the equation

$$y = -x^2 + 2x + 0.5,$$

where x, in metres, is the horizontal distance from the edge of the platform, and y, in metres, is the distance from his feet to the ground. Given that Vasi is 1.6 m tall, determine if he will be able to reach the bell if the platform is 1 metre away from the bell.

2. The profit, \$$P$ million, of a manufacturing company in its first 10 years of operation can be modelled by the equation $P = 2 - 0.1(x - 3)^2$, where x is the number of years of operation. The table shows some values of x and the corresponding values of P.

x	0	1	2	3	4	5	6	7	8	9	10
P	1.1	1.6	1.9	2	1.9	1.6	1.1	0.4	–0.5	–1.6	–2.9

(i) Using a scale of 1 cm to represent 1 year, draw a horizontal x-axis for $0 \leqslant x \leqslant 10$. Using a scale of 2 cm to represent \$1 million, draw a vertical P-axis for $-4 \leqslant P \leqslant 3$. On your axes, plot the points given in the table and join them with a smooth curve.

(ii) Use your graph to find the value of x when the profit of the company is zero.

Basic | Intermediate | Advanced

Exercise 3D

1. The perimeter of a rectangular poster is 112 cm and its breadth is x cm.

(i) Write down an expression, in terms of x, for the length of the rectangular poster.

(ii) The area of the poster is 597 cm². Write down an equation to represent this information and show that it simplifies to $x^2 - 56x + 597 = 0$.

(iii) Solve the equation $x^2 - 56x + 597 = 0$, giving both answers correct to 2 decimal places.

(iv) Identical posters are to be pasted along a wall 4 m in length. Find the maximum number of posters that can be pasted in one row on the wall, without any overlaps.

2. There are 2 printers in a library. Printer A prints 60 pages every x minutes.

(i) Write down an expression, in terms of x, for the number of pages printed by Printer A in 1 minute.

(ii) Printer B takes 2 minutes longer than Printer A to print 60 pages. Write down an expression, in terms of x, for the number of pages printed by Printer B in 1 minute.

When both printers are in use, they are able to print a total of 144 pages in 1 minute.

(iii) Write down an equation in x to represent this information and show that it simplifies to $6x^2 + 7x - 5 = 0$.

(iv) Solve the equation $6x^2 + 7x - 5 = 0$.

(v) Hence, find the time taken by Printer B to print 144 pages.

3. Rice from Brand A costs \$$x$ per kilogram. A food catering company spent \$600 on rice in January.

(i) Write down an expression, in terms of x, for the amount of rice that this food catering company ordered in January.

In February, the food catering company decides to buy rice from Brand B, which costs \$0.40 less per kilogram.

(ii) Given that the company still spent \$600 on rice in February, write down an expression, in terms of x, for the amount of rice ordered in February.

(iii) If the difference in the amount of rice ordered is 45 kg, write down an equation to represent this information and show that it simplifies to $15x^2 - 6x - 80 = 0$.

(iv) Hence, find the price of each kilogram of rice from Brand B.

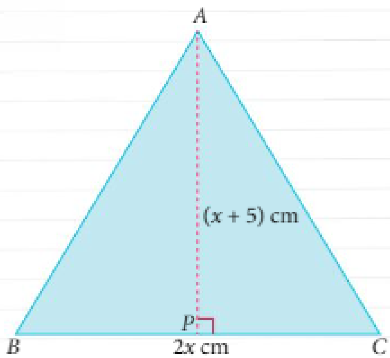
4. An object is launched from a platform. The path of the object can be modelled by the equation

$$h = -t^2 + 5t + 10,$$

where t, in seconds, is the time from the launch and h, in metres, is the height of the object from the ground. Find the difference in height between the object at its highest point and at the point from which it was thrown.

Exercise 3D

5. A model of a commemorative structure in the shape of a square-based pyramid has a scale of 1 : 100. The diagram shows one of the lateral faces of the model, where $AP = (x + 5)$ cm and $BC = 2x$ cm.



(i) Write down an expression, in terms of x, for the total surface area of the model.
(ii) Given that the total surface area of the model is 50 cm², formulate an equation in x and show that it reduces to $4x^2 + 10x - 25 = 0$.
(iii) Solve the equation $4x^2 + 10x - 25 = 0$.
(iv) Explain why one of the solutions in part **(iii)** must be rejected as the length of BC.
(v) Find the actual floor area, in m², that the commemorative structure occupies.

6. Waseem and Yasir started a 10-km hike at 8 a.m. They started walking at the same speed of x km/h. After 2 km, Waseem increased his speed by 1 km/h and walked the remaining distance at a constant speed of $(x + 1)$ km/h. Yasir maintained his speed of x km/h throughout the hike.
(i) Write down an expression, in terms of x, for the time taken by Waseem to complete the hike.
(ii) Waseem completed the hike 30 minutes earlier than Yasir. Write down an equation to represent this information and show that it simplifies to $x^2 + x - 16 = 0$.
(iii) Solve the equation $x^2 + x - 16 = 0$. Explain why one of the solutions must be rejected.
(iv) Hence, find the time when Waseem finished the hike.

7. Sara travels 700 km by coach from Islamabad to Bahawalnagar to visit her grandparents. She returns to Islamabad by car at an average speed which is 30 km/h greater than that of a coach.
(i) If the average speed of the car is x km/h and the time taken for the whole journey is 20 hours, formulate an equation in x and show that it reduces to $x^2 - 100x + 1050 = 0$.
(ii) Solve the equation $x^2 - 100x + 1050 = 0$, giving both your answers correct to 2 decimal places.
(iii) Find the time taken for the return journey.

8. A tank, when full, contains 1500 litres of water. Pump A can fill the tank with water at a rate of x litres per minute.
(i) Write down an expression, in terms of x, for the number of minutes taken by Pump A to fill the tank completely.

Pump B can fill the tank with water at a rate of $(x + 50)$ litres per minute.
(ii) Write down an expression, in terms of x, for the number of minutes taken by Pump B to fill the tank completely.
(iii) Pump A takes 30 seconds longer than Pump B to fill the tank completely. Write down an equation to represent this information and show that it simplifies to $x^2 + 50x - 150\,000 = 0$.
(iv) Solve the equation $x^2 + 50x - 150\,000 = 0$, giving both your answers correct to 2 decimal places.
(v) Find the time taken for Pump B to fill the tank completely, giving your answer in minutes and seconds, correct to the nearest second.

Basic | Intermediate | Advanced

Exercise 3D

9. Two weeks before Nadia went to New York for a holiday, she exchanged S\$2000 into US dollars (US\$) at XYZ Money Exchange at a rate of US\$1 = S\$$x$.
   (i) Write down an expression, in terms of x, for the amount of US\$ she received from XYZ Money Exchange.

   One week before her holiday, she exchanged another S\$1000 into US\$ at ABC Money Exchange at a rate of US\$1 = S\$$(x + 0.05)$.
   (ii) Write down an expression, in terms of x, for the amount of US\$ she received from ABC Money Exchange.
   (iii) If Nadia received a total of US\$2180 from the two money changers, formulate an equation in x and show that it reduces to $218x^2 - 289.1x - 10 = 0$.
   (iv) Solve the equation $218x^2 - 289.1x - 10 = 0$, giving both your answers correct to 4 decimal places.
   (v) Find the exchange rate between S\$ and US\$ offered by ABC Money Exchange.

10. During a test flight, an aircraft flies from Sandy Land to White City and back to Sandy Land. The distance between Sandy Land and White City is 450 km and the total time taken for the whole journey is 5 hours and 30 minutes. Given that there is a constant wind blowing from Sandy Land to White City and that the speed of the aircraft in still air is 165 km/h, find the speed of the wind. State the assumptions you have made to solve this problem.
    Hint: Let the speed of the wind be x km/h.

11. In a carnival game, participants throw a balloon filled with water from the top of a platform onto a sandpit. Points are allocated based on the horizontal distance from the foot of the platform to where the balloon lands.
    Cheryl throws a balloon. During the flight, its height above ground level, y cm, is represented by the equation $y = 200 + 7x - 6x^2$, where x is the horizontal distance, in metres, from the foot of the platform. The table shows some values of x and the corresponding values of y.

x	0	1	2	3	4	5	6
y	200	201	190	167	132	85	26

   (i) Using a scale of 1 cm to represent 1 m, draw a horizontal x-axis for $0 \leqslant x \leqslant 7$.
   Using a scale of 2 cm to represent 50 cm, draw a vertical y-axis for $0 \leqslant y \leqslant 250$.
   On your axes, plot the points given in the table and join them with a smooth curve.
   (ii) Use your graph to find
      (a) the maximum height the balloon reaches,
      (b) the horizontal distance from the foot of the platform to the balloon when it is 50 cm above the ground.
   (iii) Given that the flight of the balloon above ground level can only be modelled by the equation $y = 200 + 7x - 6x^2$ for $0 \leqslant x \leqslant t$, state the value of t. Explain your answer.

Looking Back

In this chapter, we extended our knowledge of quadratic equations by learning how to solve quadratic equations whose corresponding quadratic expressions cannot be easily factorised. The key is to change the quadratic equation into an **equivalent** form whose solution can be more easily obtained. For example, in the **Introductory Problem**, the quadratic expression in the equation $x^2 + 4x - 3 = 0$ cannot be easily factorised into two linear factors. However, the equation is equivalent to $(x + 2)^2 = 7$, which can be solved easily by taking square roots. The same concept applies to solving fractional equations.

Although there is no known procedure to solve a fractional equation directly, if the fractional equation can be rewritten in terms of an equivalent quadratic equation, we would be able to solve it! We see that the "trick" of changing the form of an equation into another, which is easier to solve, allows us to solve a great variety of equations. This is the foundation of many of our solution methods for solving equations.

Furthermore, the idea of equivalence also enables us to express quadratic functions in different ways, so as to obtain different information that we can use to help us sketch the graph of the function. The ability to change the equation of a function to its other equivalent forms is an important skill that will enable us to solve other kinds of equations and perhaps, rediscover ancient methods of solving such equations!

Summary

1. To **complete the square** for $x^2 + bx$, we add the number $\left(\frac{b}{2}\right)^2$, so that the completed square is $\left(x+\frac{b}{2}\right)^2$.

Therefore, $x^2 + bx = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

$$= \left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

2. A quadratic equation of the form $ax^2 + bx + c = 0$ can be solved by

(a) factorisation,

(b) completing the square,

(c) using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

(d) using a graphical method.

- Which of the above methods cannot be used to solve all quadratic equations?
- Which of the above methods do you prefer? Explain.

>

Summary

3. Equations that contain one or more algebraic fractions are known as **fractional equations**.
To solve a fractional equation that can be reduced to a quadratic equation:
   - Multiply both sides of the equation by the LCM of the denominators.
   - Reduce the equation to a quadratic equation.
   - Solve the quadratic equation, usually by factorisation or using the quadratic formula.

CHAPTER 4

Indices, Surds, Exponential Growth and Decay, and Standard Form

- According to the United Nations, the world population reached 8 000 000 000 in mid-November 2022. The world population is projected to exceed 9 700 000 000 by 2050.
- The distance travelled by light in a year, also known as a light year, is exactly 9 460 730 472 580 800 metres. The nearest known star (other than the Sun), Proxima Centauri, is about 4.2 light years away!
- The Bohr radius of a hydrogen atom is estimated to be about 0.000 000 000 053 m.

These numbers remind us of both the vastness and tininess present in this universe. However, writing and working with such big and small numbers can be very tedious. With the invention of index **notation** to represent these numbers, we can now use and work with these extreme numbers in a more concise yet precise manner. In this chapter, we will learn how to apply the index notation to express these extreme numbers using standard form, as well as to solve problems involving exponential growth and decay.

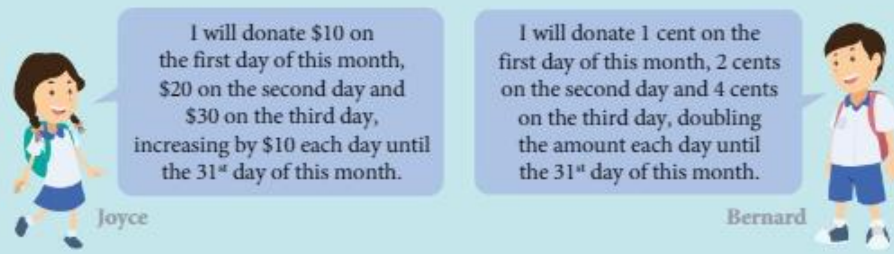
Learning Outcomes

What will we learn in this chapter?

- What the five Laws of Indices are
- How to apply the five Laws of Indices
- How to state and use the definitions of zero, negative and rational indices
- How to simplify expressions involving surds
- How to use conjugate surds to rationalise the denominator of an expression containing a surd
- How to represent very large or very small numbers using standard form
- Why indices, exponential growth and decay, and standard form have useful applications in real life

Introductory Problem

Joyce and Bernard want to donate money to a charity.



Will Joyce or Bernard donate more money in total by the end of the month?

In this chapter, we will learn more about indices and how to operate them.

4.1 Indices

We have learnt how to represent $5 \times 5 \times 5 \times 5$ as 5^4 (read as '5 to the power of 4'), where 5 is called the base and 4 is called the index (plural: **indices**). 5^4 is called the *index notation* (or index form) of $5 \times 5 \times 5 \times 5$.

5^4 ← index

↑

base

Write $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$ in index notation:

Big Idea

Notations

The index notation is used to represent a number multiplied by itself n times in a *precise and concise manner*.

Information

Numbers with a base of '5' are called powers of 5. Examples of powers of 5 are 5^1, 5^2, 5^3 and 5^4.

In the **Introductory Problem**, to find the amount Bernard donated on the 31st day of the month, we need to find the value of 2 multiplied by itself 30 times.

This is very tedious to write using the multiplication notation. Instead, the index notation 2^{30} could be used. In addition, although Bernard's donation may seem to start off as a small amount, 2^{30} cents is actually equal to 1 073 741 824 cents or about $10.7 million. This shows how 'powerful' the index 30 is!

4.2 Laws of Indices

In this section, we will learn five Laws of Indices to help us perform some operations on numbers written in index notations.

A. Law 1 of Indices

Investigation: Discovering Law 1 of Indices

Copy and complete the following.

1. $7^2 \times 7^4 = \underbrace{(7 \times 7)}_{2\text{ factors}} \times \underbrace{(\boxed{\qquad\qquad})}_{\boxed{\quad}\text{ factors}}$

$= \underbrace{7 \times 7 \times \ldots \times 7}_{6\text{ factors}}$

$= 7^{\boxed{\quad}}$

$= 7^{2+\boxed{\quad}}$

2. $(-6)^5 \times (-6)^3 = \underbrace{[\boxed{\qquad\qquad\qquad}]}_{\boxed{\quad}\text{ factors}} \times \underbrace{[\boxed{\qquad\qquad}]}_{\boxed{\quad}\text{ factors}}$

$= \underbrace{(-6) \times (-6) \times \ldots \times (-6)}_{\boxed{\quad}\text{ factors}}$

$= (-6)^{\boxed{\quad}}$

$= (-6)^{5+\boxed{\quad}}$

3. $a^3 \times a^4 = \underbrace{(a \times a \times a)}_{3\text{ factors}} \times \underbrace{(\boxed{\qquad\qquad})}_{\boxed{\quad}\text{ factors}}$

$= \underbrace{a \times a \times \ldots \times a}_{\boxed{\quad}\text{ factors}}$

$= a^{\boxed{\quad}}$

$= a^{3+\boxed{\quad}}$

4. $a^m \times a^n = \underbrace{(a \times a \times a \times \ldots \times a \times a)}_{m\text{ factors}} \times \underbrace{(a \times a \times \ldots \times a \times a)}_{\boxed{\quad}\text{ factors}}$

$= \underbrace{a \times a \times \ldots \times a}_{(m+\boxed{\quad})\text{ factors}}$

$= a^{m+\boxed{\quad}}$

Just For Fun

10^{100} is called a googol, and $10^{10^{100}}$, i.e. $10^{(10^{100})}$, is called a googolplex. Thus a googolplex has one googol (or 10^{100}) zeros. If one newspaper page can print 30 000 digits, we will need at least 10^{95} newspaper pages to print out all the zeros of a googolplex. But this is not possible because this is greater than the observable universe, which is estimated to contain about 10^{80} atoms!
If you think a googolplex is a very large number, search the Internet for 'Graham's number', which is so much larger than the googolplex that a new notation has to be used to represent it.

From the above Investigation, we observe the following:

Law 1 of Indices:

$$a^m \times a^n = a^{m+n}$$

where base a is a real number, and indices m and n are positive integers.

Attention

In order to apply Law 1, the *bases* of the two factors (i.e. a^m and a^n) must be the *same*.

Worked Example 1

Applying Law 1 of Indices

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $5^8 \times 5^9$ **(b)** $(-2)^3 \times (-2)$ **(c)** $6b^3c^5 \times 2bc^4$

*Solution

(a) $5^8 \times 5^9 = 5^{8+9}$ Law 1 of Indices
$= 5^{17}$

(b) $(-2)^3 \times (-2) = (-2)^{3+1}$ Law 1 of Indices
$= (-2)^4$

(c) $6b^3c^5 \times 2bc^4 = 12b^{3+1}c^{5+4}$ Law 1 of Indices
$= 12b^4c^9$

Problem-solving Tip

(b) You can also leave your answer as 2^4 instead of $(-2)^4$. Why? On the other hand, is $(-2)^4 = -2^4$? Explain.

(c) Since multiplication is commutative, we can rearrange the order of the factors:
$6b^3c^5 \times 2bc^4$
$= 6 \times 2 \times b^3 \times b \times c^5 \times c^4$

Practise Now 1

Similar and Further Questions
Exercise 4A
Questions 1(a)–(d), 6(a)–(c)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $7^2 \times 7^5$ **(b)** $(-3)^5 \times (-3)$

(c) $a^{12} \times a^8$ **(d)** $2xy^4 \times 3x^5y^3$

Big Idea

Notations

In Worked Example 1(**c**), bc^4 means $b \times c^4$, i.e. the index 4 is for c only. If we want the index 4 for both b and c, then we have to write it as $(bc)^4$ (see Law 4 later for more details). Understanding the convention of such notations helps us interpret what is given correctly.

B. Law 2 of Indices

Investigation Discovering Law 2 of Indices

Copy and complete the following.

1. $$5^7 \div 5^3 = \frac{\overbrace{5\times5\times5\times5\times5\times5\times5}^{7\text{ factors}}}{\underbrace{\square}_{\square\text{ factors}}}$$
$= 5^{\square}$
$= 5^{7-\square}$

2. $$\frac{(-10)^5}{(-10)^2} = \frac{\overbrace{(-10)\times(-10)\times(-10)\times(-10)\times(-10)}^{5\text{ factors}}}{\underbrace{\square}_{\square\text{ factors}}}$$
$= (-10)^{\square}$
$= (-10)^{5-\square}$

3. If $a \neq 0$, then $a^9 \div a^4 = \dfrac{\overbrace{a \times a \times a \times a \times a \times a \times a \times a \times a}^{9 \text{ factors}}}{\underbrace{\square}_{\square \text{ factors}}}$

$= a^{\square}$

$= a^{9 - \square}$

4. If $a \neq 0$ and $m > n$, then $a^m \div a^n = \dfrac{\overbrace{a \times a \times a \times \ldots \times a \times a}^{m \text{ factors}}}{\underbrace{a \times a \times \ldots \times a \times a}_{\square \text{ factors}}}$

$= \underbrace{a \times a \times \ldots \times a}_{(m - \square) \text{ factors}}$

$= a^{m - \square}$

5. What will happen to the result in Question 3 or 4 if $a = 0$?
6. What will happen to the result in Question 4 if $m \leqslant n$?

From the above Investigation, we observe the following:

Law 2 of Indices:

$$a^m \div a^n = a^{m-n} \quad \text{or} \quad \frac{a^m}{a^n} = a^{m-n}$$

where base a is a real number such that $a \neq 0$, and indices m and n are positive integers such that $m > n$.

Attention

In order to apply Law 2, the *bases* of the dividend and divisor (i.e. a^m and a^n respectively) must be the *same*.

Worked Example 2

Applying Law 2 of Indices

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $7^8 \div 7^3$ **(b)** $(-5)^6 \div (-5)$ **(c)** $12b^6c^3 \div 6b^4c^2$

Solution

(a) $7^8 \div 7^3 = 7^{8-3}$ Law 2 of Indices

$= 7^5$

(b) $(-5)^6 \div (-5) = (-5)^{6-1}$ Law 2 of Indices

$= (-5)^5$

(c) $12b^6c^3 \div 6b^4c^2 = \dfrac{12b^6c^3}{6b^4c^2}$

$= 2b^{6-4}c^{3-2}$ Law 2 of Indices

$= 2b^2c$

Problem-solving Tip

(b) You can also leave your answer as -5^5 instead of $(-5)^5$. Why? On the other hand, is $(-5)^4 = -5^4$? Explain.

Big Idea

Notations

We can interpret $12b^6c^3 \div 6b^4c^2$ as $12b^6c^3 \div (6b^4c^2)$, i.e. $\dfrac{12b^6c^3}{6b^4c^2}$.

Practise Now 2

Similar and Further Questions
Exercise 4A
Questions 2(a)–(d), 3(a), (b), 6(d)–(f)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $9^7 \div 9^3$ (b) $(-4)^8 \div (-4)$

(c) $a^{10} \div a^6$ (d) $\dfrac{27x^9y^4}{9x^6y^3}$

C. Law 3 of Indices

Investigation Discovering Law 3 of Indices

Copy and complete the following.

1. $(2^5)^2 = 2^5 \times 2^5$
$= 2^{5+\square}$ Law 1 of Indices
$= 2^{5\times\square}$

2. $[(-9)^4]^3 = (-9)^4 \times (-9)^4 \times (-9)^4$
$= (-9)^{4+\square+\square}$ Law 1 of Indices
$= (-9)^{4\times\square}$

3. $(a^m)^n = \underbrace{a^m \times a^m \times \ldots \times a^m}_{n \text{ factors}}$
$= a^{\overbrace{m+m+\ldots+m}^{\square \text{ factors}}}$ Law 1 of Indices
$= a^{m\times\square}$

Attention

$(2^5)^2$ means we evaluate 2^5 first, and then we square it.

From the above Investigation, we observe the following:

Law 3 of Indices:

$$(a^m)^n = a^{mn}$$

where base a is a real number, and indices m and n are positive integers.

Worked Example 3

Applying Laws 1 to 3 of Indices

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $(5^3)^9$ (b) $[(-h)^7]^4$ (c) $(7^p)^5 \times (7^2)^p \div (7^3)^p$

*Solution

(a) $(5^3)^9 = 5^{3\times9}$ Law 3 of Indices
$= 5^{27}$

(b) $[(-h)^7]^4 = (-h)^{7\times4}$ Law 3 of Indices
$= (-h)^{28}$

(c) $(7^p)^5 \times (7^2)^p \div (7^3)^p$
$= 7^{5p} \times 7^{2p} \div 7^{3p}$ Law 3 of Indices
$= 7^{4p}$ Laws 1 & 2 of Indices

Problem-solving Tip

(b) You can also leave your answer as h^{28}. Why?

Practise Now 3

Similar and Further Questions
Exercise 4A
Questions 4(a), (b), 7(a)–(c), 8

1. Simplify each of the following, leaving your answer in index notation where appropriate.
   (a) $(6^3)^4$ (b) $(k^5)^9$
   (c) $[(-4)^p]^3 \times [(-4)^5]^p$ (d) $\dfrac{(3^q)^6 \times (3^4)^q}{(3^3)^q}$
2. Given that $x^8 \times (x^3)^n \div (x^n)^2 = x^{10}$, find the value of n.

D. Law 4 of Indices

Investigation: Discovering Law 4 of Indices

Copy and complete the following.

1. $$\begin{aligned} 2^3 \times 7^3 &= \underbrace{(2 \times 2 \times 2)}_{3 \text{ factors}} \times \underbrace{(\square)}_{\square \text{ factors}} \\ &= \underbrace{(2 \times 7) \times (2 \times 7) \times (2 \times \square)}_{3 \text{ factors}} \\ &= (2 \times \square)^3 \end{aligned}$$

2. $$\begin{aligned} (-3)^2 \times (-4)^2 &= \underbrace{(-3) \times (-3)}_{2 \text{ factors}} \times \underbrace{(\square) \times (\square)}_{\square \text{ factors}} \\ &= \underbrace{[(-3) \times (-4)] \times [(-3) \times (\square)]}_{2 \text{ factors}} \\ &= [(-3) \times (\square)]^{\square} \end{aligned}$$

3. $$\begin{aligned} a^n \times b^n &= \underbrace{(a \times a \times \ldots \times a)}_{n \text{ factors}} \times \underbrace{(b \times b \times \ldots \times b)}_{\square \text{ factors}} \\ &= \underbrace{(a \times b) \times (a \times b) \times \ldots \times (a \times b)}_{\square \text{ factors}} \\ &= (a \times b)^{\square} \end{aligned}$$

From the above Investigation, we observe the following:

> **Law 4 of Indices:**
> $$a^n \times b^n = (a \times b)^n$$
> where base a and b are real numbers, and index n is a positive integer.

Another useful version of Law 4 of Indices can be written as $(a \times b)^n = a^n \times b^n$ or $(ab)^n = a^n b^n$ (see Worked Example 4 parts **(b)** and **(c)**).

Reflection

How is Law 4 of Indices different from Law 1?

Attention

In order to apply Law 4, the *indices* of the two factors (i.e. a^n and b^n) must be the *same*.

Big Idea

Notations
$(ab)^n = a^n \times b^n$ but $ab^n = a \times b^n$. Understanding the conventions of such notations helps us interpret what is given correctly.

Worked Example 4

Applying Laws 1 to 4 of Indices

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $2^4 \times 7^4$ **(b)** $(-2h^2)^5$ **(c)** $\dfrac{(xy^2)^3 \times (-3x^2y)^4}{(3x^3y^4)^2}$

Solution

(a) $2^4 \times 7^4 = (2 \times 7)^4$ Law 4 of Indices

$= 14^4$

(b) $(-2h^2)^5 = (-2)^5(h^2)^5$ Law 4 of Indices

$= -32h^{10}$ Law 3 of Indices

(c) $\dfrac{(xy^2)^3 \times (-3x^2y)^4}{(3x^3y^4)^2} = \dfrac{x^3y^6 \times (-3)^4x^8y^4}{9x^6y^8}$ Law 4, then Law 3, of Indices

$= \dfrac{81x^{11}y^{10}}{9x^6y^8}$ Law 1 of Indices

$= 9x^5y^2$ Law 2 of Indices

Practise Now 4

Similar and Further Questions
Exercise 4A
Questions 4(c)–(f), 7(d), (e), 9(a)–(d), 11(a)–(d), 12

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $3^7 \times 8^7$ **(b)** $(5b^4)^3$

(c) $(-2c^2d^5)^5$ **(d)** $\dfrac{(h^4k^2)^2 \times (-5hk^5)^3}{(5h^3k)^3}$

E. Law 5 of Indices

Investigation Discovering Law 5 of Indices

Copy and complete the following.

1. $8^3 \div 5^3 = \dfrac{8^3}{5^3}$

$= \dfrac{\overbrace{8 \times 8 \times 8}^{3 \text{ factors}}}{\underbrace{\square \times \square \times \square}_{\square \text{ factors}}}$

$= \underbrace{\dfrac{8}{\square} \times \dfrac{8}{\square} \times \dfrac{8}{\square}}_{3 \text{ factors}}$

$= \left(\dfrac{8}{\square}\right)^3$

2. $(-12)^4 \div (-7)^4 = \dfrac{(-12)^4}{(-7)^4}$

$$= \frac{\overbrace{(-12)\times(-12)\times(-12)\times(-12)}^{4 \text{ factors}}}{\underbrace{\square\times\square\times\square\times\square}_{\square \text{ factors}}}$$

$$= \underbrace{\frac{(-12)}{\square}\times\frac{(-12)}{\square}\times\frac{(-12)}{\square}\times\frac{(-12)}{\square}}_{4 \text{ factors}}$$

$$= \left[\frac{(-12)}{\square}\right]^{\square}$$

3. If $b \neq 0$, then $a^n \div b^n = \dfrac{\overbrace{a\times a\times \ldots \times a}^{n \text{ factors}}}{\underbrace{b\times b\times \ldots \times b}_{\square \text{ factors}}}$

$$= \underbrace{\frac{a}{\square}\times\frac{a}{\square}\times\ldots\times\frac{a}{\square}}_{\square \text{ factors}}$$

$$= \left(\frac{a}{\square}\right)^{\square}$$

From the above Investigation, we observe the following:

Law 5 of Indices:

$$a^n \div b^n = \left(\frac{a}{b}\right)^n$$

where bases a and b are real numbers such that $b \neq 0$, and index n is a positive integer.

Reflection

How is Law 5 of Indices different from Law 2?

Attention

In order to apply Law 5, the *indices* of the dividend and divisor (i.e. a^n and b^n respectively) must be the *same*.

Another useful version of Law 5 of Indices can be written as $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ (see Worked Example 5 parts **(b)** and **(c)**).

Worked Example 5

Applying Laws 1 to 5 of Indices

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $10^8 \div 2^8$ **(b)** $\left(\frac{2}{b}\right)^7$ **(c)** $\left(\frac{-3p^2}{q^3}\right)^4 \div \frac{27p^3}{pq^{16}}$

Solution

(a) $10^8 \div 2^8 = \left(\frac{10}{2}\right)^8$ Law 5 of Indices

$= 5^8$

(b) $\left(\frac{2}{b}\right)^7 = \frac{2^7}{b^7}$ Law 5 of Indices

(c) $\left(\frac{-3p^2}{q^3}\right)^4 \div \frac{27p^3}{pq^{16}} = \frac{3^4p^8}{q^{12}} \times \frac{pq^{16}}{3^3p^3}$ Law 5, then Law 3, of Indices

$= 3^{4-3}p^{8+1-3}q^{16-12}$ Laws 1 & 2 of Indices

$= 3p^6q^4$

Problem-solving Tip

(c) $(-3)^4 = 3^4$. Why?

Practise Now 5

Similar and Further Questions
Exercise 4A
Questions 5(a)–(f), 7(f), 10(a)–(d)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $21^3 \div 7^3$ (b) $\left(\frac{a}{4}\right)^3$

(c) $\left(\frac{p^2}{q}\right)^3 \div \frac{p^5}{q^7}$ (d) $\left(\frac{4x^2}{x^3}\right)^3 \div \frac{64x^7}{x^{21}}$

Reflection

1. How do I know when and which Laws of Indices to apply? (For example, to apply Law 1 of Indices, the bases of the two factors must be the same.)
2. What have I learnt in this section that I am still unclear of?

Basic | Intermediate | Advanced

Exercise 4A

1. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $2^3 \times 2^7$ (b) $(-4)^6 \times (-4)^5$

(c) $x^8 \times x^3$ (d) $(3y^2) \times (8y^7)$

2. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $5^8 \div 5^5$ (b) $(-7)^{11} \div (-7)^4$

(c) $6x^7 \div x^3$ (d) $(-15y^9) \div 5y^4$

3. (a) Express $\frac{3^3 \times 3^4}{3}$ in the form 3^n.

(b) Write $\frac{2^8 \times 2^3}{2^7}$ as a single power of 2.

4. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $(9^2)^4$ (b) $(h^2)^5$

(c) $2^3 \times 9^3$ (d) $3^{14} \times (5^2)^7$

(e) $(2k^6)^3$ (f) $(-3x^6y^2)^4$

5. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $14^{13} \div 7^{13}$ (b) $(9^5)^4 \div 3^{20}$

(c) $\left(\frac{m}{2}\right)^5$ (d) $\left(\frac{3}{n^2}\right)^3$

(e) $\left(\frac{p^4}{q}\right)^6$ (f) $\left(-\frac{x}{y^2}\right)^4$

Advanced | Intermediate | Basic

Exercise 4A

6. Simplify each of the following.

(a) $h^2k \times h^{11}k^9$ (b) $(-m^7n^3) \times 4m^{11}n^9$

(c) $11p^6q^7 \times 2p^3q^{10}$ (d) $h^9k^6 \div h^5k^4$

(e) $15m^8n^7 \div 3m^2n$ (f) $(-10x^5y^6) \div (-2xy^5)$

7. Simplify each of the following.

(a) $(a^2)^3 \times a^5$ (b) $(b^3)^7 \times (b^4)^5$

(c) $(c^6)^5 \div (-c^2)$ (d) $(-3d^3)^2 \div (2d)^3$

(e) $(e^3f^2)^5 \div (-e^2f)^4$ (f) $(4k^6)^3 \div (-2k^3)^3$

8. Given that $x^9 \times (x^3)^{2n} \div (-x^n)^2 = x^{13}$, find the value of n.

9. Simplify each of the following.

(a) $(ab^2)^3 \times (2a^2b)^3$ (b) $c^2d^2 \times (-5c^3d^3)^2$

(c) $(8e^5f^3)^2 \div (e^3f)^3$ (d) $16g^{10}h^7 \div (-2g^3h^2)^3$

10. Simplify each of the following.

(a) $\left(\frac{2a^2}{b}\right)\times\left(\frac{a}{b^2}\right)^2$ (b) $\left(\frac{c}{d^2}\right)^3\times\left(\frac{c^3}{2d}\right)^3$

(c) $\left(\frac{3e^3}{f^2}\right)^4\div\left(\frac{27e^9}{f^{11}}\right)$ (d) $\left(\frac{-3g^5}{2h^2}\right)^3\div\left(\frac{g^2}{h^3}\right)^3$

11. Simplify each of the following.

(a) $\frac{(2x^2y)^3}{(10xy^3)^2}\times\frac{(5xy^4)^3}{4xy}$ (b) $\frac{8x^8y^4}{(2xy^2)^2}\times\frac{(4x^2y^2)^2}{(3xy)^2}$

(c) $\frac{(2xy^2)^6}{(4x^2y)^2(xy^3)}$ (d) $\frac{4x^2y^4\times 8x^4y^2}{(4x^2y^2)^2}$

12. Given that $\frac{(2p^3q^4)^4}{(-3q^5)^2}\div\frac{(4p^2q)^2}{9}=\frac{p^{a+b}}{q^{a-b}}$, form a pair of simultaneous equations in a and b and hence find the value of a and of b.

4.3 Zero and negative indices

A. Zero indices

In Section 4.2, we have learnt about the five Laws of Indices, which apply when the indices are positive integers. But can an index be zero? For example, what does 3^0 mean?

Investigation — Making sense of the zero index

We know that 3^4 means 3 multiplied by itself 4 times. But 3^0 has to be interpreted in another way because it does not make sense to multiply 3 by itself 0 times.

1. Complete Table 4.1 (refer to the Problem-solving Tip).

Index form	Value	
3^4	81	÷ 3
3^3	27	÷ 3
3^2		÷ 3
3^1		÷ 3
3^0		

Table 4.1

Problem-solving Tip

- What number must you divide 81 (i.e. 3^4) by, to obtain 27 (i.e. 3^3)?
- What number must you divide 27 (i.e. 3^3) by, to obtain the value of 3^2?
- By continuing this pattern, what number must you divide 3^1 by, to obtain the value of 3^0?

2. Complete Table 4.2.

Index form	Value
$(-2)^4$	16
$(-2)^3$	–8
$(-2)^2$	
$(-2)^1$	
$(-2)^0$	

$\div(-2)$ $\div(-2)$ $\div(-2)$ $\div(-2)$

Table 4.2

3. Does the above pattern work for 0^4, 0^3, 0^2, 0^1 and 0^0? Explain your answer.
4. Complete the following:

$$\text{If } a \neq 0, \text{ then } a^n \div a^n = \frac{\overbrace{a \times a \times a \times \ldots \times a \times a}^{n \text{ factors}}}{\underbrace{a \times a \times a \times \ldots \times a \times a}_{\boxed{} \text{ factors}}}$$
$$= 1$$

But $a^n \div a^n = a^{n-\boxed{}}$ Law 2 of Indices
$= a^0$

Therefore, if $a \neq 0$, then $a^0 = \boxed{}$

The definition of a^n, where n is a positive integer, is a multiplied by itself n times. This does not make sense when $n = 0$. Therefore, we have to define a^0. The above Investigation leads us to the following definition:

Definition 1:

$$a^0 = 1$$

where base a is a real number such that $a \neq 0$.

Worked Example 6

Applying definition of zero index

Evaluate each of the following without using a calculator.

(a) 38^0 **(b)** $2x^0$
(c) $(2x)^0$ **(d)** $-7^0 \times (-7)^0 + 16^0$

*Solution

(a) $38^0 = 1$

(b) $2x^0 = 2 \times 1$
$= 2$

(c) $(2x)^0 = 1$

(d) $-7^0 \times (-7)^0 + 16^0$
$= -1 \times 1 + 1$
$= -1 + 1$
$= 0$

Reflection

What is the difference
- between $2x^0$ and $(2x)^0$,
- between -7^0 and $(-7)^0$?

Explain.

Practise Now 6

Similar and Further Questions
Exercise 4B
Questions 1(a)–(f), 2(a)–(d)

1. Evaluate each of the following without using a calculator.
   (a) 2022^0 **(b)** $(-8)^0$
   (c) $3y^0$ **(d)** $(3y)^0$
2. Find the value of each of the following without using a calculator.
   (a) $3^0 \times 3^3 \div 3^2$ **(b)** $3^0 + 3^2$

B. Negative indices

Can an index be negative? For example, what does 3^{-1} mean?

Investigation — Making sense of negative indices

Now that we know $3^0 = 1$, let us find out what 3^{-1} and 3^{-2} are equal to.

1. Continuing the same pattern in the Investigation on page 93, complete Table 4.3.

Index form	Value	
3^2	9	
3^1	3	$\div 3$
3^0		$\div 3$
3^{-1}		$\div 3$
3^{-2}		$\div 3$

Table 4.3

Problem-solving Tip

$$\frac{1}{3} \div 3 = \frac{1}{3} \times \frac{1}{3}$$
$$= \frac{1}{3 \times 3}$$
$$= \frac{1}{3^2}$$

2. Continuing the same pattern in the Investigation on page 94, complete Table 4.4.

Index form	Value	
$(-2)^2$	4	
$(-2)^1$	-2	$\div (-2)$
$(-2)^0$		$\div (-2)$
$(-2)^{-1}$		$\div (-2)$
$(-2)^{-2}$		$\div (-2)$

Table 4.4

3. What do you think 0^{-2} is equal to? Explain your answer.

4. Complete the following:

$$a^4 \div a^7 = \frac{\square}{a \times a \times a \times a \times a \times a \times a}$$
$$= \frac{1}{a \times a \times a}$$
$$= \frac{1}{a^3}$$

But $a^4 \div a^7 = a^{4-\square}$
$= a^{-3}$

Therefore, $a^{-3} = \square$

The definition of a^n, where n is a positive integer, is a multiplied by itself n times. This does not make sense when n is negative. Therefore, we have to define a^{-n}, where n is a positive integer. The above Investigation leads us to the following definition:

Definition 2:

$$a^{-n} = \frac{1}{a^n}$$

where base a is a real number such that $a \neq 0$, and n is a positive integer.

Worked Example 7

Applying definition of negative indices

Evaluate each of the following without using a calculator.

(a) 4^{-3} (b) $(-7)^{-1}$ (c) $\left(\frac{3}{5}\right)^{-2}$

*Solution

(a) $4^{-3} = \frac{1}{4^3}$

$= \frac{1}{64}$

(b) $(-7)^{-1} = \frac{1}{(-7)^1}$

$= -\frac{1}{7}$

(c) $\left(\frac{3}{5}\right)^{-2} = \frac{1}{\left(\frac{3}{5}\right)^2}$

$= 1 \div \left(\frac{3}{5}\right)^2$

$= 1 \div \frac{3^2}{5^2}$ Law 5 of Indices

$= 1 \times \frac{5^2}{3^2}$

$= \frac{25}{9}$

Attention

(c) Since $\frac{5^2}{3^2} = \left(\frac{5}{3}\right)^2$, then $\left(\frac{3}{5}\right)^{-2} = \left(\frac{5}{3}\right)^2$.

In general, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

Practise Now 7

Similar and Further Questions
Exercise 4B
Questions 3(a)–(d), 12(a)–(c)

1. Evaluate each of the following without using a calculator.

   (a) 6^{-2} (b) $(-8)^{-1}$

   (c) $\left(\frac{4}{5}\right)^{-3}$ (d) $9^0 \div \left(\frac{1}{9}\right)^{-1}$

2. Simplify $(2d)^0 \div (d^2e^{-4})^{-1}$.

C. Extension of Laws of Indices to zero and negative indices

In Section 4.2, we have learnt that all the five Laws of Indices apply when the indices are positive integers. With the definitions of zero and negative indices, we can now extend all the five Laws of Indices to include *all integer indices*.

Thinking Time

Copy and complete the following.

1. If bases a and b are real numbers, and indices m and n are **integers**, then

Law 1 of Indices: $a^m \times a^n =$ ______ if $a \neq 0$

Law 2 of Indices: $a^m \div a^n =$ ______ if $a \neq 0$

Law 3 of Indices: $(a^m)^n =$ ______ if ______

Law 4 of Indices: $a^n \times b^n =$ ______ if $a, b \neq 0$

Law 5 of Indices: $a^n \div b^n =$ ______ if ______

2. Notice that some conditions for the bases a and b are now different.
   (i) Why is it necessary for $a \neq 0$ in Law 1?
   (ii) Why is it necessary for $a, b \neq 0$ in Law 4?

3. **(i)** What happens if $m = n$ in Law 2?
   (ii) What happens if $m < n$ in Law 2?
   (iii) Why is it no longer necessary for $m > n$ in Law 2?
   (iv) What happens if $m = 0$ in Law 2?

Attention

Since zero and negative indices only apply for base $a \neq 0$, we have to be careful about the conditions under which the five laws apply.

From the above Thinking Time, we observe that the five Laws of Indices can apply to *all integer indices* only if the *bases are not equal to 0*.

Worked Example 8

Applying Laws of Indices involving integer indices

Simplify each of the following, leaving your answer in positive index form where appropriate.

(a) $a^{-7} \times a^4 \div a^{-3}$ **(b)** $8b^{-6}c^3 \div (2b^2c)^3$

Solution

(a)
$$\begin{aligned} a^{-7} \times a^4 \div a^{-3} &= a^{-7+4-(-3)} && \text{Laws 1 \& 2 of Indices} \\ &= a^{-7+4+3} \\ &= a^0 \\ &= 1 && \text{Definition 1} \end{aligned}$$

(b)
$$\begin{aligned} 8b^{-6}c^3 \div (2b^2c)^3 &= \frac{8b^{-6}c^3}{8b^6c^3} && \text{Law 3 of Indices} \\ &= b^{-6-6}c^{3-3} && \text{Law 2 of Indices} \\ &= b^{-12}c^0 \\ &= \frac{1}{b^{12}} && \text{Definitions 1 \& 2} \end{aligned}$$

Practise Now 8

Similar and Further Questions
Exercise 4B
Questions 4(a)–(d), 13(a)–(f), 14(a), (b)

1. Simplify each of the following, leaving your answer in positive index form.

   (a) $a^{-1} \times a^3 \div a^{-2}$ **(b)** $\dfrac{16d^{-2}e}{(2d^{-1}e)^3}$ **(c)** $18g^{-6} \div 3(g^{-2})^2$

2. Simplify $6h^2 \div 2h^{-2} - h \times h^3 - \dfrac{4}{h^{-4}}$.

4.4 Rational indices

So far, we have only learnt about indices that are integers. But can an index be a non-integer rational number?

For example, what does $3^{\frac{1}{4}}$ mean?

Let's first learn about the n^{th} root of a number.

Recall

A rational number can be expressed in the form $\frac{a}{b}$, where a and b are integers such that $b \neq 0$.

A. Positive n^{th} root

In Book 1, we learnt about the square root and the cube root of a number. For example, since $3^2 = 3 \times 3 = 9$, then the positive square root of 9 is $\sqrt{9} = 3$; and since $3^3 = 3 \times 3 \times 3 = 27$, then the cube root of 27 is $\sqrt[3]{27} = 3$.

We can extend the same idea to $3^4 = 3 \times 3 \times 3 \times 3 = 81$ by defining the *positive 4th root* of 81 to be $\sqrt[4]{81} = 3$.

Big Idea

Notations

Since $3^2 = 9$ and $(-3)^2 = 9$, there are *two square roots* of 9: 3 and -3. However, we use the notation $\sqrt{}$ to represent the **positive square root** so that there is no ambiguity, i.e. $\sqrt{9} = 3$ and $-\sqrt{9} = -3$.

In general,

> If a is a non-negative number such that $a = b^n$ for some non-negative number b, then b is the **positive n^{th} root** of a, and we write $b = \sqrt[n]{a}$.

An expression that involves the *radical sign* $\sqrt[n]{}$ is called a radical expression.

Worked Example 9

Finding positive n^{th} root

Evaluate each of the following without using a calculator.

(a) $\sqrt[4]{625}$ (b) $\sqrt[5]{\frac{243}{32}}$

Solution

(a) $\sqrt[4]{625} = \sqrt[4]{5\times5\times5\times5}$ prime factorisation

$= 5$

(b) $\sqrt[5]{\frac{243}{32}} = \sqrt[5]{\frac{3\times3\times3\times3\times3}{2\times2\times2\times2\times2}}$ prime factorisation

$= \sqrt[5]{\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}}$

$= \frac{3}{2}$

Practise Now 9

Similar and Further Questions Exercise 4B Questions 5(a)–(d)

Evaluate each of the following without using a calculator.

(a) $\sqrt[4]{256}$ (b) $\sqrt[5]{1024}$

(c) $\sqrt[3]{\frac{64}{125}}$ (d) $\sqrt[6]{\frac{64}{729}}$

B. Rational indices

We can now link the n^{th} root to indices that are non-integer rational numbers.

Investigation — Making sense of rational indices

Copy and complete the following.

1. What is $5^{\frac{1}{3}}$ equal to?

Let $p = 5^{\frac{1}{3}}$. Then $p^3 = \left(\square\right)^3$

$= 5^{\square \times 3}$ Law 3 of Indices

$= 5^1$

$= 5$

$\therefore p = \sqrt[3]{5}$

i.e. $5^{\frac{1}{3}} = \square$

2. What is $3^{\frac{1}{2}}$ equal to?

Let $p = 3^{\frac{1}{2}}$. Then $p^2 = \left(\square\right)^2$

$= 3^{\square \times 2}$ Law 3 of Indices

$= 3^1$

$= 3$

$\therefore p = \pm\sqrt{\square}$

There are two values of p. We choose the positive value because we want $y = a^{\frac{1}{x}}$ to be a *function*, i.e. for every value of x, there should be exactly one value of y.

Hence, $p = \sqrt{3}$

i.e. $3^{\frac{1}{2}} = \square$

The definition of a^n, where n is a positive integer, is a multiplied by itself n times. This does not make sense when n is not an integer. So we have to define $a^{\frac{1}{n}}$, where n is a positive integer.

The above Investigation leads us to the following definition:

Definition 3:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

where base a is a real number such that $a \geq 0$, and n is a positive integer.

Attention

We restrict the base to be non-negative so that Definition 3 works for any positive integer n. However, note that if n is odd, the base a can be negative. For example, if $n = 3$, then $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$. But if $n = 2$, then $(-9)^{\frac{1}{2}} = \sqrt{-9}$ is not a real number.

Thinking Time

Consider $a^{\frac{1}{n}} = \sqrt[n]{a}$.

1. What happens if $a < 0$?
2. What happens if $a = 0$?

Worked Example 10

Applying definition of rational indices

Rewrite each of the following in the radical form and hence evaluate the result without using a calculator.

(a) $16^{\frac{1}{4}}$ **(b)** $(-27)^{-\frac{1}{3}}$

Solution

(a) $$16^{\frac{1}{4}} = \sqrt[4]{16} = \sqrt[4]{2\times2\times2\times2} = 2$$

(b) $$(-27)^{-\frac{1}{3}} = \frac{1}{(-27)^{\frac{1}{3}}} \quad \text{Definition 2}$$
$$= \frac{1}{\sqrt[3]{-27}} \quad \text{Definition 3}$$
$$= \frac{1}{\sqrt[3]{(-3)\times(-3)\times(-3)}}$$
$$= \frac{1}{-3}$$
$$= -\frac{1}{3}$$

Attention

If $a = 0$, $a^{\frac{1}{n}}$ is not defined if n is negative since $0^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{0}}$.

Practise Now 10

Similar and Further Questions
Exercise 4B
Questions 6(a)–(d)

Rewrite each of the following in the radical form and hence evaluate the result without using a calculator.

(a) $36^{\frac{1}{2}}$ **(b)** $625^{\frac{1}{4}}$

(c) $243^{-\frac{1}{5}}$ **(d)** $(-1000)^{-\frac{1}{3}}$

Investigation

Discovering a result involving rational indices

Copy and complete the following.

What is $5^{\frac{2}{3}}$ equal to?

(a) $$5^{\frac{2}{3}} = 5^{2\times\frac{1}{\square}}$$
$$= (5^2)^{\frac{1}{\square}} \quad \text{Law 3 of Indices}$$
$$= \sqrt[\square]{5^2}$$

(b) $$5^{\frac{2}{3}} = 5^{\frac{1}{\square}\times 2}$$
$$= \left(5^{\frac{1}{\square}}\right)^2 \quad \text{Law 3 of Indices}$$
$$= \left(\sqrt[\square]{5}\right)^2$$

From the above Investigation, we observe the following:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad \text{or} \quad \left(\sqrt[n]{a}\right)^m$$

where base a is a real number such that $a \geqslant 0$, and m and n are positive integers.

Worked Example 11

Applying result involving rational indices

(a) Evaluate $125^{\frac{2}{3}}$ without using a calculator.

(b) Simplify $\frac{1}{\sqrt{x^n}}$, expressing your answer in index form.

Solution

(a) Method 1:

$$125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2$$
$$= 5^2$$
$$= 25$$

Method 2:

$$125^{\frac{2}{3}} = \sqrt[3]{125^2}$$
$$= \sqrt[3]{15\ 625}$$
$$= \sqrt[3]{25 \times 25 \times 25}$$
$$= 25$$

(b) $$\frac{1}{\sqrt{x^n}} = \frac{1}{x^{\frac{n}{2}}}$$
$$= x^{-\frac{n}{2}}$$

Problem-solving Tip

$125^{\frac{2}{3}}$ ← root

The denominator of the index is always the root. You can think of it as 'below the ground'.

Reflection

(a) Which method do you prefer? Why?

Practise Now 11

Similar and Further Questions
Exercise 4B
Questions 7(a)–(f), 8(a)–(f)

1. Evaluate each of the following without using a calculator.

(a) $64^{\frac{2}{3}}$ **(b)** $32^{-\frac{3}{5}}$

(c) $100^{1.5}$ **(d)** $4^{2.5} + 27^{\frac{4}{3}}$

2. Simplify each of the following, expressing your answer in index form.

(a) $\sqrt[3]{a^n}$ **(b)** $\frac{1}{\sqrt[5]{x^2}}$

C. Extension of Laws of Indices to rational indices

In Section 4.3, we have learnt that the five Laws of Indices apply to all integer indices.
With the definition of rational indices, the five Laws of Indices can be extended to include *all rational indices*.

Thinking Time

Copy and complete the following:

1. If bases a and b are real numbers, and indices m and n are **rational numbers**, then

Law 1 of Indices: $a^m \times a^n =$ ______ if $a > 0$

Law 2 of Indices: $a^m \div a^n =$ ______ if $a > 0$

Law 3 of Indices: $(a^m)^n =$ ______ if ______

Law 4 of Indices: $a^n \times b^n =$ ______ if $a, b > 0$

Law 5 of Indices: $a^n \div b^n =$ ______ if ______

2. Notice that some conditions for the bases a and b are now different.

(i) Why is it necessary for $a > 0$ in Law 1?

(ii) Why is it necessary for $a, b > 0$ in Law 4?

Attention

Although $a^{\frac{1}{n}}$, where n is a positive integer, applies only for base $a \geq 0$, zero and negative indices apply only for base $a \neq 0$. Therefore, if we want the five Laws of Indices to apply for any rational indices (including zero and negative indices), we need to combine both sets of conditions together, i.e. **base $a > 0$.**

3. What happens if you are not careful about the conditions of the bases?
The following shows a ridiculous proof that concludes that $1 = -1$.
Explain what is wrong with the proof.

$$\begin{aligned} 1 &= \sqrt{1} \\ &= \sqrt{(-1)\times(-1)} \\ &= \sqrt{-1}\times\sqrt{-1} \\ &= \left(\sqrt{-1}\right)^2 \\ &= (-1)^{\frac{1}{2}\times 2} \\ &= (-1)^1 \\ &= -1 \end{aligned}$$

Worked Example 12

Applying Laws of Indices involving rational indices

Simplify each of the following, expressing your answer in positive index form.

(a) $\sqrt[3]{m}\times\sqrt[4]{m^3}$ **(b)** $(pq)^{\frac{2}{3}}\div\left(p^{\frac{3}{4}}q^{\frac{1}{3}}\right)^2$ **(c)** $\left(\dfrac{32x^{10}}{y^{15}}\right)^{-\frac{1}{5}}$

Solution

(a)
$$\begin{aligned} \sqrt[3]{m}\times\sqrt[4]{m^3} &= m^{\frac{1}{3}}\times m^{\frac{3}{4}} && \text{Definition 3} \\ &= m^{\frac{1}{3}+\frac{3}{4}} && \text{Law 1 of Indices} \\ &= m^{\frac{13}{12}} \end{aligned}$$

(b)
$$\begin{aligned} (pq)^{\frac{2}{3}}\div\left(p^{\frac{3}{4}}q^{\frac{1}{3}}\right)^2 &= p^{\frac{2}{3}}q^{\frac{2}{3}}\div p^{\frac{3}{2}}q^{\frac{2}{3}} && \text{Law 4, then Law 3, of Indices} \\ &= p^{-\frac{5}{6}}q^0 && \text{Law 2 of Indices} \\ &= \frac{1}{p^{\frac{5}{6}}} && \text{Definitions 1 \& 2} \end{aligned}$$

(c)
$$\begin{aligned} \left(\frac{32x^{10}}{y^{15}}\right)^{-\frac{1}{5}} &= \frac{1}{\left(\frac{32x^{10}}{y^{15}}\right)^{\frac{1}{5}}} && \text{Definition 2} \\ &= \frac{1}{\frac{\left(32x^{10}\right)^{\frac{1}{5}}}{\left(y^{15}\right)^{\frac{1}{5}}}} && \text{Law 5 of Indices} \\ &= \frac{\left(y^{15}\right)^{\frac{1}{5}}}{\left(32x^{10}\right)^{\frac{1}{5}}} \\ &= \frac{y^{\frac{15}{5}}}{2^{\frac{5}{5}}x^{\frac{10}{5}}} && \text{Law 3 of Indices} \\ &= \frac{y^3}{2x^2} \end{aligned}$$

Problem-solving Tip
(c) $32 = 2\times2\times2\times2\times2 = 2^5$

Practise Now 12

Similar and Further Questions
Exercise 4B
Questions 9, 15(a)–(f), 16(a)–(f), 17(a)–(d)

1. Simplify each of the following, expressing your answer in positive index form.

(a) $(m^2)^{\frac{5}{6}} \times m^{\frac{1}{3}}$

(b) $\sqrt[5]{m} \div \sqrt[3]{m^2}$

(c) $(m^{-3}n^5)^{-\frac{1}{3}}$

(d) $\left(\dfrac{x^9y^3}{1000y^{12}}\right)^{-\frac{1}{3}}$

(e) $\dfrac{h^{-\frac{1}{3}}k^{-\frac{1}{4}}}{\left(h^2k^{-\frac{1}{3}}\right)^{-2}}$

(f) $(25p^2q^{-4})^{\frac{1}{2}}\left(p^3q^{-\frac{2}{5}}\right)^2$

2. Without using a calculator, evaluate $\left(\dfrac{64}{27}\right)^{-\frac{2}{3}}$. Give your answer as a fraction.

D. Equations involving indices

To solve an equation such as $x^2 = 100$, we take the square root on both sides to obtain $x = \pm 10$.
Similarly, to solve the equation $y^3 = 64$, we take the cube root on both sides to obtain $y = 4$.
But how do we find the value of x when given the equation $2^x = 32$?

Attention

In $x^2 = 100$ and $y^3 = 64$, the unknown is the base; while in $2^x = 32$, the unknown is the index.

Worked Example 13

Solving equations involving indices

Solve each of the following equations.

(a) $2^x = 32$ (b) $3^y = \dfrac{1}{9}$ (c) $9^z = 27$

Problem-solving Tip

Convert both sides to the same base and observe that if $a^x = a^n$, then $x = n$, provided $a \neq -1$, 0 or 1.

Solution

(a)
$$2^x = 32$$
$$2^x = 2^5$$
$$\therefore x = 5$$

(b)
$$3^y = \frac{1}{9}$$
$$= \frac{1}{3^2}$$
$$= 3^{-2} \quad \text{Definition 2}$$
$$\therefore y = -2$$

(c)
$$9^z = 27$$
$$(3^2)^z = 3^3$$
$$3^{2z} = 3^3 \quad \text{Law 3 of Indices}$$
$$2z = 3$$
$$\therefore z = \frac{3}{2}$$

Practise Now 13

Similar and Further Questions
Exercise 4B
Questions 10(a)–(d), 11(a)–(c)

1. Solve each of the following equations.

(a) $5^x = 125$ (b) $7^y = \dfrac{1}{49}$ (c) $8^z = 16$

2. Given that $9^{-2} \times 3^k = 1$, write down the value of k.

Reflection

1. How do I know when an index can be zero, negative or rational? (For example, for an index to be zero, base $a \neq 0$.)
2. How do I know when I can apply the five Laws of Indices if the indices can be zero, negative or rational?
3. What have I learnt in this section that I am still unclear of?

Basic | **Intermediate** | **Advanced**

Exercise 4B

1. Evaluate each of the following without using a calculator.

(a) 17^0 (b) $\left(-\frac{2}{7}\right)^0$

(c) $4a^0$ (d) $-8b^0$

(e) $(72cd^2)^0$ (f) $7(e^8)^0$

2. Find the value of each of the following without using a calculator.

(a) $2^0 \times 2^4$ (b) $7^2 \times 7^0 \div 7$

(c) $(-8)^0 - 8^2$ (d) $3^3 + 3^0 - 3$

3. Evaluate each of the following without using a calculator.

(a) 2^{-3} (b) $(-5)^{-1}$

(c) $\left(\frac{3}{4}\right)^{-2}$ (d) $\left(\frac{5}{3}\right)^{-1}$

4. Evaluate each of the following without using a calculator.

(a) $(7^2)^{-2} \div 7^{-4}$ (b) $5^0 - 5^{-2}$

(c) $(2^{15})^0 + \left(\frac{3}{5}\right)^{-1}$ (d) $\left(\frac{3}{4}\right)^{-2} \times 3^2 \div 2015^0$

5. Evaluate each of the following without using a calculator.

(a) $\sqrt{196}$ (b) $\sqrt[3]{125}$

(c) $\sqrt[5]{\frac{1}{32}}$ (d) $\sqrt[4]{\frac{16}{81}}$

6. Rewrite each of the following in the radical form and hence evaluate the result without using a calculator.

(a) $81^{\frac{1}{2}}$ (b) $(-27)^{\frac{1}{3}}$

(c) $32^{-\frac{1}{5}}$ (d) $(-343)^{-\frac{1}{3}}$

7. Evaluate each of the following without using a calculator.

(a) $216^{\frac{2}{3}}$ (b) $8^{-\frac{5}{3}}$

(c) $16^{1.5}$ (d) $(-1000)^{\frac{4}{3}}$

(e) $81^{\frac{3}{4}} - 9^{-\frac{3}{2}}$ (f) $625^{0.75} + 100^{2.5}$

8. Simplify each of the following, expressing your answer in index form.

(a) $\sqrt[4]{a}$ (b) $\left(\sqrt[5]{b}\right)^4$

(c) $\sqrt[3]{c^{2n}}$ (d) $\frac{1}{\sqrt[n]{d}}$

(e) $\frac{2}{\sqrt[8]{e^4}}$ (f) $\frac{1}{\left(\sqrt[3]{f}\right)^5}$

9. Without using a calculator, evaluate $\left(\frac{1000}{27}\right)^{-\frac{4}{3}}$. Give your answer as a decimal.

10. Solve each of the following equations.

(a) $11^a = 1331$ (b) $2^b = \frac{1}{128}$

(c) $9^c = 243$ (d) $10^d = 0.01$

Advanced | Intermediate | Basic

Exercise 4B

11. (a) Given that $\frac{1}{256} = 4^k$, find k.

(b) Given that $32 \times 16^{\frac{3}{4}} = 2^n$, find n.

(c) Find p if $p^{\frac{2}{3}} = 4$.

12. Simplify each of the following, expressing your answer in positive index form.

(a) $(b^{-5}c^2)^{-1}$ (b) $5f^0 \div 3f^{-4}$

(c) $(3c)^0 \div (c^{-3}d^5)^{-1}$

13. Simplify each of the following, expressing your answer in positive index form where appropriate.

(a) $5a^4 \times 3a^2 \div a^{-3}$ (b) $-24b^{-6} \div (3b^{-3})^2$

(c) $\frac{\left(4e^{-6}f^3\right)^2}{8e^{12}f^6}$ (d) $(3g^{-3}h^{-1})^2 \times (-4g^3h^{-2})^2$

(e) $(j^2k^{-1})^{-3} \times \left(\frac{j^2}{k^3}\right)^{-3}$ (f) $\frac{(m^5n^3)\times\left(m^2\right)^{-2}}{\left(m^{-1}n\right)^2}$

14. Simplify each of the following.

(a) $3a \div a^{-2} + a^2 \times a - \frac{6a^{-1}}{2a^{-4}}$

(b) $(5p)^3 - 10p \times 7p^2 + \frac{6}{p^{-3}}$

15. Simplify each of the following, expressing your answer in positive index form.

(a) $\sqrt[3]{a^2} \times \sqrt[6]{a}$ (b) $b^{\frac{4}{5}} \times b^{\frac{1}{2}} \div b^{-\frac{2}{5}}$

(c) $c^{\frac{1}{10}} \div c^{-\frac{1}{5}} \times c^{-\frac{3}{2}}$ (d) $\left(m^{\frac{1}{3}}n^{-10}\right)^{\frac{3}{5}}$

(e) $\left(p^{\frac{2}{3}}q^{-\frac{4}{5}}\right)^{\frac{3}{2}}$ (f) $\left(\frac{81x^{20}}{16x^8y^8}\right)^{-\frac{1}{4}}$

16. Simplify each of the following, expressing your answer in positive index form.

(a) $\left(a^{-2}b^3\right)^{\frac{1}{3}} \times \left(a^4b^{-5}\right)^{\frac{1}{2}}$ (b) $\left(c^{-3}d^{\frac{3}{5}}\right)^{-2} \times \left(c^{\frac{4}{5}}d^{-\frac{2}{5}}\right)^5$

(c) $\frac{e^{-\frac{1}{3}}f^{-\frac{1}{4}}}{\left(e^2f^{-\frac{1}{3}}\right)^{-2}}$ (d) $\left(\frac{g^{-2}h^2}{25}\right)^{-\frac{1}{2}}$

(e) $\left(4j^4k\right)^{\frac{1}{2}} \div 2h^3k^{-\frac{1}{2}}$

(f) $\left(m^3n^{-\frac{1}{4}}\right)^4 \div \sqrt[5]{32m^4n^{-8}}$

17. Simplify each of the following, expressing your answers in positive index form.

(a) $\left(\frac{x^{-4}y^7z^{-6}}{x^3y^{-1}z^3}\right)^3 \times \left(\frac{x^5y^2z^{-6}}{x^{-3}y^{-5}z^4}\right)^{-4}$

(b) $\left(\frac{x^3y^{-4}z^7}{x^{-5}y^2}\right)^3 \div \left(\frac{x^{-4}yz^{-5}}{x^7y^{-3}}\right)^{-2}$

(c) $\frac{ab^n}{bc} \times \frac{c^nd}{cd} \div \frac{b^{n+2}}{c^{n+3}}$

(d) $\frac{(a+b)^n}{bc^2} \div \frac{(a+b)^{n+3}}{abc}$

4.5 Surds

An *irrational number* that comprises the square root or the cube root is called a **surd**. For example, $\sqrt{2}$ and $\sqrt[3]{5}$ are surds, but $\sqrt{9}=3$ and $\sqrt[3]{8}=2$ are not surds. Why?

Other examples of surds include $3\sqrt{2}-\sqrt[3]{2}$ and $\frac{1+\sqrt{5}}{2}$.

Attention

$3\sqrt{2}$ means $3\times\sqrt{2}$ which is different from $\sqrt[3]{2}$ (cube root of 2).

In this section, we will learn three Laws of Surds and what we call *conjugate surds* to simplify expressions involving surds and the four operations.

A. Laws of surds

Is $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$? Is $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$? Let us investigate.

Investigation — Simplifying surds

1. Are the following statements true? (You can use a calculator to evaluate them.)
   (a) $\sqrt{16+9}=\sqrt{16}+\sqrt{9}$? (b) $\sqrt{16-9}=\sqrt{16}-\sqrt{9}$?
   (c) $\sqrt{16\times 9}=\sqrt{16}\times\sqrt{9}$? (d) $\sqrt{\frac{9}{16}}=\frac{\sqrt{9}}{\sqrt{16}}$?

In Section 4.4, we have learnt that $a^{\frac{1}{n}}=\sqrt[n]{a}$, where $a\geqslant 0$ and n is a positive integer.

In other words, $\sqrt{a}=a^{\frac{1}{2}}$, where $a\geqslant 0$.

2. For statements in Question 1 that are true, express them in the general form and prove them. For example, is $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$?
   Hint: Use Laws 4 and 5 of Indices. Take note of the conditions of a and b.
3. Simplify $\sqrt{a}\times\sqrt{a}$.

Recall

Laws of Indices
Law 4: $a^n\times b^n=(ab)^n$
Law 5: $\frac{a^n}{b^n}=\left(\frac{a}{b}\right)^n$

From the proofs in the above Investigation, we conclude that the following two **Laws of Surds** are true under certain conditions for a and b:

If $a>0$ and $b>0$, then **Law 1:** $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$

Law 2: $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

Special Case: **Law 3:** $\sqrt{a}\times\sqrt{a}=\sqrt{a^2}=a$ or $(\sqrt{a})^2=\sqrt{a^2}=a$

Attention

- Law 3 is a special case of Law 1 when $a=b$.
- $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$
  $\sqrt{a-b}\neq\sqrt{a}-\sqrt{b}$

Worked Example 14

Simplifying surds

Simplify each of the following without using a calculator.

(a) $\sqrt{2}\times\sqrt{8}$ (b) $\dfrac{\sqrt{216}}{\sqrt{6}}$ (c) $\sqrt{18}$

*Solution

(a) $\sqrt{2}\times\sqrt{8}$
$= \sqrt{2\times 8}$ Law 1 of Surds
$= \sqrt{16}$
$= 4$

(b) $\dfrac{\sqrt{216}}{\sqrt{6}}$
$= \sqrt{\dfrac{216}{6}}$ Law 2 of Surds
$= \sqrt{36}$
$= 6$

(c) $\sqrt{18}$
$= \sqrt{9\times 2}$
$= \sqrt{9}\times\sqrt{2}$ Law 1 of Surds
$= 3\sqrt{2}$

Practise Now 14

Similar and Further Questions
Exercise 4C
Questions 1(a)–(d)

Simplify each of the following without using a calculator.

(a) $\sqrt{27}\times\sqrt{3}$ (b) $\dfrac{\sqrt{125}}{\sqrt{5}}$ (c) $\sqrt{12}$ (d) $\dfrac{\sqrt{80}\times\sqrt{12}}{\left(\sqrt{16}\right)^2}$

Class Discussion

Mathematical fallacy: 1 = −1?

Let us consider the following 'proof' that 1 = −1.

$$\begin{aligned}1 &= \sqrt{1}\\ &= \sqrt{(-1)\times(-1)}\\ &= \sqrt{-1}\times\sqrt{-1}\\ &= -1\end{aligned}$$

What is wrong with the above 'proof'?

B. Adding and subtracting surds

Worked Example 15

Adding surds

Simplify $\sqrt{32}+\sqrt{50}$ without using a calculator.

*Solution

$$\begin{aligned}\sqrt{32}+\sqrt{50} &= \sqrt{16\times 2}+\sqrt{25\times 2}\\ &= \sqrt{16}\times\sqrt{2}+\sqrt{25}\times\sqrt{2} &&\text{Law 1 of Surds}\\ &= 4\sqrt{2}+5\sqrt{2}\\ &= 9\sqrt{2} &&\text{similar to '}4x + 5x = 9x\text{'}\end{aligned}$$

Attention

$\sqrt{32}+\sqrt{50} \neq \sqrt{32+50}$

Attention

In general,

- $p\sqrt{a}+q\sqrt{a} = (p+q)\sqrt{a}$
- $p\sqrt{a}-q\sqrt{a} = (p-q)\sqrt{a}$

Practise Now 15

Similar and Further Questions
Exercise 4C
Questions 2(a)–(f)

Simplify each of the following without using a calculator.

(a) $\sqrt{75}+\sqrt{108}$ (b) $\sqrt{80}-\sqrt{20}$

(c) $\sqrt{24}+\sqrt{54}-\sqrt{216}$

C. Multiplying surds

Worked Example 16

Multiplying surds

Simplify $(3+5\sqrt{2})(4-\sqrt{2})$ without using a calculator.

*Solution

$(3+5\sqrt{2})(4-\sqrt{2})$

$= 12-3\sqrt{2}+20\sqrt{2}-10$ since $5\sqrt{2}\times\sqrt{2}=5\times 2=10$, Law 3 of Surds

$= 2+17\sqrt{2}$

Practise Now 16

Similar and Further Questions
Exercise 4C
Questions 3(a)–(f)

Simplify each of the following without using a calculator.

(a) $(7+2\sqrt{3})(5-\sqrt{3})$ (b) $(4-3\sqrt{2})^2$

(c) $(3+2\sqrt{5})(3-2\sqrt{5})$ (d) $(3\sqrt{6}+4\sqrt{2})^2$

Recall

- $(x+y)^2=x^2+2xy+y^2$
- $(x-y)^2=x^2-2xy+y^2$

D. Conjugate surds

Class Discussion

Product of two irrational numbers

From parts **(a)**, **(b)** and **(d)** in Practise Now 16, we observe that the product of two irrational numbers is also an irrational number, e.g. $(7+2\sqrt{3})(5-\sqrt{3})=29+3\sqrt{3}$.

But in part **(c)**, the *product* of two *irrational* numbers is a *rational* number: $(3+2\sqrt{5})(3-2\sqrt{5})=-11$.

1. What patterns do you observe about the two irrational numbers in part **(c)**? Discuss why the product is a rational number.
2. Use the patterns in Question 1 to find a pair of irrational numbers whose product is a rational number.
3. What irrational number can you multiply $\sqrt{5}$ by, in order to obtain a rational number?

In general, if p, q and a are rational numbers, and $a>0$, then:

The *product* of **conjugate surds**, $p+q\sqrt{a}$ and $p-q\sqrt{a}$, is a *rational* number.

Special Case: The conjugate surd of $\sqrt{a}$ is itself or $-\sqrt{a}$.

Thinking Time

1. Use the algebraic identity $(x + y)(x - y) = x^2 - y^2$ to simplify $\left(p+q\sqrt{a}\right)\left(p-q\sqrt{a}\right)$.
2. What will you get if you substitute $p = 0$ and $q = 1$ in Question 1?
3. Is $\left(q\sqrt{a}+p\right)\left(q\sqrt{a}-p\right)$ also a rational number? Show your working.
4. What will you get if you substitute $p = 0$ and $q = 1$ in Question 3?
5. What do you think the conjugate surd of $p\sqrt{a}+q\sqrt{b}$ is? Why?

E. Rationalising the denominator

A fraction with a surd in its denominator can be simplified by making the denominator rational. For example, $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ ($\sqrt{2}\times\sqrt{2}=2$, Law 3 of Surds) and $\sqrt{2}$ is approximately 1.41, then $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 1.41 \div 2$. The process of multiplying $\frac{1}{\sqrt{2}}$ by $\frac{\sqrt{2}}{\sqrt{2}}$ is known as **rationalising the denominator**.

Big Idea

Equivalence

$\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$ are equivalent expressions because they have the same value.

Can you explain why we do not multiply $\frac{1}{\sqrt{2}}$ by $\frac{\sqrt{3}}{\sqrt{3}}$ instead?

To *rationalise the denominator*, we multiply the surd in the denominator by its *conjugate surd*.

Worked Example 17

Rationalising the denominator

Simplify each of the following by rationalising the denominator.

(a) $\frac{6}{\sqrt{5}}$ (b) $\frac{7}{2+\sqrt{3}}$

Solution

(a) $\frac{6}{\sqrt{5}} = \frac{6}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}$ multiplying by 1 will not change its value

$= \frac{6\sqrt{5}}{5}$ $\sqrt{5}\times\sqrt{5} = 5$, Law 3 of Surds

$= \frac{6}{5}\sqrt{5}$

(b) $\frac{7}{2+\sqrt{3}} = \frac{7}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$ $2+\sqrt{3}$ and $2-\sqrt{3}$ are conjugate surds

$= \frac{7\left(2-\sqrt{3}\right)}{2^2-\left(\sqrt{3}\right)^2}$ use $(x + y)(x - y) = x^2 - y^2$

$= \frac{14-7\sqrt{3}}{4-3}$

$= 14-7\sqrt{3}$

Problem-solving Tip

To rationalise the denominator, use the product of conjugate surds.

Attention

We rationalise the denominator to make an expression look simpler. Do the answers show this?

Practise Now 17

Similar and Further Questions
Exercise 4C
Questions 4(a)–(f), 5(a), (b), 6(a)–(d), 7(a)–(d), 8, 9, 12, 13

1. Simplify each of the following by rationalising the denominator.

   (a) $\dfrac{12}{\sqrt{3}}$ (b) $\dfrac{22}{4+\sqrt{5}}$ (c) $\dfrac{5}{2\sqrt{6}-3}$ (d) $\dfrac{5}{4-3\sqrt{3}}+\dfrac{7}{3\sqrt{3}+4}$

2. Express $\left(4-\sqrt{6}\right)^2-\dfrac{6}{3-\sqrt{6}}$ in the form $a+b\sqrt{6}$, where a and b are integers.

3. Given that $\sqrt{h+k\sqrt{5}}=\dfrac{4}{\left(3-\sqrt{5}\right)^2}$, where h and k are rational numbers, find the value of h and of k.

Investigation

Rational and irrational roots of quadratic equations

Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are *rational* numbers.

1. Solve the following equations by factorisation where possible.

   (a) $2x^2 + 3x - 2 = 0$ (b) $x^2 + 2x - 1 = 0$

   If the values of the roots are not exact, leave your answers in surd form.

2. (i) Which of the above quadratic equations can be solved by factorisation? Are its roots rational or irrational?

   (ii) Does the other quadratic equation have rational or irrational roots?

   (iii) The formula for the general solution is $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. When are the roots rational and when are they irrational?

3. If the roots are irrational, why must they be *conjugate surds*? Explain this using the formula for the general solution.

Information

The roots of a quadratic equation, where a, b and c can be *irrational* numbers, are usually not conjugate surds, with some exceptions. Can you explain why this is so?

Similar and Further Questions
Exercise 4C
Question 14

Worked Example 18

Problem involving surds

A triangle is such that its area is $\left(3-\sqrt{2}\right)$ cm² and the length of its base is $\left(\sqrt{2}-1\right)$ cm. Without using a calculator, find its height in the form $\left(a+b\sqrt{2}\right)$ cm, where a and b are integers.

Solution

Let the height of the triangle be h cm.

Area of triangle $=\frac{1}{2}\times$ base $\times$ height

$$3-\sqrt{2}=\frac{1}{2}\times\left(\sqrt{2}-1\right)\times h$$

$$6-2\sqrt{2}=\left(\sqrt{2}-1\right)\times h$$

$$
\begin{aligned}
h &= \frac{6-2\sqrt{2}}{\sqrt{2}-1} \\
&= \frac{6-2\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} \\
&= \frac{6\sqrt{2}+6-2\sqrt{2}\times\sqrt{2}-2\sqrt{2}}{(\sqrt{2})^2-1^2} \\
&= \frac{6\sqrt{2}+6-4-2\sqrt{2}}{2-1} \\
&= 2+4\sqrt{2}
\end{aligned}
$$

$\therefore$ the height of the triangle is $(2+4\sqrt{2})$ cm.

Practise Now 18

Similar and Further Questions
Exercise 4C
Questions 10, 11

1. A rectangle is such that its area is $(7-\sqrt{3})$ cm² and its length is $(5+\sqrt{3})$ cm. Without using a calculator, find the breadth of the rectangle in the form $(a+b\sqrt{3})$ cm, where a and b are rational numbers.
2. A cuboid has a square base. The length of each side of the base is $(2+\sqrt{3})$ cm and the volume of the cuboid is $(15+6\sqrt{3})$ cm³. Find the height of the cuboid in the form $(p+q\sqrt{3})$ cm, where p and q are integers.

Advanced | Intermediate | Basic

Exercise 4C

1. Simplify each of the following without using a calculator.

(a) $\sqrt{2}\times\sqrt{32}$ (b) $\frac{\sqrt{343}}{\sqrt{7}}$

(c) $\sqrt{63}$ (d) $\frac{\sqrt{75}\times\sqrt{72}}{\sqrt{24}}$

2. Simplify each of the following without using a calculator.

(a) $\sqrt{112}+\sqrt{28}$ (b) $\sqrt{180}-\sqrt{125}$

(c) $\sqrt{48}+\sqrt{12}-\sqrt{27}$ (d) $-\sqrt{11}-\sqrt{99}+\sqrt{44}$

(e) $\sqrt{240}-\sqrt{12}\times\sqrt{45}$ (f) $\frac{\sqrt{245}-\sqrt{20}}{\sqrt{500}}$

3. Simplify each of the following without using a calculator.

(a) $(5+\sqrt{2})(6-3\sqrt{2})$ (b) $(3+2\sqrt{6})^2$

(c) $(3\sqrt{11}-4)^2$ (d) $(7\sqrt{3}-3\sqrt{7})^2$

(e) $(9-2\sqrt{5})(9+2\sqrt{5})$

(f) $(2\sqrt{7}+3\sqrt{5})(2\sqrt{7}-3\sqrt{5})$

4. Simplify each of the following by rationalising the denominator.

(a) $\frac{10}{\sqrt{5}}$ (b) $\frac{1}{3\sqrt{10}}$

(c) $\frac{7}{2+\sqrt{3}}$ (d) $\frac{9}{8-\sqrt{6}}$

(e) $\frac{5}{4\sqrt{3}-2}$ (f) $\frac{4}{2\sqrt{7}+3}$

Basic | Intermediate | Advanced

Exercise 4C

5. Simplify each of the following by rationalising the denominator.

(a) $\dfrac{\sqrt{3}}{2\sqrt{5}+8}$ (b) $\dfrac{8}{2\sqrt{5}+3}-\dfrac{4}{2\sqrt{5}-3}$

6. Without using a calculator, express each of the following in its simplest surd form.

(a) $\dfrac{3}{\sqrt{8}}+\dfrac{5}{\sqrt{2}}-\dfrac{\sqrt{32}}{3}$ (b) $\dfrac{4}{\sqrt{27}}-\dfrac{\sqrt{18}}{4}+\dfrac{4}{\sqrt{3}}$

(c) $\dfrac{2}{\sqrt{3}}\left(\dfrac{4}{\sqrt{12}}+\dfrac{\sqrt{27}}{3}\right)$ (d) $\dfrac{6}{\sqrt{2}}\left(\dfrac{3}{\sqrt{8}}-\dfrac{\sqrt{128}}{3}\right)$

7. Simplify each of the following by rationalising the denominator.

(a) $\dfrac{13}{(\sqrt{3}+4)^2}$ (b) $\dfrac{3}{(\sqrt{2}+6)^2}+\dfrac{5}{(\sqrt{2}-6)^2}$

(c) $\dfrac{3\sqrt{2}-6}{6+3\sqrt{2}}$ (d) $\dfrac{\sqrt{48}-\sqrt{50}}{\sqrt{27}-\sqrt{8}}$

8. Express $(9-\sqrt{3})^2-\dfrac{78}{\sqrt{3}+9}$ in the form $a+b\sqrt{3}$, where a and b are integers.

9. Given that $h = 3+\sqrt{2}$, express $\dfrac{h^2+1}{h-2}$ in the form $p+q\sqrt{2}$, where p and q are integers.

10. The area of a rectangle is $2\sqrt{6}$ cm². If its breadth is $(3-\sqrt{6})$ cm, find its length in the form $(a+b\sqrt{6})$ cm, where a and b are integers.

11. A right circular cylinder has a volume of $(6+2\sqrt{3})\pi$ cm³ and a base radius of $(1+\sqrt{3})$ cm. Find its height in the form $(a+b\sqrt{3})$ cm, where a and b are integers.

12. If $a = \dfrac{1}{\sqrt{2}}$ and $b = \dfrac{1+a}{1-a}$, express in its simplest surd form,

(i) b, (ii) $b-\dfrac{1}{b}$.

13. Given that $\sqrt{a+b\sqrt{11}} = \dfrac{c}{(\sqrt{11}-3)^2}$, where a, b and c are rational numbers, find a possible set of values for a, b and c.

OPEN

14. Waseem solved a quadratic equation with rational coefficients and found that the roots are $2-\sqrt{3}$ and $1+\sqrt{3}$. Explain why his answer is wrong.

4.6 Exponential growth and decay

Exponential growth and decay have several applications in real life. One of which is compound interest, which we learnt in Book 2. In this section, we shall apply what we have learnt so far in this chapter to solve problems involving exponential growth and decay.

Worked Example 19

Solving problem involving exponential growth

The population of a small town increases by 4% each year. If the population in 2023 was 500, what will the population be in 2030, correct to the nearest whole number?

Solution

Since the population increases by 4% each year, the population in each year is 104% = 1.04 times of that in the preceding year.

After 7 years,

$$\text{population in 2030} = 500 \times 1.04^7$$
$$= 658 \text{ (to the nearest whole number)}$$

Practise Now 19

Similar and Further Questions
Exercise 4D
Questions 1, 3, 4, 6

The value of a house increases by 3.5% each year. What will its value be after 10 years if its current value is $100 000?

Worked Example 20

Solving problem involving exponential decay

Each year, the mass of a radioactive substance decreases by 30%. Its current mass is 30 g. Find its mass after 5 years.

Solution

Since the mass decreases by 30% each year, the mass in each year is 70% = 0.7 times of that in the preceding year.

After 5 years,

$$\text{mass} = 30 \times 0.7^5$$
$$= 5.04 \text{ g (to 3 s.f.)}$$

Practise Now 20

Similar and Further Questions
Exercise 4D
Questions 2, 5, 7, 8

A car was purchased for $20 000. Every year, it depreciates in value by 7.5%. Find its value after 6 years.

Advanced | Intermediate | Basic

Exercise 4D

1. The number of bacteria in a petri dish doubles every hour. If the initial number of bacteria is 50, how many bacteria are there after 6 hours?

2. The number of tourists who visited a particular attraction in 2018 was 22 000. Each year, the number declines by 12%. Find the number of tourists in 2030.

3. Initially, the population of a species of insects in a colony is 1000. It increases by 1% daily.
   (i) Write down a formula for the population, P, after d days.
   (ii) What is the population after 30 days?

Basic | Intermediate | Advanced

Exercise 4D

4. The value of a painting, \$$P$, n years after 2020 is given by $P = 2000 \times 1.1^n$.
(i) What was the initial value of the painting?
(ii) Find the value of the painting in 2030.

5. Yasir bought a car. After t years, its value, \$$P$, is given by $P = 25\ 000 \times 0.86^t$.
(i) How much did Yasir pay for the car?
(ii) Find the percentage decrease in the value of the car each year.

6. A company was founded on 1 January 2022. After t years, its number of workers, N, is given by $N = 30 \times 1.2^t$.
(i) State the initial number of workers that the company started with.
(ii) How many workers are there after 3 years?
(iii) In which year will there be 100 workers in the company?

7. In a particular experiment, the mass of a radioactive sample is halved every 1.5 hours. After 6 hours from the start of the experiment, it had a mass of 300 grams.
(i) Find the initial mass of the radioactive sample.
(ii) Calculate the mass of the radioactive sample 24 hours after the start of the experiment.
(iii) Yasir intends to end the experiment once the mass of the radioactive sample is 75 grams. Determine the duration of the experiment.

8. In a particular study, the population of a town is observed to follow a cycle every 2 years – in the first year, its population increases by 10%, and in the second year, its population decreases by 10%. If the current population is 50 000, what is the population after 40 years, correct to the nearest whole number?

4.7 Standard form

The world population, estimated to be about 8 000 000 000 in mid-November 2022, can be written as 8×10^9.
The Bohr radius of a hydrogen atom is 0.000 000 000 053 m, which can be written as 5.3×10^{-11} m.
Both 8×10^9 and 5.3×10^{-11} are examples of numbers expressed in **standard form** (or *scientific notation*).

> **Big Idea**
> **Notations**
> Standard form can be used to express very big and very small numbers in a precise and concise manner.

A. Standard form

Class Discussion **Exploring standard form**

Table 4.5 shows some examples of measurements which involve very large or very small numbers.

		Ordinary notation	Standard form
(i)	Pakistan's population in August 2023	241 500 000	2.415×10^8
(ii)	Distance between Earth and the Sun	149 600 000 km	1.496×10^8 km
(iii)	Mass of a dust particle	0.000 000 000 753 kg	7.53×10^{-10} kg
(iv)	Mass of an oxygen atom	0.000 000 000 000 000 000 000 000 026 56 kg	2.656×10^{-26} kg
(v)	Speed of light	300 000 000 m/s	$3 \times 10^{\square}$ m/s
(vi)	Wavelength of violet light	0.000 038 cm	$3.8 \times 10^{\square}$ cm
(vii)	Mass of a water molecule	0.000 000 000 000 000 000 000 0299 g	$2.99 \times 10^{\square}$ g

Table 4.5

1. The examples in **(i)**–**(ii)** involve very large numbers. What do you observe about the powers of 10 in each standard form?
2. The examples in **(iii)**–**(iv)** involve very small numbers. What do you observe about the powers of 10 in each standard form?
3. Complete the last column for **(v)**–**(vii)** in Table 4.5.

Table 4.6 shows numbers expressed in standard form and numbers not expressed in standard form. (Some of the pairs of values are not meant to be equal.)

	Standard form	Not standard form
(i)	4.5×10^4	45×10^3
(ii)	2.06×10^{-8}	0.206×10^{-7}
(iii)	1.0×10^{16}	10×10^{15}
(iv)	7×10^1	70×10^{-1}
(v)	3.71×10^{-4}	$3.71 \times 10^{-4.2}$

Table 4.6

4. For a number in the form $A \times 10^n$ to be considered a standard form, what can you say about A and n? Explain.

From the above Class Discussion, we observe the following:

> A number is said to be expressed in **standard form** when it is written as $A \times 10^n$, where $1 \leqslant A < 10$ and n is an integer.

Worked Example 21

Expressing numbers in standard form

Express each of the following numbers in standard form.

(a) 149 600 000 (b) 0.000 038

Solution

(a) $149\ 600\ 000 = 1.496 \times 100\ 000\ 000$

$= 1.496 \times 10^8$

(b) $0.000\ 038 = 3.8 \times 0.000\ 01$

$= 3.8 \times \frac{1}{100\ 000}$

$= 3.8 \times \frac{1}{10^5}$

$= 3.8 \times 10^{-5}$

Problem-solving Tip

(a) Observe that

$149\ 600\ 000 = 1.496 \times 10^8$

shift decimal point by 8 places to the left

(b) Observe that

$0.000\ 038 = 3.8 \times 10^{-5}$

shift decimal point by 5 places to the right

Practise Now 21

Similar and Further Questions
Exercise 4E
Questions 1(a)–(d), 2(a)–(d)

1. Express each of the following numbers in standard form.
   (a) 5 300 000 (b) 600 000 000
   (c) 0.000 048 (d) 0.000 000 000 167

2. Express each of the following in ordinary notation.
   (a) 1.325×10^6 (b) 4.4×10^{-3}

B. SI prefixes

Have you seen an external hard disk with a capacity of 512 *giga*bytes? Have you used a *micro*meter screw gauge in the science laboratory?

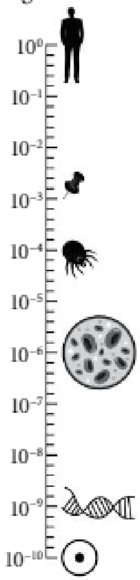
Giga and micro are called **SI prefixes** (SI stands for International System of Units). They are commonly used in our daily lives to denote certain powers of 10.

Table 4.7 lists some of the common prefixes and their symbols used for very large and very small numbers.

Power of 10	English word	SI prefix	Symbol	Numerical value
10^{12}	trillion	tera-	T	1 000 000 000 000
10^9	billion	giga-	G	1 000 000 000
10^6	million	mega-	M	1 000 000
10^3	thousand	kilo-	k	1000
10^{-3}	thousandth	milli-	m	$0.001 = \frac{1}{1000}$
10^{-6}	millionth	micro-	μ	$0.000\ 001 = \frac{1}{1\ 000\ 000}$
10^{-9}	billionth	nano-	n	$0.000\ 000\ 001 = \frac{1}{1\ 000\ 000\ 000}$
10^{-12}	trillionth	pico-	p	$0.000\ 000\ 000\ 001 = \frac{1}{1\ 000\ 000\ 000\ 000}$

Table 4.7

512 gigabytes is actually 512×10^9 bytes or 512 billion bytes. How many bytes are there in 64 megabytes?
A micrometer screw gauge can measure up to an accuracy of 10 micrometres, which is 10×10^{-6} metres or 10 millionth metres. How many metres are there in 2.4 nanometres?
Fig. 4.1 shows a range of SI prefixes used in our daily lives.



The height of a man is about 1.65 m.

The head of a pin is about 2×10^{-3} m or 2 mm (*milli*metre) in diameter.

The typical dust mite is about 0.4×10^{-3} m or 0.4 mm long.

A red blood cell is about 6×10^{-6} m or 6 μm (*micro*metre) wide.

The width of a strand of DNA is about 2.4×10^{-9} m or 2.4 nm (*nano*metre).

The diameter of a hydrogen atom is about 0.1×10^{-9} m or 0.1 nm.

Fig. 4.1

Information

Do you know that one billion is not always 10^9 or one thousand million? In the 'long scale' used in most European countries (including the UK) throughout most of the 19th and 20th centuries, one billion means 10^{12} or one million million. In 1974, the UK switched to the 'short scale', used in the USA. In the 'short scale', one billion is 10^9 or one thousand million. Search the Internet for 'long and short scales' to find out more.

Internet Resources

Search the Internet for 'powers of 10' to view a video starting from a picnic in Chicago to our galaxy 10^{24} metres away and then back to the picnic to within a proton at 10^{-16} metres.

Worked Example 22

Expressing SI prefixes in daily lives in standard form

For each of the following, give your answer in standard form.

(a) A steam power plant has a capacity of 250 megawatts. Given that 1 megawatt = 10^6 watts, express this capacity in watts.

(b) The average lifespan of a certain molecule is 0.5 nanoseconds. Given that 1 nanosecond = 10^{-9} seconds, express this time in seconds.

***Solution**

(a)
$$
\begin{aligned}
250 \text{ megawatts} &= 250 \times 10^6 \text{ watts} \\
&= 2.5 \times 10^2 \times 10^6 \text{ watts} \\
&= 2.5 \times 10^{2+6} \text{ watts} && \text{Law 1 of Indices} \\
&= 2.5 \times 10^8 \text{ watts}
\end{aligned}
$$

(b)
$$
\begin{aligned}
0.5 \text{ nanoseconds} &= 0.5 \times 10^{-9} \text{ seconds} \\
&= 5 \times 10^{-1} \times 10^{-9} \text{ seconds} \\
&= 5 \times 10^{-1+(-9)} \text{ seconds} && \text{Law 1 of Indices} \\
&= 5 \times 10^{-10} \text{ seconds}
\end{aligned}
$$

Practise Now 22

Similar and Further Questions
Exercise 4E
Questions 3–6

For each of the following, give your answer in standard form.

(a) An external hard drive has a capacity of 4.0 terabytes. Given that 1 terabyte = 10^{12} bytes, express this capacity in bytes.

(b) The diameter of a human hair is 25.4 micrometres. Given that 1 micrometre = 10^{-6} metres, express this diameter in metres.

(c) A rain gauge measures rainfall over a period of time. The average annual rainfall in Pakistan is 494 mm. Express this measurement in metres.

Performance Task

Does a thumbdrive with a capacity of 1 gigabyte (1 GB) have exactly 1 billion bytes of storage space? Search the Internet to find out how many bytes 1 GB is actually equal to. Why is it not possible for the manufacturer to produce a thumbdrive with exactly 1 billion bytes?

Hint: How does the number 2 in 2^{30} play a part here?

Search the Internet to find out why computer storage systems always come in the form of 128 MB, 256 MB, 512 MB and so on. Present your findings to your class.

Worked Example 23

Using calculator to apply orders of operation in standard form

Evaluate each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(1.35 \times 10^3 + 4.5 \times 10^{-1}) \times (2.37 \times 10^4)$

(b) $\dfrac{2.58\times10^{-3}-4.19\times10^{-4}}{3.17\times10^{2}}$

Solution

(a) Sequence of calculator keys:

(1 . 3 5 ×10ˣ 3
+ 4 . 5 ×10ˣ (–) 1)
× 2 . 3 7 ×10ˣ 4 =

$(1.35 \times 10^3 + 4.5 \times 10^{-1}) \times (2.37 \times 10^4) = 32\,005\,665$

$= 3.200\,5665 \times 10^7$

$= 3.20 \times 10^7$ (to 3 s.f.)

(b) Sequence of calculator keys:

▭/▭ 2 . 5 8 ×10ˣ (–) 3 – 4 . 1 9 ×10ˣ (–) 4 ▶
3 . 1 7 ×10ˣ 2 =

$\dfrac{2.58\times10^{-3}-4.19\times10^{-4}}{3.17\times10^{2}} = 0.000\,006\,817\,03$

$= 6.817\,03 \times 10^{-6}$

$= 6.82 \times 10^{-6}$ (to 3 s.f.)

Attention

The buttons on calculators vary with each model. Refer to the manual of your calculator.

Practise Now 23

Similar and Further Questions
Exercise 4E
Questions 7(a)–(h), 8(a)–(f), 9–12

Calculate each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(1.14 \times 10^5) \times (4.56 \times 10^4)$

(b) $(4.2 \times 10^{-4}) \times (2.6 \times 10^2)$

(c) $(2.4 \times 10^8) \div (6 \times 10^4)$

(d) $\dfrac{3.5\times10^{-5}}{1.4\times10^{8}}$

(e) $1.14 \times 10^5 + 4.56 \times 10^4$

(f) $2.6 \times 10^6 - 4 \times 10^4$

(g) $\dfrac{2.37\times10^{-3}+3.25\times10^{-4}}{4.1\times10^{5}}$

(h) $\dfrac{6.3\times10^{6}}{1.5\times10^{2}-3\times10^{-1}}$

Worked Example 24

Applying standard form in real-world contexts

The approximate mass of the moon is 7×10^{19} tonnes while the mass of the Earth is approximately 6×10^{24} tonnes. Calculate the number of times that the Earth is as heavy as the moon, giving your answer correct to the nearest 1000.

Solution

Number of times the Earth is as heavy as the moon

$= \dfrac{6\times10^{24}}{7\times10^{19}}$

$= \dfrac{6}{7} \times 10^{5}$

$= 86\ 000$ (to the nearest 1000)

Information

The short form of tonne is t.
1 t = 1000 kg.
Do you need to use this conversion rate in Worked Example 24? Explain.

Practise Now 24

Similar and Further Questions
Exercise 4E
Questions 13–15

A Secure Digital (SD) memory card has a capacity of 512 megabytes. Each photograph has a size of 640 kilobytes. Assuming that 1 MB = 10^6 bytes and 1 kB = 10^3 bytes, how many photographs can this memory card store?

Reflection

1. How do I convert a number from ordinary notation to standard form and vice versa?
2. What have I learnt in this section or chapter that I am still unclear of?

Advanced | Intermediate | Basic

Exercise 4E

1. Express each of the following numbers in standard form.
   (a) 85 300 (b) 52 700 000
   (c) 0.000 23 (d) 0.000 000 0904

2. Express each of the following in ordinary notation.
   (a) 9.6×10^3 (b) 4×10^5
   (c) 2.8×10^{-4} (d) 1×10^{-6}

3. A male African elephant can weigh as heavy as 7000 kilograms. Express this weight in grams, giving your answer in standard form.

4. Microwaves are a form of electromagnetic radiation with frequencies between 300 000 000 Hz and 300 GHz. Giving your answer in standard form, express
   (i) 300 000 000 Hz in MHz,
   (ii) 300 GHz in MHz.
   (1 MHz = 10^6 Hz and 1 GHz = 10^9 Hz)

5. (i) A nitrogen atom has an atomic radius of a picometres (pm), where $a = 70$ and 1 pm = 10^{-12} m. Express this radius in metres. Give your answer in standard form.

Basic | Intermediate | Advanced

Exercise 4E

(ii) An oxygen atom has an atomic radius of b nanometres (nm), where $b = 0.074$ and $1 \text{ nm} = 10^{-9}$ m. Express this radius in metres. Give your answer in standard form.

(iii) Express $a : b$ as a ratio of two integers in its simplest form.

6. The mean distance from the Earth to the Sun is c gigametres (Gm), where $c = 150$ and $1 \text{ Gm} = 10^9$ m. The mean distance from Pluto to the Sun is d terametres (Tm), where $d = 5.91$ and $1 \text{ Tm} = 10^{12}$ m. Express d as a percentage of c. Give your answer in standard form.

7. Calculate each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(2.34 \times 10^5) \times (7.12 \times 10^{-4})$

(b) $(5.1 \times 10^{-6}) \times (2.76 \times 10^{-3})$

(c) $(13.4 \times 10^4) \div (4 \times 10^5)$

(d) $\dfrac{3\times10^{-4}}{9\times10^{-8}}$

(e) $2.54 \times 10^3 + 3.11 \times 10^4$

(f) $3.1 \times 10^7 - 6 \times 10^5$

(g) $\dfrac{4.37\times10^{-4}+2.16\times10^{-5}}{3\times10^{-3}}$

(h) $\dfrac{2.4\times10^{-10}}{7.2\times10^{-6}-3.5\times10^{-8}}$

8. Calculate each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(1.35 \times 10^{-4})^3$ **(b)** $6(3.4 \times 10^3)^2$

(c) $\sqrt{1.21\times10^8}$ **(d)** $\sqrt[3]{9.261\times10^6}$

(e) $\dfrac{2.3\times10^{-2}\times4.7\times10^3}{2\times10^3}$ **(f)** $\dfrac{8\times10^2+2.5\times10^3}{2\times10^{-2}-3.4\times10^{-3}}$

9. Given that $P = 7.5 \times 10^3$ and $Q = 5.25 \times 10^4$, express each of the following in standard form.

(a) $2P \times 4Q$ **(b)** $Q - P$

10. Given that $x = 2 \times 10^{-3}$ and $y = 7 \times 10^{-4}$, evaluate $x + 8y$, giving your answer in standard form.

11. Given that $M = 3.2 \times 10^6$ and $N = 5.0 \times 10^7$, find the value of each of the following, giving your answer in standard form.

(a) MN **(b)** $\dfrac{M}{N}$

12. Given that $R = \dfrac{M}{EI}$, find the value of R when $M = 6 \times 10^4$, $E = 4.5 \times 10^8$ and $I = 4 \times 10^2$. Give your answer in standard form.

13. Light travels at a speed of 300 000 000 m/s.

(i) Express this speed in standard form.

(ii) Given that the mean distance from the Sun to Jupiter is 778.5 million kilometres, find the time taken, in minutes and seconds, for light to travel from the Sun to Jupiter.

14. On a journey from Planet P to Venus, a rocket is travelling at a constant speed. During this journey, the rocket travels past Planet Q in 4 days. The distance from Planet P to Planet Q is 4.8×10^5 km.

(i) Find the distance travelled by the rocket in 12 days. Give your answer in standard form.

(ii) Given that the distance between Planet P and Venus is 4.8×10^7 km, find the time taken, in days, for the journey.

15. The table shows the approximate population of the world in the past centuries.

Year	World population
1549	4.20×10^8
1649	5.45×10^8
1749	7.28×10^8
1849	1.17×10^9

Find

(i) the increase in population from 1549 to 1649,

(ii) the number of times that the population in 1849 is as large as that in 1649,

(iii) the number of times that the population of China in 2020 is as large as that of the world in 1749, given that the population of China in year 2020 was approximately 1.44 billion, where 1 billion = 10^9.

Looking Back

Mathematical **notations** have come a long way since the ancient days of using tally marks to represent numbers (one tally to represent one unit). In this chapter, we learnt more about index notations and their purposes of expressing very big and very small numbers in standard form. Knowing how to use these notations appropriately helps us communicate mathematical results in a precise and concise manner. The invention of the index notation has enabled us to express measurements of different units in relation to the magnitude (size) of the measurements. This makes it easier for us to work with numbers in many real-world contexts, especially in the sciences.

Operations involving surds have a rich and interesting historical background. Although surds can now be easily computed using a calculator, doing so by hand helps us to better appreciate the development of mathematical ideas. More importantly, working with surds allows us to see the structure of the roots of quadratic equations.
For example, when solving quadratic equations such as $x^2 - x - 1 = 0$, we obtain the following solutions expressed in surd notation:

$$x = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}$$

This surd notation gives us more information about the structure of the roots than its decimal representation. In fact, we see that the solutions of a quadratic equation are of the form

$$x = \frac{A \pm B}{C}$$

or more precisely,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for any quadratic equation $ax^2 + bx + c = 0$.

Therefore, while we can now determine these values using calculators, the use of surds provides insights into the relationship between the roots and the form of the equation. Moreover, leaving the answer in surd form will also give an exact answer.

Summary

1. The index notation of $\underbrace{a \times a \times a \times \ldots \times a \times a}_{n \text{ factors}}$, where a is a real number and n is a positive integer, is a^n.

2. **Zero indices**

   We define $a^0 = 1$, where base a is a real number such that $a \neq 0$.
   - What happens if $a = 0$?

3. **Negative indices**

   We define $a^{-n} = \frac{1}{a^n}$, where base a is a real number such that $a \neq 0$, and n is a positive integer.
   - What happens if $a = 0$?

Summary

4. **Rational indices**

We define $a^{\frac{1}{n}} = \sqrt[n]{a}$, where base a is a real number such that $a \geq 0$, and n is a positive integer.

In addition, $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$, where base a is a real number such that $a \geq 0$, and m and n are positive integers.

- What happens if $a < 0$?

5. **Laws of Indices**

If bases a and b are real numbers, and indices m and n are *rational numbers*, then

Law 1 of Indices: $a^m \times a^n = a^{m+n}$, if $a > 0$

Law 2 of Indices: $a^m \div a^n = a^{m-n}$, if $a > 0$

Law 3 of Indices: $(a^m)^n = a^{mn}$, if $a > 0$

Law 4 of Indices: $a^n \times b^n = (a \times b)^n$, if $a, b > 0$

Law 5 of Indices: $a^n \div b^n = \left(\frac{a}{b}\right)^n$, if $a, b > 0$

- What happens if indices m and n are *integers*? Will all the Laws of Indices work if the bases a and b are negative or zero?

6. An *irrational number* comprising the square root or the cube root is called a **surd**.

7. **Laws of Surds**

If $a, b > 0$, then **Law 1:** $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

Law 2: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Special Case: **Law 3:** $\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$ or $\left(\sqrt{a}\right)^2 = \sqrt{a^2} = a$

8. **Conjugate surds**

- The *product* of conjugate surds, $p + q\sqrt{a}$ and $p - q\sqrt{a}$, is a *rational* number.

  Special Case: The conjugate surd of $\sqrt{a}$ is itself or $-\sqrt{a}$.

- To *rationalise the denominator*, we multiply the surd in the denominator by its *conjugate surd*.

9. **Standard form**

A number is said to be expressed in standard form when it is written as $A \times 10^n$, where $1 \leq A < 10$ and *n is an integer*.

CHAPTER 5

Coordinate Geometry

The Cartesian coordinate system specifies the location of any point on a plane using an ordered pair of numbers (x, y), also known as coordinates. The invention of the Cartesian coordinate system by René Descartes was said to be one of the greatest mathematical achievements. His invention made it possible to relate geometrical objects such as points, lines, curves and shapes, to algebraic expressions and equations. This "marriage" between algebra and geometry is a classic example of how mathematicians have developed new and powerful ideas by connecting different concepts in mathematics. In this chapter, we will begin to relate some of the geometrical objects to algebraic expressions and equations through the seemingly simple **notation** of the coordinate system.

Learning Outcomes

What will we learn in this chapter?

- How to find the length and the midpoint of a line segment given the coordinates of its endpoints
- How to find the gradient of a straight line given the coordinates of two points on it, or given that it is parallel or perpendicular to a given line
- How to interpret and find the equation of a straight line graph in the form $y = mx + c$
- How to solve geometry problems involving the use of coordinates

Introductory Problem

Joyce was given the following problem to solve:

Given that $x + y = 5$, find the minimum value of $\sqrt{x^2 + y^2}$.

Joyce tried to solve this problem algebraically but could not do so. Ali looked at the problem for a few minutes and said, "That's a geometric problem and the answer is $\frac{5}{2}\sqrt{2}$."

Can you figure out how Ali solved the problem?

In this chapter, we will be using some tools of the coordinate system to solve this problem. Let us turn our attention to one of the simplest geometric objects — a straight line.

5.1 Length of a line segment

A. Cartesian coordinates (Recap)

In Book 2, we learnt that a rectangular or **Cartesian** plane consists of two number lines intersecting at right angles at the point O, known as the **origin**. The horizontal and vertical axes are called the ***x*-axis** and the ***y*-axis** respectively.

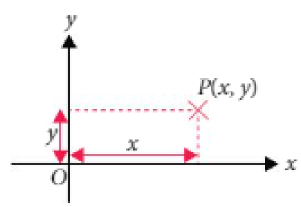


Fig. 5.1

Just For Fun

René Descartes, a French philosopher in the early 17th century, invented the coordinate system. His use of (x, y) as ordered pairs enhanced the inter-relationship between geometrical curves and algebraic equations. He was also the first person to coin the dictum, "I think, therefore I am."

Each point P in the plane is located by an ordered pair (x, y). We say that P has coordinates (x, y).

Class Discussion — Determining the length of a line segment in a Cartesian plane

Let us explore how we can determine the length of a line segment in a Cartesian plane.

1. **(a)** Consider the line segment formed by joining the points (1, 1) and (7, 1). The length of this line segment is 6 units. Why?
   (b) How do you find the length of a horizontal line segment?
2. **(a)** Consider the vertical line segment formed by joining the points (1, 1) and (1, 9). Find the length of this line segment.
   (b) How do you find the length of a vertical line segment?

3. Now, consider the line segment formed by joining the points $A(1, 1)$ and $B(7, 9)$. How is this line segment different from the previous two line segments? How can we find the length of this line segment AB? Discuss your answer with your classmates.

For the points $A(1, 1)$ and $B(7, 9)$, let us look at the diagram in Fig. 5.2.

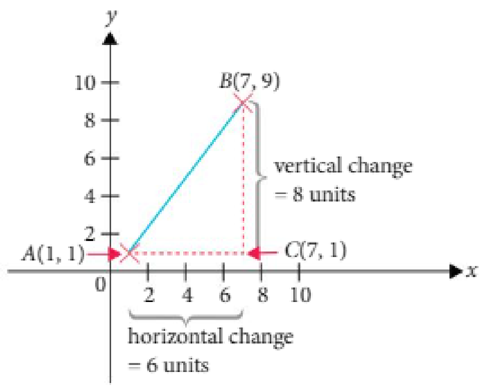


Fig. 5.2

Attention

Recall that a line segment is part of a line with two endpoints. A line has no endpoints so it does not have a definite length. However, a line segment has two endpoints and so it has length.

Attention

BC is vertical, i.e. x-coordinate of C = x-coordinate of B.
AC is horizontal, i.e. y-coordinate of C = y-coordinate of A.

Big Idea

Diagrams

The coordinate system helps us to visualise algebraic equations in terms of graphs and to relate geometric objects in terms of algebraic expressions and equations. This offers us a more powerful tool to solve problems than algebra or geometry alone.

$\triangle ABC$ is formed by drawing AC parallel to the x-axis and BC parallel to the y-axis. The coordinates of the point C are given by $(7, 1)$.

Hence, $AC = 7 - 1 = 6$ units

and $\quad BC = 9 - 1 = 8$ units.

Using Pythagoras' Theorem,

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= 6^2 + 8^2 \\ &= 100 \text{ units}^2 \\ AB &= \sqrt{100} \\ &= 10 \text{ units} \end{aligned}$$

Consider any two points P and Q with coordinates (x_1, y_1) and (x_2, y_2) respectively. By completing the right-angled $\triangle PQR$, we have the coordinates of R as (x_2, y_1).

Hence, $PR = x_2 - x_1$ and $QR = y_2 - y_1$.

Using Pythagoras' Theorem,

$$\begin{aligned} PQ^2 &= PR^2 + QR^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$

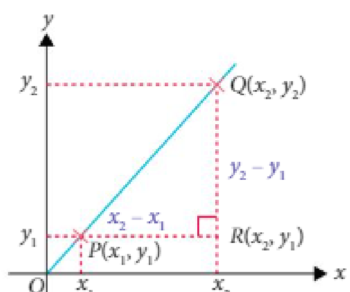


Fig. 5.3

In general, the length of any line segment PQ, where the coordinates of the points P and Q are (x_1, y_1) and (x_2, y_2) respectively, is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Thinking Time

Instead of writing the length of line segment PQ as $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, can we also write it as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$?

Is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$? Explain your answer.

Worked Example 1

Finding the length of a line segment

Given that the coordinates of the points A and B are $(-4, 1)$ and $(6, -5)$ respectively, find the length of the line segment AB.

Solution

$$\begin{aligned}\text{Length of line segment } AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[6-(-4)]^2 + (-5-1)^2} \\ &= \sqrt{10^2 + (-6)^2} \\ &= \sqrt{100 + 36} \\ &= \sqrt{136} \\ &= 11.7 \text{ units (to 3 s.f.)}\end{aligned}$$

Practise Now 1

Similar and Further Questions
Exercise 5A
Questions 1(a)–(d)

Find the length of the line segment joining each of the following pairs of points.

(a) $C(6, 2)$ and $D(3, -2)$
(b) $M(-1, 5)$ and $N(6, -4)$
(c) $P(2, 7)$ and $Q(8, 7)$

Worked Example 2

Using the length to determine the coordinates of a point on the line

Given that the coordinates of the points A and B are $(-3, 2)$ and $(1, -6)$ respectively, find the coordinates of the point C that lies on the y-axis such that $AC = BC$. Hence, find the area of $\triangle ACO$, where O is the origin.

Solution

Let the coordinates of C be $(0, k)$.

$$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[0-(-3)]^2 + (k-2)^2}$$
$$= \sqrt{(0+3)^2 + (k-2)^2}$$
$$= \sqrt{9 + (k-2)^2} \text{ units}$$

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(0-1)^2 + [k-(-6)]^2}$$
$$= \sqrt{(-1)^2 + (k+6)^2}$$
$$= \sqrt{1 + (k+6)^2} \text{ units}$$

Since $AC = BC$,

$$\sqrt{9+(k-2)^2} = \sqrt{1+(k+6)^2}$$
$$\left[\sqrt{9+(k-2)^2}\right]^2 = \left[\sqrt{1+(k+6)^2}\right]^2 \qquad \text{square both sides}$$
$$9 + (k-2)^2 = 1 + (k+6)^2$$
$$9 + k^2 - 4k + 4 = 1 + k^2 + 12k + 36$$
$$k^2 - 4k + 13 = k^2 + 12k + 37$$
$$-16k = 24$$
$$k = -\frac{24}{16}$$
$$= -\frac{3}{2}$$
$$= -1\frac{1}{2}$$

$\therefore$ the coordinates of C are $\left(0, -1\frac{1}{2}\right)$.

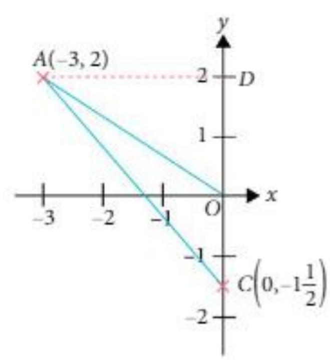
Problem-solving Tip

Since C lies on the y-axis, its x-coordinate is 0.

Problem-solving Tip

It is helpful to draw a diagram to visualise the relative positions of the points A and C in relation to O. Then we can find the base and height of $\triangle ACO$ easily.

$$\text{Area of } \triangle ACO = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times OC \times AD$$
$$= \frac{1}{2} \times 1\frac{1}{2} \times 3$$
$$= \frac{9}{4}$$
$$= 2\frac{1}{4} \text{ units}^2$$



Practise Now 2

Similar and Further Questions
Exercise 5A
Questions 2–8

Given that the coordinates of the points C and D are $(4, -1)$ and $(-2, 7)$ respectively, find

(a) the coordinates of the point E that lies on the y-axis such that $CE = DE$,

(b) the coordinates of the point F that lies on the x-axis such that $CF = DF$.

Hence, find the area of $\triangle OEF$, where O is the origin.

Worked Example 3

Using the length to show that a triangle is right-angled

A triangle has vertices $A(0, -5)$, $B(-2, 1)$ and $C(10, 5)$. Show that $\triangle ABC$ is a right-angled triangle and identify the right angle.

Solution

$$\begin{aligned} AB^2 &= (-2 - 0)^2 + [1 - (-5)]^2 \\ &= (-2)^2 + 6^2 \\ &= 4 + 36 \\ &= 40 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} BC^2 &= [10 - (-2)]^2 + (5 - 1)^2 \\ &= 12^2 + 4^2 \\ &= 144 + 16 \\ &= 160 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} AC^2 &= (10 - 0)^2 + [5 - (-5)]^2 \\ &= 10^2 + 10^2 \\ &= 100 + 100 \\ &= 200 \text{ units}^2 \end{aligned}$$

Since $AB^2 + BC^2 = 40 + 160$
$= 200$
$= AC^2$,

the triangle is a right-angled triangle with $\angle ABC = 90°$. converse of Pythagoras' Theorem

Problem-solving Tip

Check that the sum of the squares of the two shorter sides is equal to the square of the longest side.

Problem-solving Tip

The hypotenuse (longest side) of a right-angled triangle is always opposite the right angle (largest angle).

Practise Now 3

Similar and Further Questions
Exercise 5A
Questions 9–11

1. A triangle has vertices $D(6, 1)$, $E(2, 3)$ and $F(-1, -3)$. Show that $\triangle DEF$ is a right-angled triangle and identify the right angle.
2. A triangle has vertices $P(-3, 1)$, $Q(6, 3)$ and $R(1, 8)$. Determine if $\triangle PQR$ is a right-angled triangle.

Reflection

1. What do I already know about the Cartesian plane that could help me find the length of a line segment?
2. What have I learnt in this section that I am still unclear of?

Exercise 5A

1. Find the length of the line segment joining each of the following pairs of points.

(a) $A(2, 3)$ and $B(9, 7)$
(b) $C(3, 6)$ and $D(-5, 9)$
(c) $E(-1, 4)$ and $F(8, -3)$
(d) $G(-10, 2)$ and $H(-4, -7)$

2. If the distance between the points $A(p, 0)$ and $B(0, p)$ is 10 units, find the possible values of p.

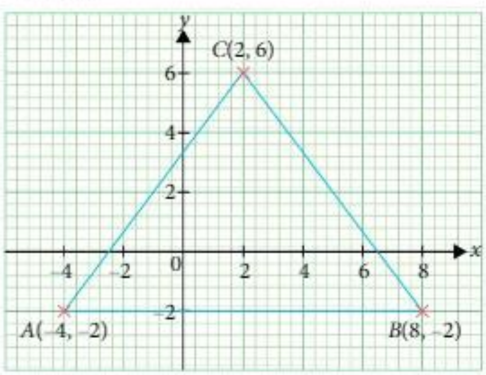
3. Given that the coordinates of the points P and Q are $(-2, 6)$ and $(9, 3)$ respectively, find

(a) the coordinates of the point R that lies on the y-axis such that $PR = QR$,
(b) the coordinates of the point S that lies on the x-axis such that $PS = QS$.

4. A line segment has two endpoints $M(3, 7)$ and $N(11, -6)$. Find the coordinates of the point W that lies on the y-axis such that W is equidistant from M and from N.

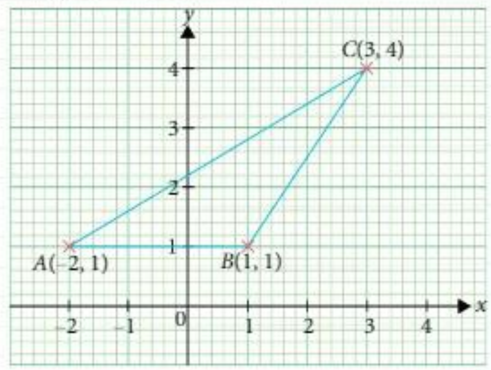
Hint: The term 'equidistant' means 'same distance'.

5. The vertices of $\triangle ABC$ are $A(-4, -2)$, $B(8, -2)$ and $C(2, 6)$.



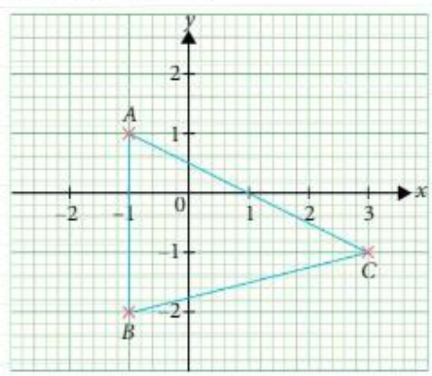
(i) Find the perimeter and the area of $\triangle ABC$.
(ii) Hence, find the length of the perpendicular from A to BC.

6. The diagram shows $\triangle ABC$ with vertices $A(-2, 1)$, $B(1, 1)$ and $C(3, 4)$.



(i) Find the area of $\triangle ABC$.
(ii) Find the length of AC, giving your answer correct to 2 decimal places.
(iii) Given that $ABCD$ is a parallelogram, find the coordinates of D.
(iv) Given that K is the point $(t, 4)$ and the area of $\triangle BCK$ is 12 units², find the possible values of t.

7. The diagram shows $\triangle ABC$ with vertices $A(-1, 1)$, $B(-1, -2)$ and $C(3, -1)$.



(i) Find the lengths of AB, BC and AC.
(ii) Find the area of $\triangle ABC$.

The coordinates of a point E are $(3, k)$ and the area of $\triangle BCE$ is 14 units².

(iii) Find the possible values of k.

Basic Intermediate Advanced

Exercise 5A

8. The distance between the points $(1, 2t)$ and $(1 - t, 1)$ is $\sqrt{11 - 9t}$ units. Find the possible values of t.

9. **(i)** Show that the points $A(-1, 2)$, $B(5, 2)$ and $C(2, 5)$ are the vertices of an isosceles triangle.
(ii) Find the area of $\triangle ABC$.

10. By showing that the points $P(3, 4)$, $Q(3, 1)$ and $R(8, 4)$ are the vertices of a right-angled triangle, find the length of the perpendicular from P to QR.

11. The vertices of $\triangle PQR$ are $P(1, 3)$, $Q(5, 4)$ and $R(5, 15)$. Find the length of the perpendicular from Q to PR.

5.2 Gradient of a straight line

A. Calculation of gradient of a straight line (Recap)

Let us look at another attribute of a straight line — gradient. In Book 2, we learnt that the gradient of a straight line is the ratio of the vertical change to the horizontal change.

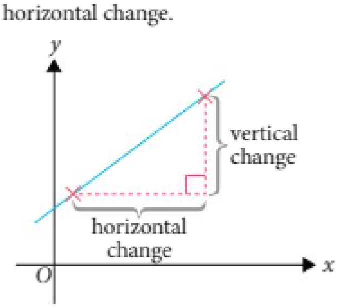
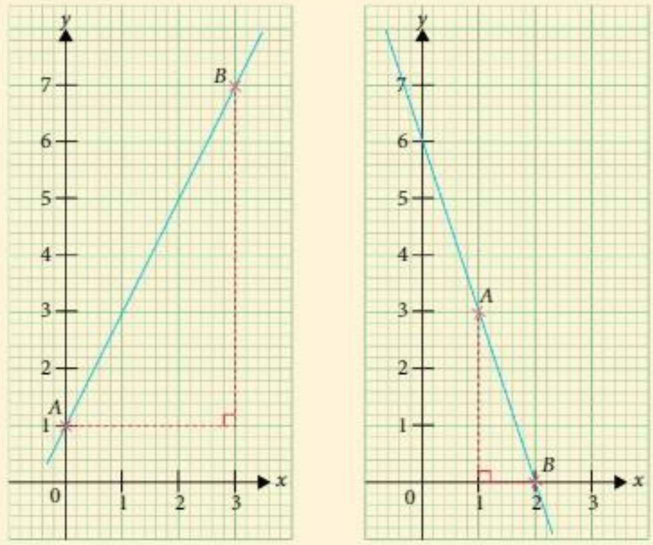


Fig. 5.4

Now let us examine how the gradient of a straight line can be computed on a Cartesian plane.

Class Discussion — Finding the gradient of a straight line

1. In Fig. 5.5(**a**) and (**b**), A and B are two points on each line.



(**a**) (**b**)

Fig. 5.5

For each of the two lines shown in Fig. 5.5:

(i) Find the vertical change from point A to point B.

(ii) Find the horizontal change from point A to point B.

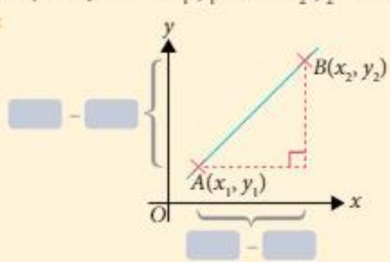
(iii) Find the gradient of the line segment AB.

Hint: Gradient = $\dfrac{\text{vertical change}}{\text{horizontal change}}$

(iv) Choose two other points C and D that lie on the line and calculate the gradient of the line segment CD.
Compare your answers with those obtained in part **(iii)**.
What do you notice? Explain your answer.

2. Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, how would you find the gradient of the line passing through A and B?

Hint:



3. Using your answer in Question 2, find the gradient of the line passing through each of the following pairs of points.
   (a) $(-1, 4)$ and $(3, 7)$ (b) $(-4, -3)$ and $(2, -11)$
   (c) $(6, 3)$ and $(-4, 3)$ (d) $(2, -1)$ and $(2, 8)$
   Compare your answers with those obtained by your classmates.

From the above Class Discussion, we observe that if $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a line, then

gradient of $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$.

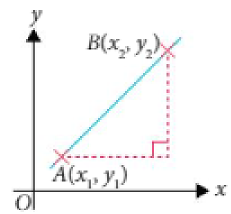


Fig. 5.6

Thinking Time

Instead of writing the gradient of AB as $\dfrac{y_2 - y_1}{x_2 - x_1}$, can we also write it as $\dfrac{y_1 - y_2}{x_1 - x_2}$?

Is $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$? Explain your answer.

Investigation — Gradient of a straight line

1. Using a suitable geometry software, draw a line segment with the endpoints as $A(-2, 1)$ and $B(0, 5)$.
2. Find the gradient of the line segment AB and record it in Table 5.1. Describe the gradient of the line segment AB using one of the following terms: positive, negative, zero or undefined.
3. Write down the value of $y_2 - y_1$ and of $x_2 - x_1$ in Table 5.1.
4. Copy and complete Table 5.1.

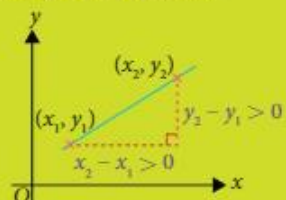
	Coordinates of endpoints	Gradient of line segment	Sign of gradient	$y_2 - y_1$	$x_2 - x_1$
(a)	$A(-2, 1)$ and $B(0, 5)$		positive	$5 - 1 =$ ☐	$0 - (-2) =$ ☐
(b)	$C(7, 5)$ and $D(4, 8)$				
(c)	$E(-2, 6)$ and $F(-4, 3)$				
(d)	$G(1, -2)$ and $H(2, -3)$				
(e)	$I(1, 1)$ and $J(3, 1)$				
(f)	$K(-4, 3)$ and $L(-4, 6)$				

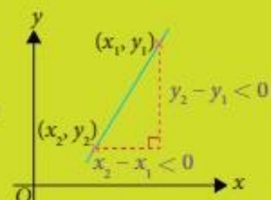
Table 5.1

5. **(a)** When $y_2 - y_1 > 0$ and $x_2 - x_1 < 0$, what do you notice about the sign of the gradient?
   (b) When $y_2 - y_1 < 0$ and $x_2 - x_1 > 0$, what do you notice about the sign of the gradient?
   (c) When the signs of $y_2 - y_1$ and $x_2 - x_1$ are the same, what do you notice about the sign of the gradient?
   (d) When $y_2 - y_1 = 0$, what do you notice about the gradient of the line?
   (e) When $x_2 - x_1 = 0$, what do you notice about the gradient of the line?

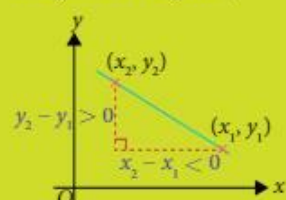
From the above Investigation, we observe that

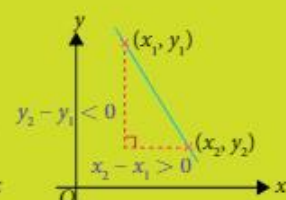
- the gradient of a straight line can be positive, negative, zero or undefined,
- if $y_2 - y_1$ and $x_2 - x_1$ have the same sign, the gradient of the straight line is positive,



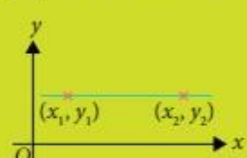


- if $y_2 - y_1$ and $x_2 - x_1$ have opposite signs, the gradient of the straight line is negative,

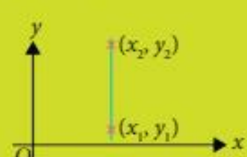




- if $y_2 - y_1 = 0$ or $y_2 = y_1$, the gradient of a horizontal line is zero,

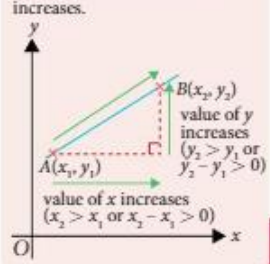


- if $x_2 - x_1 = 0$ or $x_2 = x_1$, the gradient of a vertical line is undefined.



Attention

When the gradient of the line is positive, as the value of x increases (from point A to point B), the value of y also increases.



Worked Example 4

Finding gradient given two points

Find the gradient of the line passing through each of the following pairs of points.

(a) $A(2, 3)$ and $B(7, 5)$

(b) $P(-2, 8)$ and $Q(1, -1)$

Solution

(a) Gradient of $AB = \frac{y_2 - y_1}{x_2 - x_1}$ let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (7, 5)$

$= \frac{5-3}{7-2}$

$= \frac{2}{5}$

Alternatively,

Gradient of $AB = \frac{y_1 - y_2}{x_1 - x_2}$

$= \frac{3-5}{2-7}$

$= \frac{-2}{-5}$

$= \frac{2}{5}$

Reflection

Does it matter which points we assign as (x_1, y_1) and (x_2, y_2)? Why?

(b) Gradient of $PQ = \frac{y_2 - y_1}{x_2 - x_1}$ let $(x_1, y_1) = (-2, 8)$ and $(x_2, y_2) = (1, -1)$

$= \frac{-1-8}{1-(-2)}$

$= \frac{-9}{3}$

$= -3$

Practise Now 4

Similar and Further Questions
Exercise 5B
Questions 1(a)–(f), 2, 9

Find the gradient of the line passing through each of the following pairs of points.

(a) $C(3, 1)$ and $D(6, 3)$

(b) $H(5, -7)$ and $K(0, -2)$

(c) $M(-4, 1)$ and $N(16, 1)$

Worked Example 5

Using the gradient to determine the coordinates of a point on the line

If the gradient of the line joining the points $(k, 5)$ and $(2, k)$ is -2, find the value of k.

Solution

Gradient of line, $\frac{y_2 - y_1}{x_2 - x_1} = -2$

$\frac{k-5}{2-k} = -2$ let $(x_1, y_1) = (k, 5)$ and $(x_2, y_2) = (2, k)$

$k - 5 = -2(2 - k)$

$k - 5 = -4 + 2k$

$-1 = k$

$\therefore k = -1$

Practise Now 5

Similar and Further Questions
Exercise 5B
Questions 3–8

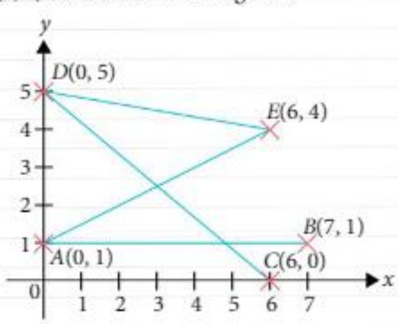
1. If the gradient of the line joining the points $(4, -9)$ and $(-3, h)$ is -3, find the value of h.
2. The points $A(m, -18)$, $B(2, -9)$ and $C(4, m)$ are collinear, i.e. they lie on a straight line. Find the possible values of m.

Advanced | Intermediate | Basic

Exercise 5B

1. Find the gradient of the line passing through each of the following pairs of points.
   (a) $A(0, 0)$ and $B(-2, 1)$
   (b) $C(2, -3)$ and $D(1, 7)$
   (c) $E(-2, 4)$ and $F(-5, 8)$
   (d) $G(-4, 7)$ and $H(1, -8)$
   (e) $I(-2, -5)$ and $J(2, 6)$
   (f) $K(-7, 9)$ and $L(6, 9)$

2. The points $A(0, 1)$, $B(7, 1)$, $C(6, 0)$, $D(0, 5)$ and $E(6, 4)$ are shown in the diagram.



   Find the gradient of each of the line segments AB, AE, DC and DE.

3. If the gradient of the line joining the points $(-3, -7)$ and $(4, p)$ is $\frac{3}{5}$, find the value of p.

4. The coordinates of A and B are $(3k, 8)$ and $(k, -3)$ respectively. Given that the gradient of the line segment AB is 3, find the value of k.

5. The gradient of the line joining the points $(9, a)$ and $(2a, 1)$ is $\frac{2}{a}$, where $a \neq 0$. Find the possible values of a.

6. The points P, Q and R have coordinates $(6, -11)$, $(k, -9)$ and $(2k, -3)$ respectively. If the gradient of PQ is equal to the gradient of PR, find the value of k.

7. The points $P(2, -3)$, $Q(3, -2)$ and $R(8, z)$ are collinear, i.e. they lie on a straight line. Find the value of z.

8. The line joining the points $A(2, t)$ and $B(7, 2t^2 + 7)$ has a gradient of 2. Find the possible values of t.

9. The coordinates of the vertices of a square $ABCD$ are $A(0, 6)$, $B(2, 1)$, $C(7, 3)$ and $D(5, 8)$.
   (i) Find the gradient of all 4 sides of $ABCD$.
   (ii) What do you observe about the gradients of the opposite sides of a square?

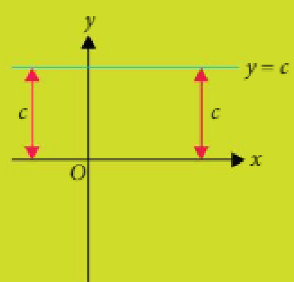
5.3 Equation of a straight line

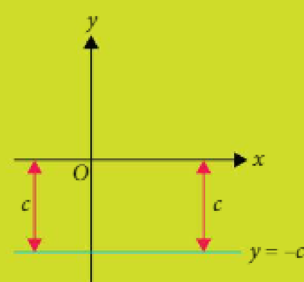
A. Equation of a straight line (Recap)

Let us now consider how the Cartesian coordinate system can express geometric objects as algebraic equations and expressions. We need to first recall some ideas we learnt in Book 2, and then use these ideas to solve problems set in the Cartesian coordinate system.

Horizontal lines

If a line is parallel to the x-axis and its distance from the x-axis is c, then every point on the line has the same y-coordinate.

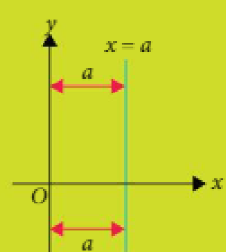


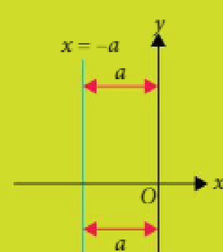


The gradient of a horizontal line is 0.
The equation of a horizontal line is $y = c$ or $y = -c$.

Vertical lines

If a line is parallel to the y-axis and its distance from the y-axis is a, then every point on the line has the same x-coordinate.





The gradient of a vertical line is undefined.
The equation of a vertical line is $x = a$ or $x = -a$.

B. Equation of a straight line $y = mx + c$

In Book 2, we learnt that the equation of a straight line is in the form $y = mx + c$, where the constant m is the **gradient** of the line and the constant c is the **y-intercept**.

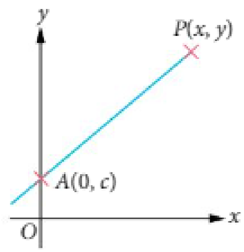


Fig. 5.7

In Fig. 5.7, the straight line passes through the points $A(0, c)$ and $P(x, y)$. If the gradient of the line is m, then

Gradient of line = m

$$\frac{y-c}{x-0} = m$$
$$y - c = mx$$
$$\therefore y = mx + c$$

In general,

> for a straight line passing through the point $(0, c)$ and with gradient m, the equation is $y = mx + c$.

The equation $y = mx + c$ is known as the **gradient-intercept form** of the equation of a straight line. In this equation, m gives the gradient of the straight line, c gives the intercept on the y-axis and $(0, c)$ is the point where the line cuts the y-axis.

Worked Example 6

Finding the y-intercept given the gradient and coordinates of a point

Given that the line $y = 3x + c$ passes through the point (3, 1), find the value of c.

Solution

Since (3, 1) lies on the line $y = 3x + c$, the coordinates (3, 1) must satisfy the equation,

i.e. $1 = 3(3) + c$ substitute $x = 3$ and $y = 1$

$= 9 + c$

$\therefore c = -8$

Practise Now 6

Similar and Further Questions
Exercise 5C
Questions 1, 2

1. Given that the line $y = 5x + a$ passes through the point $(-1, 2)$, find the value of a.
2. The point (6, 8) lies on the line $y = -4x + b$. Find the value of b.

Worked Example 7

Finding the equation of a straight line given the coordinates of 2 points

Find the equation of the straight line passing through each of the following pairs of points.

(a) $A(1, 2)$ and $B(3, 7)$
(b) $C(2, 3)$ and $D(7, 3)$
(c) $E(5, 1)$ and $F(5, 6)$

Solution

(a) Gradient of $AB = \frac{7-2}{3-1}$

$= \frac{5}{2}$

Equation of AB is in the form $y = \frac{5}{2}x + c$

Since (1, 2) lies on the line,

$2 = \frac{5}{2}(1) + c$

$c = -\frac{1}{2}$

$\therefore$ the equation of AB is $y = \frac{5}{2}x - \frac{1}{2}$.

> **Attention**
> We can also substitute (3, 7) into the equation of AB to find the value of c.

(b) $C(2, 3)$ and $D(7, 3)$ have the same y-coordinate of value 3.
$\therefore$ CD is a horizontal line with equation $y = 3$.

(c) $E(5, 1)$ and $F(5, 6)$ have the same x-coordinate of value 5.
$\therefore$ EF is a vertical line with equation $x = 5$.

Practise Now 7

Similar and Further Questions
Exercise 5C
Questions 3(a)–(h), 4(a)–(f), 5, 6(a)–(d), 7–17

Find the equation of the straight line passing through each of the following pairs of points.

(a) $A(-2, 1)$ and $B(5, 3)$
(b) $C(6, 4)$ and $D(-4, 4)$
(c) $E(-3, 5)$ and $F(-3, 8)$

Journal Writing

What information do we need to find the equation of a straight line?

Consider each of the cases below.

Case 1: Given the gradient m and the y-intercept c
Case 2: Given the gradient m and the coordinates of a point (a, b)
Case 3: Given the coordinates of two points (a, b) and (c, d)

For each case, describe how you would find the equation of the straight line.

Introductory Problem Revisited

Let us take another look at the **Introductory Problem**.

Although the problem of finding the minimum value of $\sqrt{x^2+y^2}$ may seem like an algebraic question, the same question can be modelled as a geometry problem using ideas about coordinate geometry!

First, can you see that $x + y = 5$ is actually the equation of a straight line with x-intercept $(5, 0)$ and y-intercept $(0, 5)$? This can be represented in Fig. 5.8 below:

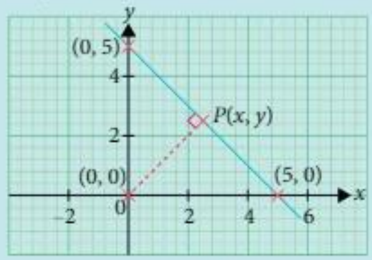


Fig. 5.8

Big Idea

Diagrams and Models

Mathematical diagrams such as graphs in the Cartesian plane can help us to visualise the relationships between variables. More importantly, a graph can represent an abstract algebraic equation or expression visually so that we can look at the same problem in a different way. This allows us to model geometric objects using equations and then apply algebraic methods to obtain an algebraic solution, before we interpret the solution geometrically.

Given that $P(x, y)$ lies on the line $x + y = 5$, we see that the distance from P to the origin is given by $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$ units. Therefore, the minimum value of $\sqrt{x^2+y^2}$ is actually the shortest possible distance from P to the origin.

(i) Explain why the coordinates of P, when it is at a minimum distance from the origin, are $(2.5, 2.5)$.

(ii) Show that the minimum distance from P to the origin, and hence the minimum value of $\sqrt{x^2+y^2}$, is $\frac{5}{2}\sqrt{2}$.

In summary, this example demonstrates how an algebraic question can sometimes be recast as an **equivalent** coordinate geometry problem! Hence, coordinate geometry can be seen as the bridge connecting algebra and geometry.

Reflection

1. What components do I require to determine the equation of a straight line?
2. What have I learnt in this section that I am still unclear of?

Advanced | Intermediate | Basic

Exercise 5C

1. Given that the line $y = -x + c$ passes through the point $(1, 2)$, find the value of c.

2. The point $(-3, 3)$ lies on the line $y = 4x + k$. Find the value of k.

Exercise 5C

3. Find the equation of the straight line passing through each of the following pairs of points.

(a) $A(0, 0)$ and $B(1, -1)$
(b) $C(1, 3)$ and $D(2, 5)$
(c) $E(2, 4)$ and $F(-2, 3)$
(d) $G(-6, -5)$ and $H(4, 4)$
(e) $I(-2, -4)$ and $J(1, -7)$
(f) $K(-7, -5)$ and $L(-1, -1)$
(g) $M(8, 0)$ and $N(-9, 0)$
(h) $O(0, 0)$ and $P(0, 7)$

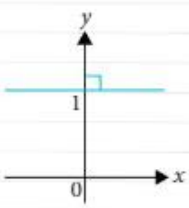
4. Find the equation of each of the following straight lines, given the gradient and the coordinates of a point that lies on it.

(a) $\frac{1}{3}$, $(0, 0)$ **(b)** 3, $(1, 1)$
(c) -3, $(2, -5)$ **(d)** $-\frac{1}{2}$, $(5, 7)$
(e) 0, $(5, 4)$ **(f)** a, $(0, a)$

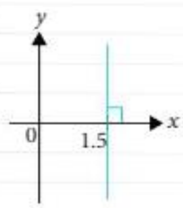
5. Write down the equation of the straight line which passes through the origin and with gradient 2.

6. In each of the following diagrams, find the gradient and the y-intercept of the line where possible. State the equation of each line.

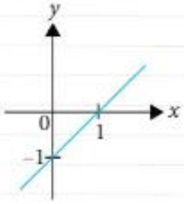
(a)



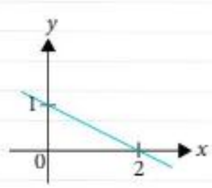
(b)



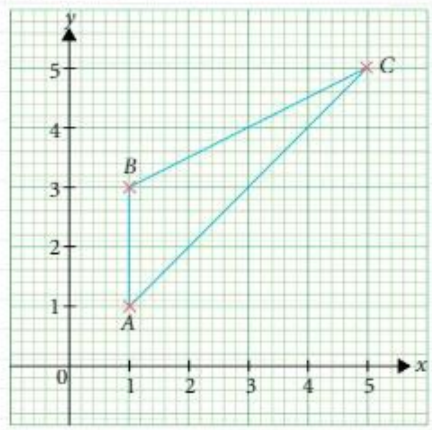
(c)



(d)



7. The diagram shows $\triangle ABC$ with vertices $A(1, 1)$, $B(1, 3)$ and $C(5, 5)$.



Find
(i) the area of $\triangle ABC$,
(ii) the gradient of the line passing through B and C,
(iii) the equation of the line passing through A and C.

8. The lines $2x - 5 = ky$ and $(k + 1)x = 6y - 3$ have the same gradient. Find the possible values of k.

9. Given the line $\frac{x}{3} + \frac{y}{2} = 1$,
(i) make y the subject of the formula $\frac{x}{3} + \frac{y}{2} = 1$,
(ii) find the gradient of the line,
(iii) find the coordinates of the point at which the line cuts the x-axis.

10. **(i)** Find the equation of the straight line which passes through the point $(-3, 5)$ and with gradient $-\frac{2}{3}$.
(ii) Given that the line in part **(i)** also passes through the point $(p, 3)$, find the value of p.

Advanced Intermediate Basic

Exercise 5C

11. Find the equation of the straight line passing through the point $(3, -2)$ and having the same gradient as the line $2y = 5x + 7$.

12. **(i)** Find the equation of the straight line which passes through the point $(3, 1)$ and with gradient 3.
(ii) Hence, find the coordinates of the point of intersection of the line in part **(i)** with the line $y = x$.

13. The line l has equation $5x + 6y + 30 = 0$. Given that P is the point $(3, -1)$, find
(i) the coordinates of the point where l crosses the x-axis,
(ii) the coordinates of the point of intersection of l with the line $x = 2$,
(iii) the equation of the line passing through P and having the same gradient as l,
(iv) the equation of the line passing through P and having a gradient of 0.

14. A straight line l passes through the points $A(0, 3)$ and $B(3, 12)$.
(a) Find
(i) the gradient of the line l,
(ii) the equation of the line l.
(b) The line $x = 3$ is the line of symmetry of $\triangle ABC$. Find the coordinates of C.

15. If the line $mx = ny + 2$ has the same gradient as the x-axis, find the value of m. State the condition for the line to be parallel to the y-axis instead.

16. The equation of the line l is $3x + 4y = 24$. It crosses the x-axis at the point A and the y-axis at the point B. Find
(i) the coordinates of A and of B,
(ii) the length of the line segment AB,
OPEN **(iii)** the equation of the line OC, where O is the origin and the point C is equidistant from the coordinate axes,
OPEN **(iv)** the coordinates of the point C given that C also lies on the line l.

17. The coordinates of the points P and Q are $(2, 3)$ and $(9, 5)$ respectively.
(i) Find the coordinates of the point where the line passing through P and Q intersects the x-axis.
(ii) Given that $y = 5$ is the line of symmetry of $\triangle PQR$, find the coordinates of R.
(iii) Calculate the length of PQ.
(iv) Hence, find the length of the perpendicular from R to PQ.

5.4 Midpoint of a line segment

In this section, we will learn how to determine the coordinates of the midpoint of a line segment, when the coordinates of its two end points are given.

For a number line in one dimension, the number exactly halfway between -1 and 4 is $\frac{-1+4}{2} = 1.5$. This number 1.5 is the mid-value of -1 and 4. It is found by taking the *average* of -1 and 4.

Recall

What is the difference between a line and a line segment (which you have learnt in Sections 5.1–5.3)? Can you find the midpoint of a line? How about the midpoint of a line segment? Explain your answer.

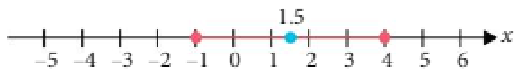


Fig. 5.9

Therefore, a number which is the mid-value of x_1 and x_2 is $\frac{x_1+x_2}{2}$.

Let us extend this idea to two dimensions.

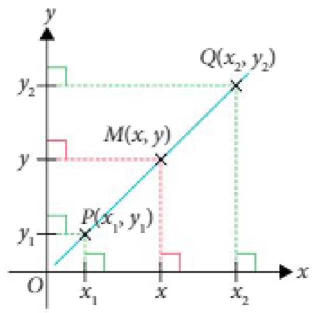


Fig. 5.10

Big Idea

Diagrams
We see the usefulness of a graph to visualise the given information to find a formula for the midpoint of a line segment.

Big Idea

Functions
The midpoint of a line segment is a function with 4 variables for the input (i.e. x_1, x_2, y_1 and y_2) and two variables for the output (i.e. $\frac{x_1+x_2}{2}$ and $\frac{y_1+y_2}{2}$).

The coordinates of P and Q are (x_1, y_1) and (x_2, y_2) respectively. Let $M(x, y)$ be the **midpoint** of the line segment PQ. The x-coordinate of M is the average of x_1 and x_2. The y-coordinate of M is the average of y_1 and y_2.

Coordinates of the midpoint M on the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Worked Example 8

Midpoint of line segment

(a) Find the coordinates of the midpoint of the line segment joining the points
  (i) $A(2, 5)$ and $B(-6, -1)$,
  (ii) $C(-3, 4)$ and $D(5, 4)$,
  (iii) $D(5, 4)$ and $E(5, 1)$.

(b) If the coordinates of the midpoint of the line segment joining $P(-5, 3)$ and $Q(a, b)$ are $(3, 4)$, find the value of a and of b.

Big Idea

Notations
We use an ordered pair (x, y) to represent the coordinates of a point in a Cartesian plane.
The ordered pair tells us that the first number is the x-coordinate and the second number is the y-coordinate.

***Solution**

(a) (i) Midpoint of $AB = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$= \left(\frac{2+(-6)}{2}, \frac{5+(-1)}{2}\right)$

$= (-2, 2)$

(ii) Midpoint of $CD = \left(\frac{x_1+x_2}{2}, 4\right)$

$$= \left(\frac{-3+5}{2}, 4\right)$$

$$= (1, 4)$$

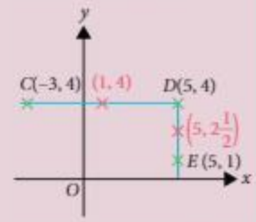
(iii) Midpoint of $DE = \left(5, \frac{y_1+y_2}{2}\right)$

$$= \left(5, \frac{4+1}{2}\right)$$

$$= \left(5, 2\frac{1}{2}\right)$$

Problem-solving Tip

In **(a)(ii)**, since both the y-coordinates of C and D are equal to 4, CD is a horizontal line segment and the y-coordinate of the midpoint of CD is 4.



In **(a)(iii)**, is the line segment DE horizontal, vertical or slanting?

(b) $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (3, 4)$

$$\left(\frac{-5+a}{2}, \frac{3+b}{2}\right) = (3, 4)$$

i.e. $\frac{-5+a}{2} = 3$ and $\frac{3+b}{2} = 4$

$-5 + a = 6$ $\qquad$ $3 + b = 8$

$a = 11$ $\qquad$ $b = 5$

Practise Now 8

Similar and Further Questions
Exercise 5D
Questions 1(a)–(f), 2(a)–(d), 8

(a) Find the coordinates of the midpoint of the line segment joining the points

(i) $A(5, 3)$ and $B(-1, 7)$, **(ii)** $C(2, 3)$ and $D(6, 3)$,

(iii) $E(1, 4)$ and $F(1, -2)$.

(b) If $(2, 0)$ is the midpoint of the line segment joining $A(8, -3)$ and $B(p, q)$, find the value of p and of q.

Worked Example 9

Midpoint of line segment

Explain why $P(5, 8)$ is not the midpoint of the line segment joining the points $G(1, 7)$ and $H(9, 7)$.

Solution

Method 1:

Since both the y-coordinates of G and H are equal to 7, GH is a horizontal line segment, i.e. the y-coordinate of the midpoint of GH must also be equal to 7.
Since the y-coordinate of $P(5, 8)$ is 8 ($\neq 7$), then $P(5, 8)$ is not the midpoint of GH.

Method 2:

y-coordinate of midpoint of $GH = \frac{y_1+y_2}{2}$

$$= \frac{7+7}{2}$$

$$= 7$$

Since the y-coordinate of $P(5, 8)$ is 8 ($\neq 7$), then $P(5, 8)$ is not the midpoint of GH.

Practise Now 9

Similar and Further Questions
Exercise 5D
Questions 4(a), (b)

1. Explain why $Q(2, 9)$ is not the midpoint of the line segment joining the points $A(0, 6)$ and $B(4, 6)$.
2. Explain why $R(-3, 7)$ is not the midpoint of the line segment joining the points $C(-5, 10)$ and $D(-5, 4)$.

Thinking Time

Are the following statements always true, sometimes true or never true? Give reasons for your answer.

(a) If M is the midpoint of line segment AB, then M is a point equidistant from points A and B.

(b) If M is a point equidistant from points A and B, then M is the midpoint of the line segment AB.

Worked Example 10

Finding 4th vertex of parallelogram

If $A(2, -4)$, $B(7, 1)$ and $C(-1, 5)$ are three vertices of a parallelogram $ABCD$, find the midpoint of AC. Hence, find

(i) the coordinates of D, (ii) the length of AD.

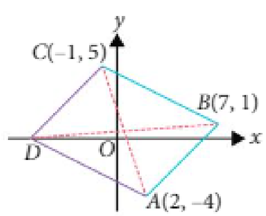
Solution

$$\text{Midpoint of } AC = \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right) = \left(\frac{2+(-1)}{2}, \frac{-4+5}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Problem-solving Tip

Regardless of the shape and size of a parallelogram, the diagonals of a parallelogram always bisect each other.

(i) Let the coordinates of D be (h, k).



Big Idea

Diagrams
We can sketch the parallelogram to help us visualise that the diagonals are AC and BD (not AB and CD).

Midpoint of AC = Midpoint of BD

$$\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}\right)$$

i.e. $\frac{1}{2} = \frac{7+h}{2}$ and $\frac{1}{2} = \frac{1+k}{2}$

$1 = 7 + h$ $\quad$ $1 = 1 + k$

$h = -6$ $\quad$ $k = 0$

$\therefore D(-6, 0)$

(ii)
$$AD = \sqrt{(x_4-x_1)^2+(y_4-y_1)^2} = \sqrt{(-6-2)^2+[0-(-4)]^2} = \sqrt{80}$$
$= 8.94$ units (to 3 s.f.)

Recall

Length of line segment joining (x_1, y_1) and (x_2, y_2) = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Practise Now 10

Similar and Further Questions
Exercise 5D
Questions 3, 5–7, 9, 10

1. $ABCD$ is a parallelogram. The coordinates of A, B and C are $(-3, 5)$, $(2, 7)$ and $(4, 6)$ respectively. Find
   (i) the coordinates of D,
   (ii) the length of the diagonal BD.

2. The line $5x + y = 17$ intersects the curve $5x^2 + y^2 = 49$ at the points P and Q.
   (i) Find the coordinates of the midpoint of PQ.
   (ii) Calculate the length of PQ.

Class Discussion — **Finding 4th vertex of quadrilateral**

1. Use another method in Worked Example 10 part **(i)** to find the coordinates of D. Which method is more efficient?
2. Given any three vertices of a parallelogram, we can use the formula for the midpoint to obtain the fourth vertex. Can this rule be applied to a square, a rectangle, a rhombus, a trapezium and a kite? Explain.

Advanced | Intermediate | Basic

Exercise 5D

1. Find the coordinates of the midpoint of the line segment joining each of the following pairs of points.
(a) $(1, 1)$ and $(7, 3)$
(b) $(3, -2)$ and $(-2, 7)$
(c) $(4, 4)$ and $(8, 4)$
(d) $(0, -2)$ and $(0, 6)$
(e) $(2a + b, 3b - a)$ and $(b - a, 2a - b)$
(f) $(ah^2, 2ah)$ and $(ak^2, 4ak)$

2. In each of the following, M is the midpoint of PQ. Find the coordinates of Q.
(a) $M(5, 0)$ and $P(-5, 0)$
(b) $M(3, -7)$ and $P(3, -10)$
(c) $M(6, 2)$ and $P(1, 1)$
(d) $M(0, -3)$ and $P(4, -1)$

3. Three of the vertices of a parallelogram $ABCD$ are $A(-3, 1)$, $B(4, 9)$ and $C(11, -3)$. Find
(i) the midpoint of the diagonal AC,
(ii) the fourth vertex D.

4. **(a)** Explain why $E(k, 4)$ is not the midpoint of the line segment joining the points $P(k, 5.5)$ and $Q(k, 7)$. What can you say about the points E, P and Q?
(b) Use two different methods to explain why $F(-6, 1)$ is not the midpoint of the line segment joining the points $P(-7, -1)$ and $Q(-5, 5)$.

5. Three of the vertices of a rhombus $PRQS$ are $P(1, -2)$, $R(5, 0)$ and $Q(7, 4)$. Find the fourth vertex S.

6. The line $y = x + 2$ intersects the curve $y = x^2 + 5x - 3$ at the points P and Q. Find the coordinates of the midpoint of PQ.

7. Given that the line $x + 2y = 5$ meets the curve $5x^2 + 4y^2 = 29 - 12x$ at the points A and B,
(i) find the coordinates of the midpoint of AB,
(ii) calculate the length of AB.

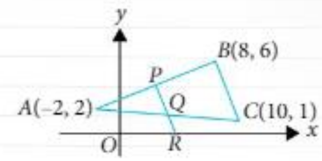
8. $ABCD$ is a rectangle.
(a) How will the midpoint of AC change when the rectangle is moved 4 units in the positive direction of the x-axis?

Basic Intermediate Advanced

Exercise 5D

(b) The coordinates of A and C are (h, k) and (m, n) respectively. Given that the rectangle is moved k units in the negative direction of the y-axis and then k units in the negative direction of the x-axis, find an expression, in terms of h, k, m and n, for the midpoint of AC.

9. The diagram shows a triangle ABC with vertices at $A(-2, 2)$, $B(8, 6)$ and $C(10, 1)$. P and Q are the midpoints of AB and AC respectively. PQ is parallel to BC.



(i) Given that PQ produced meets the x-axis at R, find the coordinates of R.

(ii) Is Q the midpoint of PR? Explain your answer.

10. In $\triangle PQR$, the midpoints of the sides PQ, QR and PR are $A(-2, 3)$, $B(5, -1)$ and $C(-4, -7)$ respectively. Find the coordinates of P, Q and R.

5.5 Parallel and perpendicular lines

In Book 2, we learnt that the **gradient** of a line is a **measure** of how steep the line is. It is the ratio of the vertical change (or rise) to the horizontal change (or run): gradient = $\dfrac{\text{vertical change}}{\text{horizontal change}}$ or $\dfrac{\text{rise}}{\text{run}}$.

In Section 5.2, we have also learnt how to calculate the gradient of a line segment joining (x_1, y_1) and (x_2, y_2): $\dfrac{y_2 - y_1}{x_2 - x_1}$.

In this section, we will explore the relationship between the gradient of a line and the *angle of inclination* θ made by the line with the positive direction of the x-axis.

Investigation — Angle of inclination

Fig. 5.11 shows two lines l_1 and l_2.

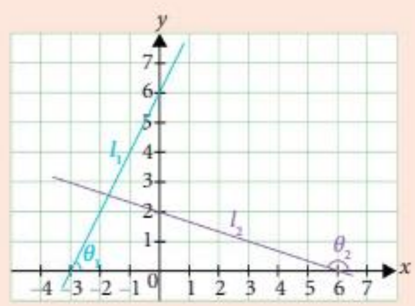


Fig. 5.11

Big Idea

Diagrams
The graph is useful to help us visualise the relationship between the gradient of a line and its angle of inclination.

1. θ_1 is the angle made by the line l_1 with the positive direction of the x-axis.
   θ_1 is called the *angle of inclination* of the line l_1.
   (a) Measure θ_1 with a protractor.
   (b) Use a calculator to evaluate $\tan \theta_1$, giving your answer to 2 significant figures.
2. Use $\dfrac{\text{rise}}{\text{run}}$ to find the gradient m_1 of the line l_1. Is $m_1 = \tan \theta_1$? Explain your answer.
   Hint: $\tan \theta_1 = \dfrac{\text{opposite side}}{\text{adjacent side}}$
3. θ_2 is the *angle of inclination* of the line l_2. Is θ_2 acute or obtuse?
4. **(a)** Measure θ_2 with a protractor.
   (b) Use a calculator to evaluate $\tan \theta_2$, giving your answer to 2 significant figures. Is $\tan \theta_2$ positive or negative?
5. Use $\dfrac{\text{rise}}{\text{run}}$ to find the gradient m_2 of the line l_2. Is $m_2 = \tan \theta_2$? Explain your answer.
   Hint: $\tan \theta_2 = -\dfrac{\text{opposite side}}{\text{adjacent side}}$
6. State the relationship between the gradient m of a line and its angle of inclination θ.
   Does this relationship apply if the line is horizontal or vertical? Explain your answer.

Information

You will learn about trigonometric ratios of obtuse angles in Book 4.

In general, if θ is the angle of inclination of a line, then:

Gradient of line = $\tan \theta$

B. Parallel lines

Class Discussion **Parallel lines**

Work in pairs. One student is to refer to Fig. 5.12 and the other student is to refer to Fig. 5.13.

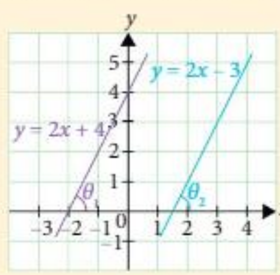


Fig. 5.12

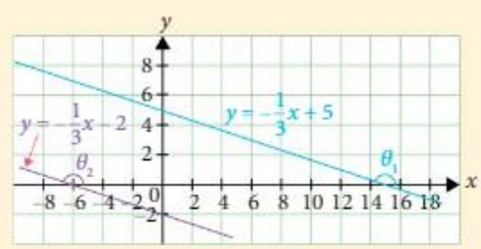


Fig. 5.13

1. Do the two lines look parallel to each other?
2. How do you know that the two lines are really parallel to each other? Use a protractor to measure θ_1 and θ_2.
   Is $\theta_1 = \theta_2$? Does this relationship tell you whether the two lines are parallel?
   Hint: Corresponding angles formed by two parallel lines and a transversal are equal.
3. Look at the equation of each of the two lines. State the gradient of each line. Do they have the same gradient?
4. **(a)** Compare your answers to Question 2 with those of your classmate. Hence, explain if the two lines in Fig. 5.12 are parallel to those in Fig. 5.13.

(b) Compare your answers to Question 3 with those of your classmate. What does that tell you about the gradient of parallel lines?

5. Copy and complete the following statements.
   (a) If two lines are parallel to each other, then they have the same ______.
   (b) If two lines have the same gradient, then they are ______ to each other.

In general, for two lines l_1 and l_2 with gradients m_1 and m_2 respectively,

l_1 is parallel to l_2 $\Leftrightarrow$ their gradients are equal i.e. $m_1 = m_2$.

We use the **notation //** to denote 'is parallel to'. For example, AB // CD means AB is parallel to CD.

Big Idea

Notations
The notation // helps to convey the idea that the lines are parallel in a concise and precise manner.

Thinking Time

1. Given three points $A(2, 3)$, $B(3, 5)$ and $C(5, 9)$, find the gradients of AB, BC and AC. What can you say about the three points?
2. Given that the gradients of XY and YZ are equal, what can you say about the points X, Y and Z?

From the above Thinking Time, we see that:

Three points are collinear if the gradient of any two pairs of the points is the same.

Attention
Three points are said to be **collinear** if they lie on the same line.

Worked Example 11

Parallel lines and collinear points

Given four points $O(0, 0)$, $A(2, k)$, $B(2k, 9)$ and $C(3k, 2k + 7)$, find the value(s) of k if

(i) OA is parallel to BC,

(ii) the points O, A and B are collinear.

Solution

(i) Since OA // BC,
then gradient of OA = gradient of BC.

$$\text{i.e.}\quad \frac{k-0}{2-0} = \frac{2k+7-9}{3k-2k}$$
$$\frac{k}{2} = \frac{2k-2}{k}$$
$$k^2 = 4k - 4$$
$$k^2 - 4k + 4 = 0$$
$$(k - 2)^2 = 0$$
$$k = 2$$

Recall
$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

(ii) If O, A and B are collinear,
then gradient of OA = gradient of OB.

i.e. $$\frac{k-0}{2-0} = \frac{9-0}{2k-0}$$
$$\frac{k}{2} = \frac{9}{2k}$$
$$2k^2 = 18$$
$$k^2 = 9$$
$$k = \pm 3$$

Attention

In **(ii)**, to find the values of k, we can also use gradient of OB = gradient of AB, or gradient of OA = gradient of AB.

Practise Now 11

Similar and Further Questions
Exercise 5E
Questions 1, 7, 8, 15

The coordinates of four points are $A(0, 9)$, $B(k + 1, k + 4)$, $C(2k, k + 3)$ and $D(2k + 2, k + 6)$. Find the value(s) of k if

(i) AB is parallel to CD, **(ii)** the points A, B and C are collinear.

C. Perpendicular lines

Class Discussion — **Perpendicular lines**

Work in pairs. One student is to refer to Fig. 5.14 and the other student is to refer to Fig. 5.15.

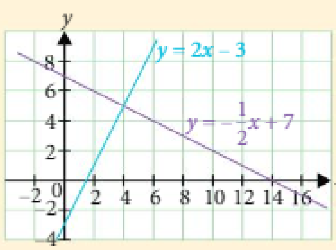


Fig. 5.14

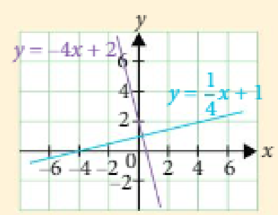


Fig. 5.15

1. Use a protractor to measure the angle between the two lines. Are the two lines perpendicular?
2. Look at the equation of each of the two lines. State the gradient of each line.
3. Calculate the product of the gradients of the two perpendicular lines.
4. Together with your classmate, compare the answers to Question 3 for Fig. 5.14 and for Fig. 5.15. What do you notice? Is this always true?
5. Does the relationship in Question 3 apply if one line is horizontal and the other is vertical? Explain your answer.

In general, for two lines l_1 and l_2 with gradients m_1 and m_2 respectively,

l_1 is **perpendicular** to l_2 $\Leftrightarrow$ product of their gradients $m_1m_2 = -1$.

However, there is an exception to the above result: if one of the lines is vertical, then its gradient is undefined and so we will not be able to calculate m_1m_2.
We use the **notation** $\perp$ to denote 'is perpendicular to'. For example, $AB \perp CD$ means AB is perpendicular to CD.

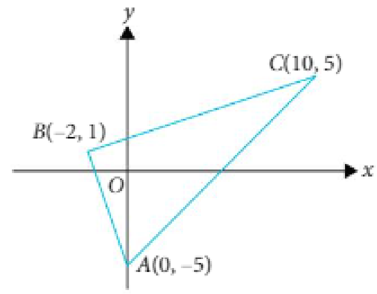
Big Idea

Notations
The notation $\perp$ helps to convey the idea that the lines are perpendicular in a concise and precise manner.

Worked Example 12

Perpendicular lines

The vertices of $\triangle ABC$ are at $A(0, -5)$, $B(-2, 1)$ and $C(10, 5)$. Is $\triangle ABC$ a right-angled triangle? Give reasons for your answer.



Solution

Method 1:

Let m_1 = gradient of AB, and m_2 = gradient of BC.

$$\text{Then } m_1 m_2 = \left(\frac{1-(-5)}{-2-0}\right)\times\left(\frac{5-1}{10-(-2)}\right)$$
$$= -1.$$

$\therefore AB \perp BC$ and so $\triangle ABC$ is a right-angled triangle.

Method 2:

$AB = \sqrt{[1-(-5)]^2+(-2-0)^2} = \sqrt{40}$ units

$BC = \sqrt{[10-(-2)]^2+(5-1)^2} = \sqrt{160}$ units

$AC = \sqrt{(10-0)^2+[5-(-5)]^2} = \sqrt{200}$ units

Since $AB^2 + BC^2 = AC^2$, by the converse of Pythagoras' Theorem, $\triangle ABC$ is a right-angled triangle.

Reflection

Which method do you prefer? Why?

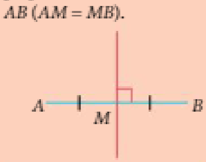
Practise Now 12

Similar and Further Questions
Exercise 5E
Questions 2–5, 9–13, 16

1. The vertices of $\triangle ABC$ are at $A(0, 1)$, $B(-1, -2)$ and $C(2, -2)$. Is $\triangle ABC$ a right-angled triangle? Give reasons for your answer.
2. The coordinates of three points are $A(1, 3)$, $B(5, 1)$ and $C(k, -1)$.
   (i) Find the coordinates of the midpoint, M, of AB.
   (ii) If MC is the perpendicular bisector of AB, find the value of k.

Recall

The perpendicular bisector of a line segment AB is a line that is perpendicular to AB and bisects AB ($AM = MB$).



Worked Example 13

Perpendicular lines

OPEN

The coordinates of three points are $O(0, 0)$, $P(m, n)$ and $Q(6, 3)$. Find a possible set of values of m and n if OP is perpendicular to OQ.

Solution

Gradient of OP × gradient of OQ = –1

$$\left(\frac{n-0}{m-0}\right)\times\left(\frac{3-0}{6-0}\right) = -1$$
$$\frac{n}{m} = -2$$
$$n = -2m$$

$\therefore$ a possible set of values is $m = 4$ and $n = -8$.

Practise Now 13

Similar and Further Questions
Exercise 5E
Questions 6, 14

OPEN The coordinates of three points are $O(0, 0)$, $A(5, 1)$ and $B(p, q)$. Find a possible set of values of p and q if OA is perpendicular to OB.

Advanced | Intermediate | Basic

Exercise 5E

1. The coordinates of four points are $O(0, 0)$, $A(2, 3k)$, $B(4k, 6)$ and $C(10k, 7)$. Find the value(s) of k if
   (i) OA is parallel to BC,
   (ii) the points O, A and B are collinear.

2. The coordinates of three points are $A(1, 1)$, $B(-1, 4)$ and $C(6, k)$. Find the value of k if AB is perpendicular to BC.

3. The vertices of $\triangle ABC$ are at $A(-1, -3)$, $B(2, 3)$ and $C(k + 5, k)$. Find the value of k if AB is perpendicular to BC.

4. The coordinates of three points are $A(-1, -6)$, $B(3, -12)$ and $C(k, 6)$. Find the value of k if
   (i) AB is perpendicular to AC,
   (ii) A, B and C are collinear.

5. Explain whether the following pairs of lines are parallel, perpendicular or neither.
   (a) $y = 7x + 14$ and $y = 7x + 16$
   (b) $y = 4x - 7$ and $y = -\frac{1}{4}x + 5$
   (c) $y = -2x + 2$ and $2y = x + 6$
   (d) $x + 2y = 6$ and $y = 3x - 8$

6. OPEN The coordinates of three points are $A(m, n)$, $B(4, 6)$ and $C(5, 3)$. Find an example of the values of m and n if AB is perpendicular to BC.

7. A point P is equidistant from $R(-2, 4)$ and $S(6, -4)$ and its x-coordinate is twice its y-coordinate.
   (i) Find the coordinates of P.
   (ii) Hence, explain why P, R and S are not collinear.

8. Explain whether the two lines with equations $3y - 2x = 4$ and $4x = 6y - 8$ are parallel to each other.

9. Show that $P(-1, 3)$, $Q(6, 8)$ and $R(11, 1)$ are the vertices of an isosceles triangle. Is $\triangle PQR$ a right-angled triangle? Give reasons for your answer.

10. Given that A is the point $(0, 4)$ and B is $(6, 6)$, find
    (a) the coordinates of point C on the x-axis such that $AB = BC$,
    (b) the coordinates of point D on the y-axis such that $\angle ABD = 90°$.

11. The line $x + 3y = 1$ intersects the curve $5y = 20 - 3x - x^2$ at the points P and Q. Given that the equations of AB and CD are $y - 3x = 4$ and $3y + x = 4$, what can you say about the lines
    (a) AB and PQ? (b) CD and PQ?

12. Give two reasons why $3x + y = 8$ is not the perpendicular bisector of the line segment joining $(-4, 1)$ and $(8, 7)$.

13. The coordinates of three points are $A(-1, -3)$, $B(2, 3)$ and $C(6, k)$. If AB is perpendicular to BC, find
    (i) the value of k,
    (ii) the gradient of AC,
    (iii) the acute angle that AC makes with the x-axis.

14. OPEN The coordinates of three points are $O(0, 0)$, $P(a, b)$ and $Q(c, d)$. Find an example of the values of a, b, c and d if OP is perpendicular to OQ.

Basic | Intermediate | Advanced

Exercise 5E

15. Use two methods to determine whether the four points (2, 1), (−1, −5), (1, 5) and (−2, −1) are the vertices of a parallelogram, showing your working clearly.

16. Use two methods to determine whether the four points (5, 8), (7, 5), (3, 5) and (5, 2) are the vertices of a rhombus, showing your working clearly.

5.6 Equation of a straight line involving parallel and perpendicular lines

In this section, we shall apply what we have learnt in Sections 5.3 and 5.5 to find the equation of a straight line, which is either parallel or perpendicular to another line.

Worked Example 14

Equation of non-vertical line

Find the equation of the line l that passes through the points $A(1, 2)$ and $B(3, 8)$. Hence, write down a possible equation of a line that is perpendicular to l.

Solution

$$\text{Gradient of } AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{3-1} = 3$$

Equation of line l is in the form $y = 3x + c$.

Since (1, 2) lies on the line, $2 = 3(1) + c$

$c = -1$

Hence, equation of l is $y = 3x - 1$.

$\therefore$ a possible equation of a line that is perpendicular to l is $y = -\frac{1}{3}x + 2$.

Practise Now 14

Similar and Further Questions
Exercise 5F
Questions 1(a), (b), 2(a)–(d), 3(a)–(d), 4, 6, 7

1. Find the equation of the line l with gradient 2 and passing through (1, 4). Hence, write down a possible equation of a line that is perpendicular to l.

2. Find the equation of the line passing through the point
   (a) (2, 7) and is perpendicular to $y = x + 5$,
   (b) (−2, 3) and is perpendicular to $2y + x = 2$,
   (c) (4, −9) and is parallel to $3y + 2x + 5 = 0$,
   (d) (−5, −2) and is parallel to $6x = 2y - 7$.

Worked Example 15

Equation of perpendicular bisector of line segment

Find the equation of the perpendicular bisector of the line segment joining $A(6, 3)$ and $B(2, 15)$.

Solution

Gradient of $AB = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{15-3}{2-6}$

$= -3$

Gradient of perpendicular bisector of $AB = \frac{1}{3}$

Midpoint of $AB = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{6+2}{2}, \frac{3+15}{2}\right)$

$= (4, 9)$

Equation of perpendicular bisector of AB is in the form

$y = \frac{1}{3}x + c.$

Since (4, 9) lies on the line,

$9 = \frac{1}{3}(4) + c$

$c = \frac{23}{3}.$

$\therefore$ equation of perpendicular bisector of AB is $y = \frac{1}{3}x + \frac{23}{3}$.

Recall

The product of the gradients of two perpendicular lines is –1.

Problem-solving Tip

The perpendicular bisector of AB passes through the midpoint of AB.

Practise Now 15

Similar and Further Questions
Exercise 5F
Questions 5(a)–(d), 8, 13

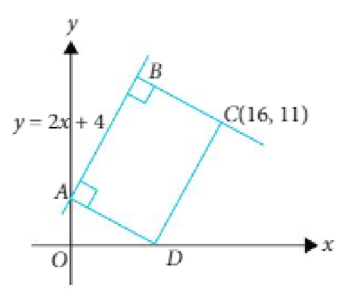
Find the equation of the perpendicular bisector of the line segment joining $C(5, 7)$ and $D(-7, 1)$.

Worked Example 16

Problem involving geometrical figure

The diagram shows a quadrilateral $ABCD$ in which the points A and D lie on the y-axis and x-axis respectively. C is the point (16, 11) and the equation of AB is $y = 2x + 4$. If AD and BC are perpendicular to AB, find

(i) the coordinates of A and of D,
(ii) the equation of BC,
(iii) the coordinates of B,
(iv) the length of CD.



Solution

(i) At A, $x = 0$, so $y = 2(0) + 4 = 4$.
$\therefore$ the coordinates of A are $(0, 4)$.
Let D be the point $(k, 0)$.
Since $AB \perp AD$ and gradient of $AB = 2$, then

gradient of $AD = -\frac{1}{2}$

$$\frac{4-0}{0-k} = -\frac{1}{2}$$

$$k = 8$$

$\therefore$ the coordinates of D are $(8, 0)$.

Problem-solving Tip

The y-coordinate of a point on the x-axis is 0 and the x-coordinate of a point on the y-axis is 0.

(ii) Gradient of BC = Gradient of $AD = -\frac{1}{2}$

Equation of BC is in the form $y = -\frac{1}{2}x + c$

Since $(16, 11)$ lies on the line,

$$11 = -\frac{1}{2}(16) + c$$

$$c = 19$$

$\therefore$ equation of BC is $y = -\frac{1}{2}x + 19$

(iii) $2x + 4 = -\frac{1}{2}x + 19$

$$4x + 8 = -x + 38$$

$$5x = 30$$

$$x = 6$$

Substitute $x = 6$ into $y = 2x + 4$:

$$y = 2(6) + 4 = 16$$

$\therefore$ the coordinates of B are $(6, 16)$.

Problem-solving Tip

To find the coordinates of B, we solve $y = 2x + 4$ and $y = -\frac{1}{2}x + 19$ simultaneously.

(iv) Length of $CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(16-8)^2 + (11-0)^2}$$

$$= \sqrt{185}$$

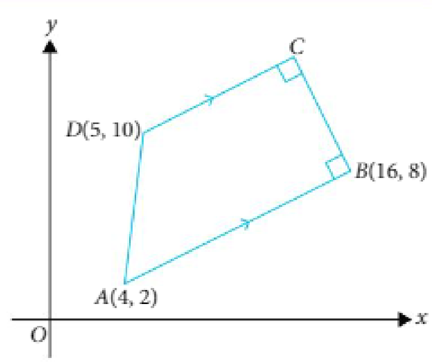
$$= 13.6 \text{ units (to 3 s.f.)}$$

Practise Now 16

Similar and Further Questions
Exercise 5F
Questions 9–12, 14

The diagram shows a trapezium $ABCD$ in which AB is parallel to DC and BC is perpendicular to both AB and DC. The coordinates of A, B and D are $(4, 2)$, $(16, 8)$ and $(5, 10)$ respectively. Find

(i) the equation of DC and of BC,
(ii) the coordinates of C,
(iii) the length of CD.



Reflection

What information do I need to find the equation of a straight line? Can I describe the strategy that I would use based on the type of information given?

Advanced | Intermediate | Basic

Exercise 5F

1. OPEN
(a) Find the equation of the line l with gradient 3 and passing through $(-2, 8)$. Hence, write down a possible equation of a line that is perpendicular to l.
(b) Find the equation of the line l with gradient $-\frac{1}{2}$ and passing through $(7, -3)$. Hence, write down a possible equation of a line that is parallel to l.

2. Find the equation of the line passing through
(a) $A(2, -1)$ and $B(5, 5)$,
(b) $C(1, -4)$ and $D(0, 6)$,
(c) $E(7, 3)$ and $F(-3, 3)$,
(d) $G(-9, -2)$ and $H(-9, 8)$.

3. Find the equation of the line passing through the point
(a) $(-2, 5)$ and is parallel to the line $3y + 7 = 29$,
(b) $(-1, -6)$ and is perpendicular to the line $42 - 7y = 5$,
(c) $(4, 8)$ and is parallel to the line $3x + y = 17$,
(d) $(2, -3)$ and is perpendicular to the line $y + 2x = 13$.

4. Find the equation of the line segment joining the points whose x-coordinates on the curve $y = 2x^2 - 3$ are -1 and 1.

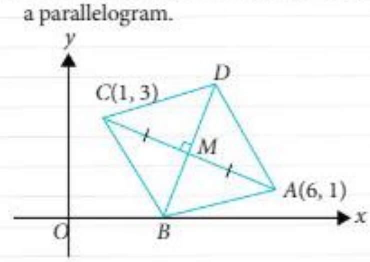
5. Find the equation of the perpendicular bisector of the line segment joining the points
(a) $(0, -2)$ and $(2, 0)$, **(b)** $(1, 8)$ and $(2, 3)$,
(c) $(0, -7)$ and $(6, -7)$, **(d)** $(5, -4)$ and $(5, 9)$.

6. Find the equation of the line that is parallel to $2y - x = 7$ and bisects the line segment joining the points $(3, 1)$ and $(1, -5)$.

7. Given that the x-intercept of a line is twice its y-intercept and that the line passes through the point of intersection of the lines $3y + x = 3$ and $4y - 3x = 5$, find the equation of this line.

8. A rhombus $ABCD$ is such that the coordinates of A and C are $(0, 2)$ and $(12, 8)$ respectively. Find the equation of the perpendicular bisector of AC and of BD.

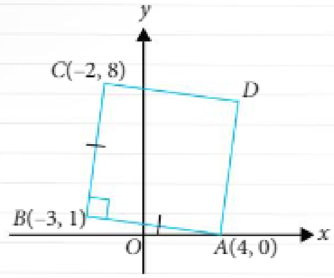
9. The diagram shows a quadrilateral $ABCD$ where A is $(6, 1)$, B lies on the x-axis and C is $(1, 3)$. The diagonal BD bisects AC at right angles at M. Find
(i) the equation of BD,
(ii) the coordinates of B,
(iii) the coordinates of D such that $ABCD$ is a parallelogram.



Basic Intermediate Advanced

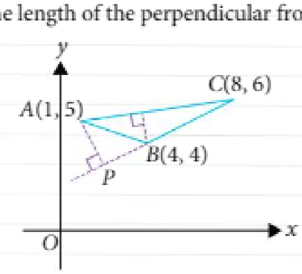
Exercise 5F

10. The coordinates of three points are $A(4, 0)$, $B(-3, 1)$ and $C(-2, 8)$.

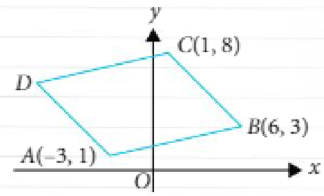


(i) Find the equation of the line segment joining the points $A(4, 0)$ and $C(-2, 8)$.
(ii) Explain why $AB = BC$ and that $\angle ABC = 90°$, showing your working clearly.
(iii) If A, B and C are the three vertices of a square, find the coordinates of the fourth vertex, D.
(iv) Hence, or otherwise, calculate the area of the square.

11. The vertices of $\triangle ABC$ are at $A(1, 5)$, $B(4, 4)$ and $C(8, 6)$. Given that P is the foot of the perpendicular from A to CB produced, find
(i) the equation of AP,
(ii) the coordinates of P,
(iii) the lengths of AP, BC and AC,
(iv) the area of $\triangle ABC$,
(v) the length of the perpendicular from B to AC.



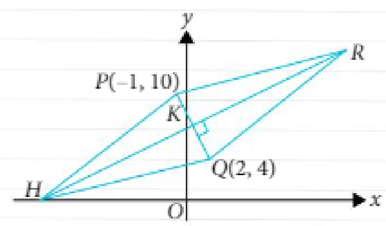
12. In the diagram, the coordinates of three points A, B and C are $A(-3, 1)$, $B(6, 3)$ and $C(1, 8)$.



(i) Find the gradient of BC and of AB.
(ii) If H is the point on the y-axis such that B, C and H are collinear, find the coordinates of H.
(iii) Find the coordinates of the point D such that $ABCD$ is a parallelogram.
(iv) Find the equation of the perpendicular bisector of BC.

13. The coordinates of the vertices of a triangle ABC are $A(1, 2)$, $B(6, 7)$ and $C(7, 2)$. Find the equations of the perpendicular bisectors of
(a) AB, **(b)** BC.
Hence, find the coordinates of the point equidistant from A, B and C.

14. The coordinates of the points P and Q are $P(-1, 10)$ and $Q(2, 4)$.



(i) Find the equation of the perpendicular bisector of the line joining P and Q.
(ii) If the perpendicular bisector of PQ cuts the x- and y-axes at points H and K respectively, calculate the length of HK, giving your answer correct to one decimal place.
(iii) Find the coordinates of R such that $PRQH$ is a parallelogram.
(iv) Show that $PRQH$ is a rhombus.

Looking Back

In this chapter, we recapped some ideas which we previously learnt in Book 2. We also learnt how the Cartesian coordinate system can be used to represent a geometric problem as an algebraic problem, and vice versa. These show us how important it is to develop good mathematical **notations** to communicate ideas in mathematics, as well as how useful mathematical **diagrams** can be in helping us solve problems. In advanced mathematics courses, you will see how geometric objects such as circles, ellipses, rectangles and 3-D objects, such as spheres and other surfaces, can be represented using algebraic equations with the use of this simple invention.
Coordinate geometry also helps us to quantify properties of geometric objects to analyse and compare them, such as the **measure** of gradient. Gradients of parallel lines are equal. The product of gradients of perpendicular lines is always equal to –1 (with an exception), even when the perpendicular lines are rotated by the same amount, which is an example of **invariance**.

Summary

1. Gradient of a line segment

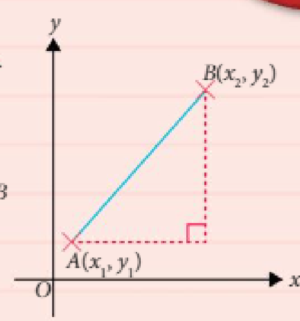
If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a line, then gradient of AB is $\frac{y_2 - y_1}{x_2 - x_1}$.

- Find the gradient of a line passing through the points (–7, 8) and (5, –5).

Length of a line segment

The length of any line segment AB, where the coordinates of the points A and B are (x_1, y_1) and (x_2, y_2) respectively is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

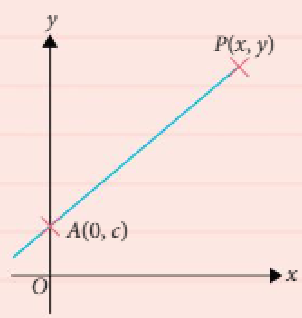
- Find the length of a line segment with endpoints (–7, 8) and (5, –5).



2. Equation of a straight line

The equation of a straight line passing through the point $(0, c)$ and with gradient m is $y = mx + c$.

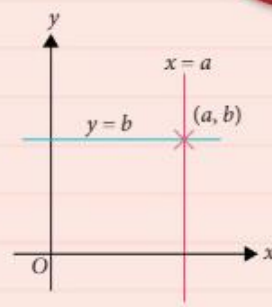
- Find the equation of a line passing through the points (–7, 8) and (5, –5).



Summary

3. **Equation of a horizontal line**
The equation of a straight line that is parallel to the x-axis and which passes through the point (a, b) is $y = b$.
It has a gradient of **0**.

Equation of a vertical line
The equation of a straight line that is parallel to the y-axis and which passes through the point (a, b) is $x = a$.
Its gradient is **undefined**.



4. If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two distinct points, then:

- **Midpoint** of $AB = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- **Gradient** of $AB = \tan\theta$, where θ is the angle of inclination

5. For two lines l_1 and l_2 with gradients m_1 and m_2 respectively,

- l_1 is **parallel** to $l_2 \Leftrightarrow$ their gradients are equal, i.e. $m_1 = m_2$
- l_1 is **perpendicular** to $l_2 \Leftrightarrow$ product of their gradients $m_1 m_2 = -1$

CHAPTER 6

Graphs of Functions and Graphical Solution

In the preceding chapter, we saw how the invention of the Cartesian coordinates system provided the link between algebra and geometry. Graphs provide us a way to visualise functions that we use to model many real-world situations.

For example, we can model the average cost, A, of producing x units of smartphones using a reciprocal function of the form $A = px + \frac{q}{x} + r$, where p, q and r are known constants. By drawing a graph of A against x, we can determine the number of units to produce for the minimum average cost.

In this chapter, we will explore the connection between the graphs and equations of cubic, reciprocal, exponential and rational functions. More specifically, we will see how the graphs of these functions can provide an alternative to solving an equation algebraically.

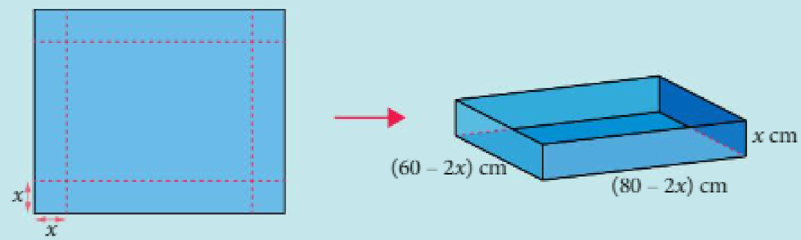
Learning Outcomes

What will we learn in this chapter?

- How to draw the graphs of simple sums of power functions $y = ax^n$, where $n = 3, 2, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1$ and -2
- How to draw the graphs of exponential functions $y = ka^x + b$, where a is a positive integer, $a \neq 1$, and k and b are rational numbers
- How to draw the graphs of rational functions and find their asymptotes
- How to sketch the graphs of cubic, reciprocal and exponential functions
- How to solve an equation graphically
- How to estimate the gradient of a curve by drawing a tangent
- How to interpret and analyse data from tables and graphs, including distance-time and speed-time graphs

Introductory Problem

To reduce wastage of paper-based packaging, a company makes open cardboard boxes (no cover) from recycled rectangular cardboard pieces, each with dimensions of 80 cm by 60 cm. Squares of sides x cm are cut from each corner of the cardboard and the resulting flaps are folded to make an open box as shown:



(a) Can the same rectangular cardboard be used to build a box with a volume of 50 000 cm³? Why or why not?

(b) What should x be so that the volume of the box is a maximum?

How did you solve the **Introductory Problem**? Did you write a function that describes the relationship between the volume of the box and the length of the square, x cm?

To do so, we need to show that the volume of the box is equal to $x(80 - 2x)(60 - 2x)$. This can be expressed as $4x^3 - 280x^2 + 4800x$, which is an example of a **cubic function**. We have not learnt to solve this algebraically. How can we approach this problem graphically?

6.1 Graphs of cubic functions

A. Graphs of cubic functions

In Chapter 2, we have learnt to draw the graphs of $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$. We have also learnt to sketch graphs of the form $y = (x - h)(x - k)$ and $y = -(x - h)(x - k)$, where h and k are constants, as well as $y = (x - p)^2 + q$ and $y = -(x - p)^2 + q$, where p and q are constants.

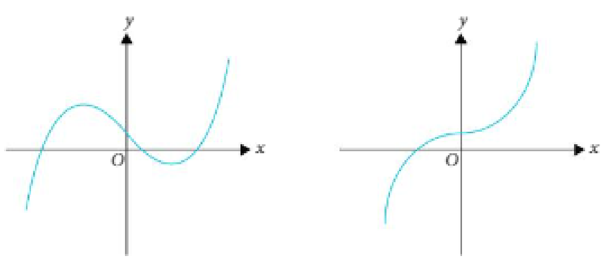
Let us now learn how to draw the graphs of cubic functions.

In general, cubic functions are of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants and $a \neq 0$.

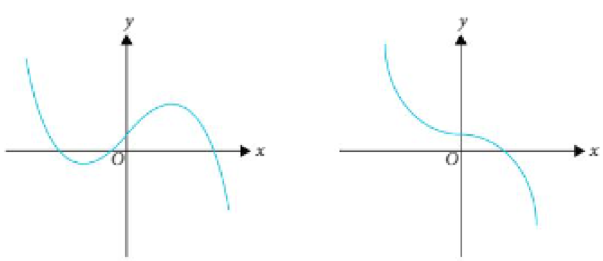
Investigation — Graphs of cubic functions

1. Using a graphing software, draw each of the following graphs.
   (a) $y = x^3$ (b) $y = 2x^3$
   (c) $y = 5x^3$ (d) $y = -x^3$
   (e) $y = -2x^3$ (f) $y = -5x^3$
2. Using your graphs in Question 1, answer each of the following questions.
   (a) Describe the shape of the graph of $y = ax^3$, where a is a constant.
   (b) Describe the effect of the value of a on the shape of the graph of $y = ax^3$.
   (c) How do you think the graph of $y = 0.5x^3$ would look like when compared to the graph of $y = x^3$?
3. The general form of the equation of a cubic function is $y = ax^3 + bx^2 + cx + d$. Using a graphing software, explore how the graph of a cubic function may look like by drawing the graphs of the following functions.
   (a) $y = x^3 + x^2$ (b) $y = x^3 + x^2 + x$
   (c) $y = x^3 + x^2 + x + 1$ (d) $y = x^3 - x^2$
   (e) $y = x^3 - x^2 + x$ (f) $y = x^3 - x^2 - x$
   (g) $y = x^3 - x^2 - x - 1$
   Describe what you notice about the general shape of the graph of a cubic function.
4. For each of the graphs in Question 3, change the coefficient of x^3 to -1. How does this affect the shape of the graphs?
5. Write down your observations about the graphs of a cubic function.

From the above Investigation, we observe that the graph of a cubic function, i.e. a function of the form $y = ax^3 + bx^2 + cx + d$, would take the shapes shown in Fig. 6.1.
When will the graph pass through the origin?



(a) $y = ax^3 + bx^2 + cx + d$, where $a > 0$



(b) $y = ax^3 + bx^2 + cx + d$, where $a < 0$

Fig. 6.1

Big Idea

Diagrams
The graph of a function expresses important information about the function in a visual way. The shape of the graph, the intersection points with the axes and the number of turning points are crucial for understanding the algebraic behaviour of the function. We need to pay attention to these features when drawing the graph so as to express the algebraic properties of the function accurately.

Worked Example 1

Drawing the graph of a cubic function

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis, draw the graph of $y = x^3 + 3$ for $-3 \leqslant x \leqslant 3$.

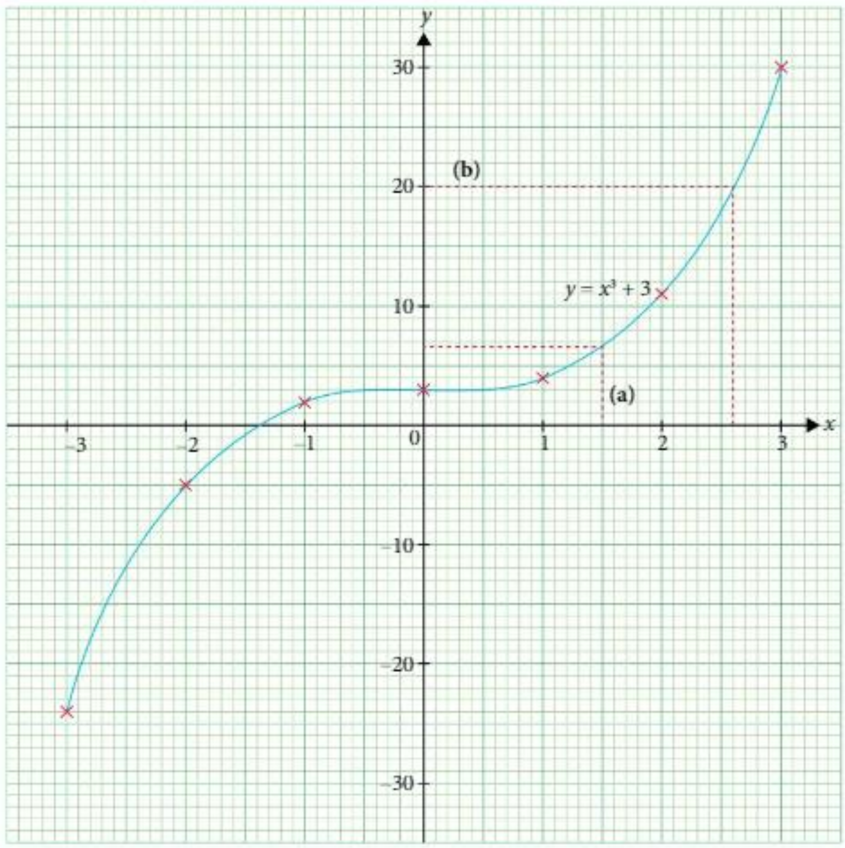
Find

(a) the value of y when $x = 1.5$,

(b) the value of x when $y = 20$.

*Solution

x	-3	-2	-1	0	1	2	3
$y = x^3 + 3$	-24	-5	2	3	4	11	30



(a) From the graph, when $x = 1.5$, $y = 6.5$.

(b) From the graph, when $y = 20$, $x = 2.6$.

Practise Now 1

Similar and Further Questions
Exercise 6A
Questions 1, 8

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis, draw the graph of $y = -x^3 - 2$ for $-3 \leqslant x \leqslant 3$.

Find

(a) the value of y when $x = 2.5$,

(b) the value of x when $y = 15$.

Introductory Problem Revisited

Can you solve the **Introductory Problem** now? The graph of a function provides a way for us to understand the algebraic properties of the given function. This also enables us to use a graph of a function to approximate or find the solutions to an algebraic equation that represents the function.

B. Sketching graphs of cubic functions

In Section 6.1A, we have learnt about the shapes of the graphs of cubic functions and how to plot them. We shall now learn to sketch them. Recall that in Chapter 2, to sketch a graph, we only need to plot the critical points which the curve passes through:

- the point where the curve cuts the y-axis,
- the points where the curve cuts the x-axis (if any), and
- the turning point.

However, finding the turning point of the graph of a cubic function is beyond the scope of this book. As such, we shall only look at the x- and y-intercepts in sketching its graph.

Worked Example 2

Sketching graph of a cubic function with a single x-intercept

Sketch the graph of $y = x^3 - 27$.

Solution

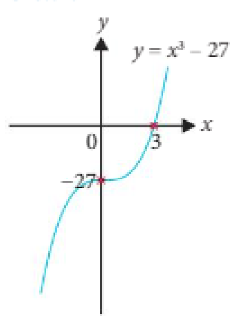
Find y-intercept:

When $x = 0$, $y = x^3 - 27$
$= 0^3 - 27$
$= -27$ y-intercept

Find x-intercept(s):

When $y = 0$, $x^3 - 27 = 0$
$x^3 = 27$
$x = 3$ x-intercept

Sketch:



Problem-solving Tip

Since the coefficient of x^3 is positive, the graph has a shape similar to Fig. 6.1(**a**).

Practise Now 2

Similar and Further Questions
Exercise 6A
Questions 2(a)-(d)

Sketch the graph of $y = -x^3 - 64$.

Problem-solving Tip

Since the coefficient of x^3 is negative, the graph has a shape similar to Fig. 6.1**(b)**.

Worked Example 3

Sketching graph of a cubic function with multiple x-intercepts

Sketch the graph of $y = x^3 - 2x^2 - 8x$.

*Solution

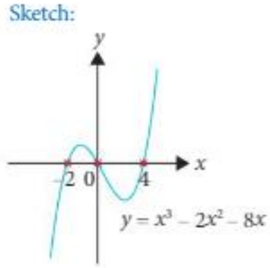
Find y-intercept:

When $x = 0$, $y = x^3 - 2x^2 - 8x$
$= 0^3 - 2(0)^2 - 8(0)$
$= 0$ y-intercept

Find x-intercept(s):

When $y = 0$, $x^3 - 2x^2 - 8x = 0$
$x(x^2 - 2x - 8) = 0$ factorise by extracting common factor x
$x(x + 2)(x - 4) = 0$ factorise using multiplication frame
$x = 0$ or $x = -2$ or $x = 4$ x-intercepts

Sketch:



Practise Now 3

Similar and Further Questions
Exercise 6A
Questions 9(a)-(d)

Sketch the graph of $y = -x^3 - 6x^2 + 9x$.

C. Graphical solution

Investigation **Graphical solution to equations**

1. Find the value(s) of x which satisfy the following simultaneous equations by plotting the graphs using a graphing software.
   $y = 2x^3$ and $y = x + 1$
2. Find the value(s) of x which satisfy the following simultaneous equations by plotting the graphs using a graphing software.
   $y = 2x^3 - 1$ and $y = x$
3. What did you notice about the equations in Questions 1 and 2?
4. Without plotting the graph, write down the solution of $2x^3 - x - 1 = 0$. Explain how you figured out the solution without drawing the graph.
5. What did you observe about how simultaneous equations can be solved using the graphical method?

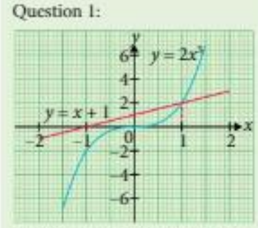
As seen from the above Investigation, the key to solving equations by the graphical method is to identify the graphs to be drawn. For example, to solve the equation $2x^3 - 1 = x$, you will need to draw the graphs of $y = 2x^3 - 1$ and $y = x$ and then find the x-coordinate(s) of the point(s) of intersection.

In general, for a single-variable equation in terms of x of the form Expression 1 = Expression 2, we will need to draw the graphs of y = Expression 1 and y = Expression 2. The solution(s) will be given by the x-coordinate(s) of the intersection point(s) of these two graphs. Furthermore, we note that there are a few ways to decide on the graphs to draw. The given equation can be solved graphically by drawing other equivalent sets of equations, obtained by algebraic manipulations that preserve the **equivalence** of the original equation.

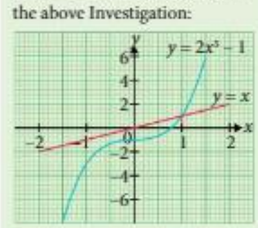
Big Idea

Equivalence

To solve a given equation such as $2x^3 - x - 1 = 0$ graphically, we can use different equivalent equations such as $2x^3 = x + 1$ in Question 1:



or $2x^3 - 1 = x$ in Question 2 in the above Investigation:



What other equivalent equations can be used (or what other graphs can be drawn) to solve $2x^3 - x - 1 = 0$?

Worked Example 4

Graphical solution involving a cubic function

The variables x and y are connected by the equation $y = -x^3 - 3$.
Some corresponding values of x and y are given in the table below.

x	-3	-2	-1	0	1	2	3
y	p	5	-2	-3	-4	-11	-30

(a) Find the value of p.

(b) Using a scale of 2 cm to represent 1 unit, draw a horizontal x-axis for $-3 \leqslant x \leqslant 3$.
Using a scale of 2 cm to represent 10 units, draw a vertical y-axis for $-30 \leqslant y \leqslant 30$.
On your axes, plot the points given in the table and join them with a smooth curve.

(c) The equation $-x^3 = 10$ has only one solution. Explain how this can be seen from your graph.

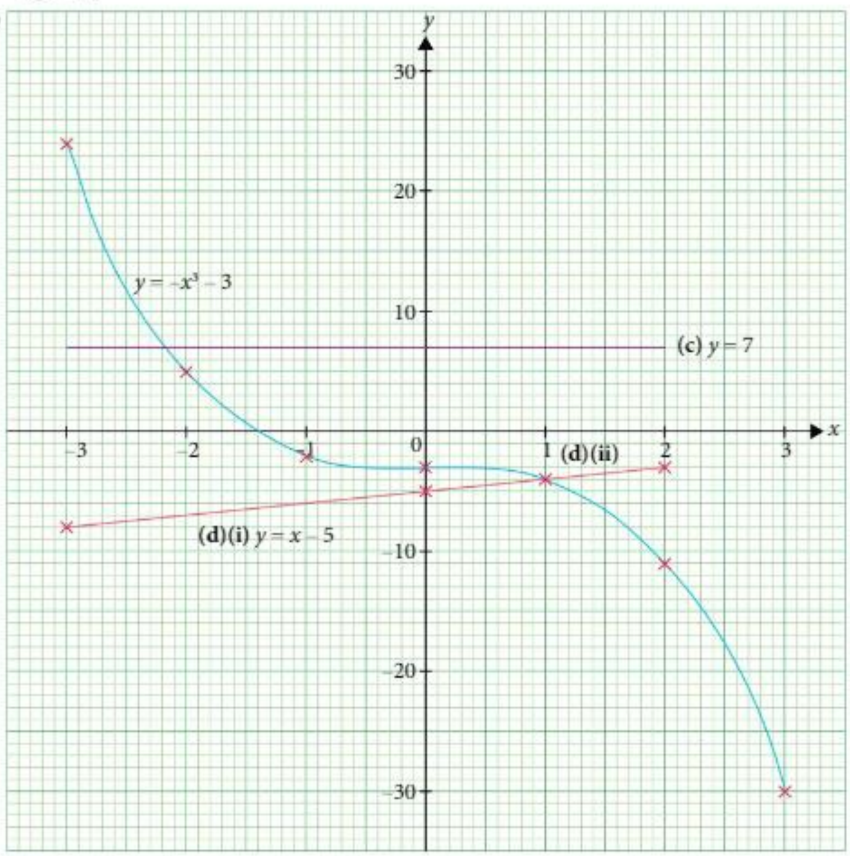
(d) **(i)** On the same axes, draw the line $y = x - 5$ for $-3 \leqslant x \leqslant 2$.
(ii) Write down the x-coordinate of the point where this line intersects the curve.
(iii) This value of x is a solution of the equation $x^3 + Ax + B = 0$.
Find the value of A and of B.

*Solution

(a) When $x = -3$, $y = -(-3)^3 - 3$
$= 24$

$\therefore p = 24$

(b)



(c) $-x^3 = 10$

$-x^3 - 3 = 10 - 3$

$-x^3 - 3 = 7$

From the graph, the line $y = 7$ only intersects the curve $y = -x^3 - 3$ at one point.

Hence, $-x^3 = 10$ has only one solution.

(d) (i)

x	-3	0	2
$y = x - 5$	-8	-5	-3

(ii) From the graph, the line $y = x - 5$ intersects the curve $y = -x^3 - 3$ at $x = 1$.

(iii) At the point of intersection,

$x - 5 = -x^3 - 3$

$x^3 + x - 2 = 0$

Comparing $x^3 + x - 2 = 0$ with $x^3 + Ax + B = 0$,

$A = 1$ and $B = -2$.

Problem-solving Tip

In (c), manipulate the equation $-x^3 = 10$ to an equivalent equation in which the LHS or RHS corresponds to the expression $-x^3 - 3$. Then we would obtain a pair of simultaneous equations that we can solve graphically.

Big Idea

Equivalence

The coordinates of the points of intersection of the graphs of $y = -x^3 - 3$ and $y = x - 5$ will satisfy the equations of both graphs simultaneously so that they will also satisfy the cubic equation $-x^3 - 3 = x - 5$ and any other equivalent equations such as $x^3 + x - 2 = 0$.

Practise Now 4

Similar and Further Questions
Exercise 6A
Questions 13, 14

The variables x and y are connected by the equation $y = x^3 + 2$.
Some corresponding values of x and y are given in the table below.

x	-3	-2	-1	0	1	2	3
y	-25	-6	1	2	3	10	q

(a) Find the value of q.

(b) Using a scale of 2 cm to represent 1 unit, draw a horizontal x-axis for $-3 \leqslant x \leqslant 3$.
Using a scale of 2 cm to represent 10 units, draw a vertical y-axis for $-30 \leqslant y \leqslant 30$.
On your axes, plot the points given in the table and join them with a smooth curve.

(c) (i) On the same axes, draw the line $y = 5 - 2x$ for $0 \leqslant x \leqslant 3$.

(ii) Write down the x-coordinate of the point where this line intersects the curve.

(iii) This value of x is a solution of the equation $x^3 + Ax + B = 0$.
Find the value of A and of B.

Reflection

1. When drawing the graph (manually or using a graphing software), it is important to consider the scale of the axes so that the important features of the graph are clearly shown. How do I decide on the scale of the axes if they are not given?
2. When solving equations by the graphical method, how do I decide on the equations of the graphs to be drawn?

6.2 Graphs of reciprocal functions

A. Graphs of $y = \dfrac{a}{x}$

Consider the following scenarios:

(a) David wants to make a rectangular flowerbed with an area of 16 m^2. If the length of the rectangular flowerbed is x m, write down an expression for the breadth of the flowerbed.

(b) The movement of an autonomous vehicle can be programmed to travel at a constant speed of y km/h on the roads. Given that the distance to be travelled is 200 km, find the time (in hours) needed to complete the journey.

(c) Boyle's Law states that the product of pressure P and volume V is a constant k for a given mass of confined gas and this holds as long as the temperature is constant. Express

  (i) P in terms of V,

  (ii) V in terms of P.

What is common in each of these scenarios?

Each of these scenarios can be described by the **reciprocal function**, i.e., $y = \dfrac{a}{x}$.

Investigation — Graphs of $y = \dfrac{a}{x}$

Using a graphing software, draw the graph of $y = \dfrac{a}{x}$ for

(a) $a = 1$, **(b)** $a = 5$,

(c) $a = -1$ **(d)** $a = -5$.

1. **(i)** For $a > 0$, which quadrants do the graphs lie in? Can you explain why?

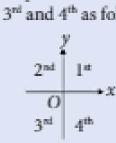
  (ii) For $a < 0$, which quadrants do the graphs lie in? Can you explain why?

2. What can you say about the rotational symmetry of each of the graphs?

3. Do the graphs intersect the x-axis and the y-axis? Explain your answer.

Information

The four quadrants on the Cartesian plane are labelled 1st, 2nd, 3rd and 4th as follows:

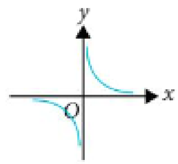


Recall

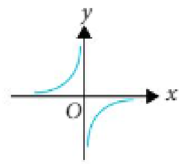
The order of rotational symmetry about a particular point is the number of distinct ways in which a figure can map onto itself by rotation in 360°.

From the Investigation on page 168, we observe that for the graph of $y = \frac{a}{x}$,

- when $x = 0$, the function $y = \frac{a}{x}$ is not defined, i.e. there is a break in the graph when $x = 0$,
- there is rotational symmetry of order 2 about the origin, i.e. it maps onto itself twice by rotation in 360°,
- if $a > 0$, the graph would take the shape in Fig. 6.2(**a**),
  if $a < 0$, the graph would take the shape in Fig. 6.2(**b**).



(**a**) $y = \frac{a}{x}$, where $a > 0$



(**b**) $y = \frac{a}{x}$, where $a < 0$

Fig. 6.2

Consider the graph $y = \frac{a}{x}$ in Fig. 6.2(**a**), where $a > 0$. The graph consists of two parts; the first part lies in the 1st quadrant because when $x > 0$, $y > 0$; the second part lies in the 3rd quadrant because when $x < 0$, $y < 0$. In the 1st quadrant, we observe that:

- as x increases, y decreases;
- as x approaches zero, y becomes very large;

  e.g. for $a = 1$, $y = \frac{1}{x}$, if $x = 0.000\,001$, $y = \frac{1}{0.000\,001} = 1\,000\,000$;
- as x becomes very large, y approaches zero;

  e.g. for $a = 1$, $y = \frac{1}{x}$, if $x = 1\,000\,000$, $y = \frac{1}{1\,000\,000} = 0.000\,001$;
- the curve gets very close to the x-axis and y-axis but never touches them.

Can you describe the part of the graph that is in the 3rd quadrant?

Can you describe the graph of $y = \frac{a}{x}$, where $a < 0$?

Information

For $y = \frac{a}{x}$ where $a > 0$, we see that as the positive x-value approaches 0, the y-value of the function gets arbitrarily large. We say that the value of y approaches *infinity* as x approaches 0, written as $y \to \infty$ as $x \to 0$. In this case, the graph approaches the x-axis as close as possible without touching it, and the x-axis is called an *asymptote*. We will learn more about them on the next page.

From what we have observed, as x becomes very large, the curve gets very close to the x- and y-axes but never touches them. A line in which a curve approaches to but will never meet it is called an *asymptote*. Hence, there are two asymptotes here, the horizontal x-axis and the vertical y-axis, which are also known as the *horizontal and vertical asymptotes* of the curve respectively.

The equation of a horizontal asymptote is of the form $y = a$ and that of a vertical asymptote is of the form $x = a$, where a is a real number. Thus, the asymptotes of the graph $y = \frac{a}{x}$ are $y = 0$ (x-axis) and $x = 0$ (y-axis).

Information

An asymptote that is neither horizontal nor vertical is called an *oblique asymptote*, which is not in the scope of this book. You may search the Internet to learn more about it.

Attention

The equation representing the x-axis is $y = 0$ and *not* $x = 0$. Similarly, the equation representing the y-axis is $x = 0$ and *not* $y = 0$.

Thinking Time

What are the equations of the lines of symmetry of the graph $y = \frac{a}{x}$ when

(a) $a > 0$? **(b)** $a < 0$?

Worked Example 5

Drawing the graph of $y = \frac{a}{x}$

Using a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis, draw the graph of $y = \frac{6}{x}$ for $-5 \leqslant x \leqslant 5$, $x \neq 0$.

(a) Find

(i) the value of y when $x = 1.4$,

(ii) the value of x when $y = -8$.

(b) The equation $\frac{6}{x} - x + 1 = 0$ can be solved by drawing a suitable straight line on the same axes.

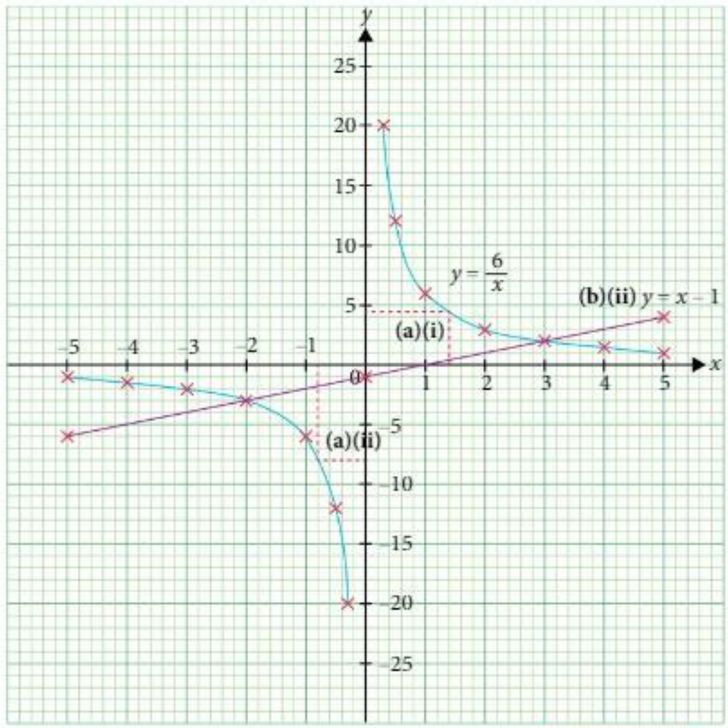
(i) Find the equation of the straight line.

(ii) By drawing this straight line, solve the equation $\frac{6}{x} - x + 1 = 0$.

Solution

(a)

x	−5	−4	−3	−2	−1	−0.5	−0.3	0.3	0.5	1	2	3	4	5
y	−1.2	−1.5	−2	−3	−6	−12	−20	20	12	6	3	2	1.5	1.2



(i) From the graph, when $x = 1.4$, $y = 4.5$.

(ii) From the graph, when $y = -8$, $x = -0.8$.

(b) (i) $\frac{6}{x} - x + 1 = 0$

$$\frac{6}{x} = x - 1$$

$\therefore$ the straight line to be drawn is $y = x - 1$.

(ii)

x	−5	0	5
$y = x - 1$	−6	−1	4

From the graph, the line $y = x - 1$ and the curve $y = \frac{6}{x}$ intersect at (3, 2) and (−2, −3).

$\therefore x = 3$ or $x = -2$

Attention

For **(a)(i)**, although the answer is 4.2857… by calculation, the answer obtained from the graph can only be accurate up to half of a small square grid, which is 0.5. Similarly, for **(a)(ii)**, although $x = -0.75$ by calculation, the answer obtained from the graph is accurate to half of a small square grid, i.e. 0.1.

Practise Now 5

Similar and Further Questions
Exercise 6A
Questions 3, 10, 11, 15–17

Using a scale of 1 cm to represent 1 unit on both axes, draw the graph of $y = \frac{3}{x}$ for $-5 \leqslant x \leqslant 5$, $x \neq 0$.

(a) Find

(i) the value of y when $x = 2.5$,

(ii) the value of x when $y = -1.2$.

(b) The equation $\frac{3}{x} - 10x - 1 = 0$ can be solved by drawing a suitable straight line on the same axes.

(i) Find the equation of the straight line.

(ii) By drawing this straight line, solve the equation $\frac{3}{x} - 10x - 1 = 0$.

B. Sketching graphs of $y = \frac{a}{x} + b$

In Section 6.2A, we have learnt about the shapes of the graphs of reciprocal functions $y = \frac{a}{x}$ and how to plot them. We see that each graph has two asymptotes, $y = 0$ (x-axis) and $x = 0$ (y-axis). What about the graph of a reciprocal function in the form $y = \frac{a}{x} + b$? The asymptotes are $y = b$ and $x = 0$ (y-axis). How can we sketch them?

Worked Example 6

Sketching graph of a reciprocal function

Sketch the graph of $y = -\frac{8}{x} + 2$.

Solution

Find asymptote(s):

When $x = 0$, y is undefined.

$\therefore$ $x = 0$ is an asymptote of the graph.

When $y = 2$, x is undefined.

$\therefore$ $y = 2$ is an asymptote of the graph.

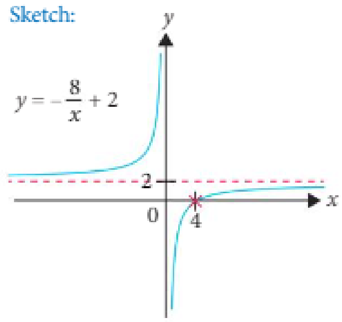
Find x-intercept(s):

When $y = 0$, $-\frac{8}{x} + 2 = 0$

$$2 = \frac{8}{x}$$

$$x = 4 \qquad x\text{-intercept}$$

Sketch:



Attention

If the asymptotes are not the x- or y-axes, draw a dotted line to show them.

Practise Now 6

Similar and Further Questions
Exercise 6A
Questions 4(a)-(d)

Sketch the graph of $y = \frac{1}{x} - 2$.

C. Graphs of $y = \frac{a}{x^2}$

There is another class of reciprocal functions that frequently occurs in nature.

For example, the intensity of light, I, decreases with the distance from a light source, d. However, instead of a simple inverse proportion between I and d, I is inversely proportional to d^2.

Similarly, the gravitational pull F between two masses is inversely proportional to r^2, where r is the distance between the centres of the two masses.

The same relationship can be found in the intensity of sound and intensity of radiation.

This relationship is broadly classified as an inverse square law, or written as $y = \frac{a}{x^2}$.

Investigation — Graphs of $y = \frac{a}{x^2}$

Using a graphing software, draw the graph of $y = \frac{a}{x^2}$ for

(a) $a = 2$,
(b) $a = 4$,
(c) $a = -1$,
(d) $a = -3$.

1. **(i)** For $a > 0$, which quadrants do the graphs lie in? Can you explain why?
(ii) For $a < 0$, which quadrants do the graphs lie in? Can you explain why?

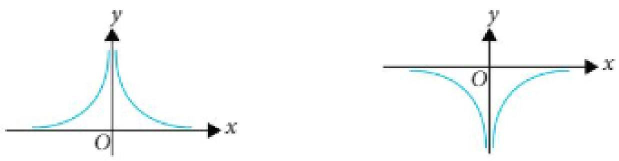
2. What can you say about the line of symmetry of each of the graphs?

3. Do the graphs intersect the x-axis and the y-axis? Explain your answer.

From the Investigation on page 173, we observe that for the graph of $y = \frac{a}{x^2}$,

when $x = 0$, the function $y = \frac{a}{x^2}$ is not defined, i.e. there is a break in the graph when $x = 0$,

- if $a > 0$, the values of y are always positive, i.e. the graph lies entirely above the x-axis;
  if $a < 0$, the values of y are always negative, i.e. the graph lies entirely below the x-axis.
- the graph is symmetrical about the y-axis, i.e. the y-axis is the line of symmetry,
- if $a > 0$, the graph would take the shape in Fig. 6.3(**a**),
  if $a < 0$, the graph would take the shape in Fig. 6.3(**b**).



(**a**) $y = \frac{a}{x^2}$, where $a > 0$ (**b**) $y = \frac{a}{x^2}$, where $a < 0$

Fig. 6.3

Consider the graph $y = \frac{a}{x^2}$ in Fig. 6.3(**a**), where $a > 0$. The graph consists of two parts; the first part lies in the 1st quadrant because when $x > 0$, $y > 0$; the second part lies in the 2nd quadrant because when $x < 0$, $y > 0$.

In the 1st quadrant, we observe that:

- as x increases, y decreases;
- as x approaches zero, y becomes very large;

  e.g. for $a = 1$, $y = \frac{1}{x^2}$, if $x = 0.001$, $y = \frac{1}{0.001^2} = 1\ 000\ 000$;
- as x becomes very large, y approaches zero;

  e.g. for $a = 1$, $y = \frac{1}{x^2}$, if $x = 1000$, $y = \frac{1}{1000^2} = 0.000\ 001$;
- the curve gets very close to the x-axis and y-axis but never touches them.

Can you describe the part of the graph that is in the 2nd quadrant?

Can you describe the graph of $y = \frac{a}{x^2}$, where $a < 0$?

Worked Example 7

Drawing the graph of $y = \frac{a}{x^2}$

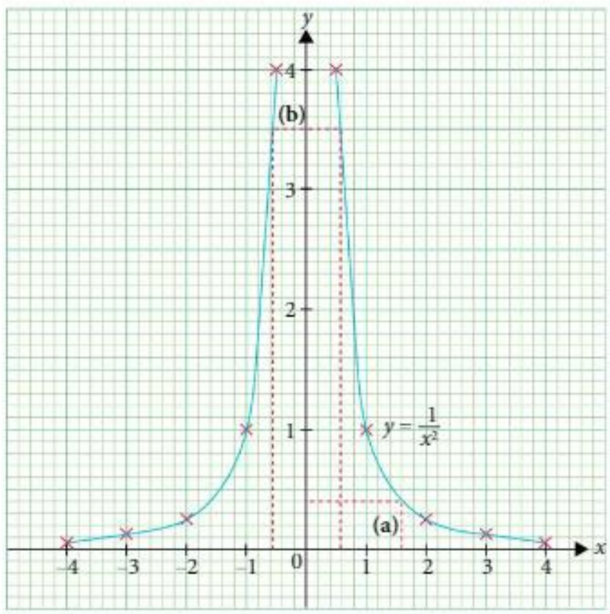
Using 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = \frac{1}{x^2}$ for $-4 \leqslant x \leqslant 4$, $x \neq 0$. Find

(a) the value of y when $x = 1.6$,

(b) the values of x when $y = 3.5$.

*Solution

x	–4	–3	–2	–1	–0.5	0.5	1	2	3	4
y	0.06	0.11	0.25	1	4	4	1	0.25	0.11	0.06



Problem-solving Tip

Based on the scale of the y-axis, we can only plot y-values to the nearest 0.05. Hence, we calculate the y-values correct to 2 d.p. in the table of values.

(a) From the graph, when $x = 1.6$, $y = 0.4$.

(b) From the graph, when $y = 3.5$, $x = -0.6$ or $x = 0.6$.

Attention

The answer can only be accurate up to half of a small square grid.

Practise Now 7

Similar and Further Questions
Exercise 6A
Questions 5, 12

Using 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = -\frac{2}{x^2}$ for $-4 \leqslant x \leqslant 4$, $x \neq 0$. Find

(a) the value of y when $x = 1.5$,

(b) the values of x when $y = -3.2$.

6.3 Graphs of functions involving $\sqrt{x}$

A. Graphs of $y = a\sqrt{x}$

The square root function $\sqrt{\ }$ has several useful applications in mathematics, which includes kinematics, statistics, trigonometry and finances. Let us now learn to draw graphs in the form $y = a\sqrt{x}$, where a is a constant.

Investigation Graphs of $y = a\sqrt{x}$

Using a graphing software, draw the graph of $y = a\sqrt{x}$ for

(a) $a = 1$, **(b)** $a = 6$,
(c) $a = -1$, **(d)** $a = -6$.

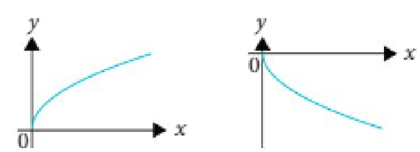
1. **(i)** For $a > 0$, which quadrant does the graph lie in? Explain.
(ii) For $a < 0$, which quadrant does the graph lie in? Explain.

2. **(a)** Describe the shape of the graph of $y = a\sqrt{x}$, where a is a constant.
(b) Describe the effect of the value of a on the shape of the graph of $y = a\sqrt{x}$.

3. Do the graphs intersect the origin? Explain.

From the above Investigation, we observe that for the graph of $y = a\sqrt{x}$,

- the graph intersects the origin,
- the function is only defined for values of $x \geqslant 0$,
- if $a > 0$, the graph would take the shape in Fig. 6.4(**a**),
  if $a < 0$, the graph would take the shape in Fig. 6.4(**b**).



(a) $y = a\sqrt{x}$, where $a > 0$ **(b)** $y = a\sqrt{x}$, where $a < 0$

Fig. 6.4

Worked Example 8

Drawing the graph of $y = a\sqrt{x}$

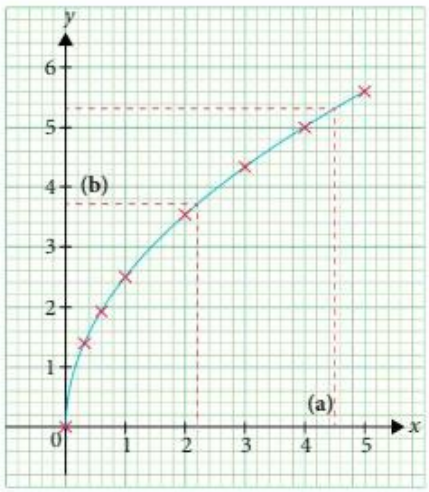
Using 1 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 2.5\sqrt{x}$ for $0 \leqslant x \leqslant 5$. Find

(a) the value of y when $x = 4.5$,

(b) the value of x when $y = 3.7$.

*Solution

x	0	0.3	0.6	1	2	3	4	5
y	0	1.4	1.9	2.5	3.5	4.3	5	5.6



Problem-solving Tip

Based on the scale of the y-axis, we can only plot y-values to the nearest 0.1. Hence, we calculate the y-values correct to 1 d.p. in the table of values.

(a) From the graph, when $x = 4.5$, $y = 5.3$.

(b) From the graph, when $y = 3.7$, $x = 2.2$.

Practise Now 8

Similar and Further Questions
Exercise 6A
Question 6

Using 1 cm to represent 2 units on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = -3\sqrt{x}$ for $0 \leqslant x \leqslant 10$. Find

(a) the value of y when $x = 8.3$,

(b) the value of x when $y = -7.7$.

B. Graphs of $y = \frac{a}{\sqrt{x}}$

Investigation Graphs of $y = \frac{a}{\sqrt{x}}$

Using a graphing software, draw the graph of $y = \frac{a}{\sqrt{x}}$ for

(a) $a = 2$, **(b)** $a = 5$,
(c) $a = -1$, **(d)** $a = -4$.

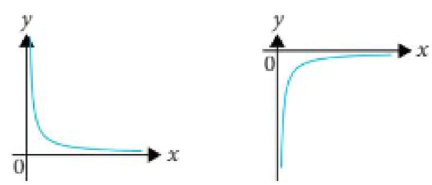
1. **(i)** For $a > 0$, which quadrant does the graph lie in? Explain.
(ii) For $a < 0$, which quadrant does the graph lie in? Explain.

2. **(a)** Describe the shape of the graph of $y = \frac{a}{\sqrt{x}}$, where a is a constant.
(b) Describe the effect of the value of a on the shape of the graph of $y = \frac{a}{\sqrt{x}}$.

3. Do the graphs intersect the x-axis and the y-axis? Explain your answer.

From the above Investigation, we observe that for the graph of $y = \frac{a}{\sqrt{x}}$,

- if $a > 0$, the values of y are always positive, i.e. the graph lies entirely above the x-axis; if $a < 0$, the values of y are always negative, i.e. the graph lies entirely below the x-axis,
- the function is only defined for values of $x > 0$,
- if $a > 0$, the graph would take the shape in Fig. 6.5(**a**), if $a < 0$, the graph would take the shape in Fig. 6.5(**b**).



(a) $y = \frac{a}{\sqrt{x}}$, where $a > 0$ **(b)** $y = \frac{a}{\sqrt{x}}$, where $a < 0$

Fig. 6.5

Worked Example 9

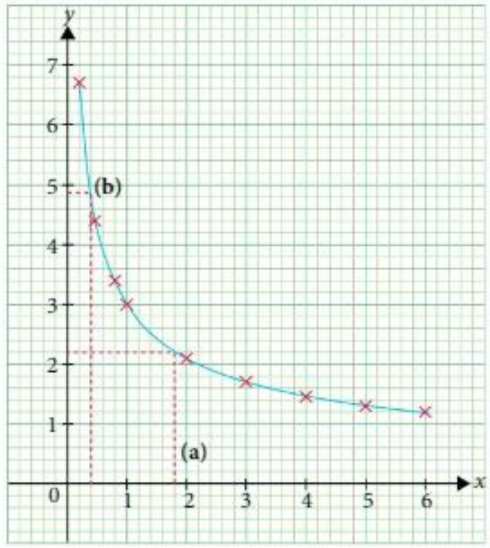
Drawing the graph of $y = \frac{a}{\sqrt{x}}$

Using 1 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = \frac{3}{\sqrt{x}}$ for $0.2 \leqslant x \leqslant 6$. Find

(a) the value of y when $x = 1.8$,
(b) the value of x when $y = 4.9$.

Solution

x	0.2	0.5	0.8	1	2	3	4	5	6
y	6.7	4.2	3.4	3	2.1	1.7	1.5	1.3	1.2



Problem-solving Tip

Based on the scale of the y-axis, we can only plot y-values to the nearest 0.1. Hence, we calculate the y-values correct to 1 d.p. in the table of values.

(a) From the graph, when $x = 1.8$, $y = 2.2$.

(b) From the graph, when $y = 4.9$, $x = 0.4$.

Practise Now 9

Similar and Further Questions Exercise 6A Question 7

Using 2 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = -\frac{1.5}{\sqrt{x}}$ for $0.5 \leqslant x \leqslant 5$. Find

(a) the value of y when $x = 3.45$,

(b) the value of x when $y = -0.85$.

Advanced Intermediate Basic

Exercise 6A

1. The table below shows some values of x and the corresponding values of y, where $y = x^3 - 3x^2 + 1$.

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3	3.5	4
y			1		-1		-3				17

(i) Copy and complete the table above, leaving your answers to 1 decimal place where necessary.

(ii) Using a scale of 2 cm to represent 1 unit, draw a horizontal x-axis for $-1 \leqslant x \leqslant 4$. Using a scale of 4 cm to represent 5 units, draw a vertical y-axis for $-5 \leqslant y \leqslant 20$. On your axes, plot the points given in the table and join them with a smooth curve.

(iii) Use your graph to find

(a) the value of y when $x = 1.2$,

(b) the value(s) of x when $y = 0$.

2. Sketch the graphs of the following.

(a) $y = 64 - x^3$

(b) $y = 8 + x^3$

(c) $y = 27x^3 - 1$

(d) $y = -0.5x^3 - 62.5$

Basic | Intermediate | Advanced

Exercise 6A

3. The table below shows some values of x and the corresponding values of y, where $y = \frac{4}{x}$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4	5
y	16			2		1	

(i) Copy and complete the table, leaving your answers to 1 decimal place where necessary.
(ii) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = \frac{4}{x}$ for $\frac{1}{4} \leqslant x \leqslant 5$.
(iii) Use your graph to find
(a) the value of y when $x = 3.6$,
(b) the value of x when $y = 1.5$.

4. Sketch the graphs of the following.
(a) $y = -\frac{2}{x}$
(b) $y = \frac{3}{x} + 4$
(c) $y = \frac{2}{x} - 5$
(d) $y = -\frac{1}{x} - 6$

5. The table below shows some values of x and the corresponding values of y, where $y = \frac{10}{x^2}$.
The values of y are correct to 1 decimal place, where necessary.

x	1	2	3	4	5
y	10	2.5	a	0.6	b

(i) Find the value of a and of b.
(ii) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = \frac{10}{x^2}$ for $1 \leqslant x \leqslant 5$.
(iii) Use your graph to find
(a) the value of y when $x = 2.8$,
(b) the value of x when $y = 4.4$.

6. Using 1 cm to represent 2 units on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = 6\sqrt{x}$ for $0 \leqslant x \leqslant 8$. Find
(a) the value of y when $x = 5.2$,
(b) the value of x when $y = 12.1$.

7. Using 1 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = -\frac{5}{\sqrt{x}}$ for $0.1 \leqslant x \leqslant 7$. Find
(a) the value of y when $x = 6.4$,
(b) the value of x when $y = -3.3$.

8. Using a suitable scale, draw the graph of $y = x^3 - 6x^2 + 13x$ for $0 \leqslant x \leqslant 5$. Use your graph to find
(i) the value of y when
(a) $x = 1.5$,
(b) $x = 3.5$,
(c) $x = 4.45$.
(ii) the value of x when
(a) $y = 7$,
(b) $y = 15$,
(c) $y = 22$.

9. Sketch the graphs of the following.
(a) $y = x^3 - x^2 - 2x$
(b) $y = -x^3 - 2x^2 + 15x$
(c) $y = -\frac{1}{3}x^3 + \frac{4}{3}x^2 - x$
(d) $y = 0.5x^3 - x^2 - 4x$

10. Using a scale of 4 cm to represent 1 unit on both axes, draw the graph of $y = -\frac{2}{x} - 1$ for $\frac{1}{2} \leqslant x \leqslant 4$.
Use your graph to find
(i) the value of y when $x = 2.5$,
(ii) the value of x when $y = -1.6$.

11. The table below shows some values of x and the corresponding values of y, where $y = x - \frac{3}{x}$.
The values of y are correct to 1 decimal place, where necessary.

Exercise 6A

x	0.5	1	2	3	4	5	6
y	–5.5	–2	0.5	h	3.3	4.4	k

(i) Find the value of h and of k.
(ii) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = x - \frac{3}{x}$ for $0.5 \leqslant x \leqslant 6$.
(iii) Use your graph to find
(a) the value of y when $x = 1.6$,
(b) the value of x when $y = -2.5$.

12. Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = x + \frac{2}{x^2}$ for $1 \leqslant x \leqslant 6$.
Use your graph to find
(i) the value of y when $x = 5.4$,
(ii) the value(s) of x when $y = 3$.

13. Using a suitable scale, draw the graph of $y = x^3 - 2x - 1$ for $-3 \leqslant x \leqslant 3$.
(a) Use your graph to find the x-coordinates of the points of intersection of the curve with the x-axis.
(b) On the same axes, draw the straight line $y = x$ for $-3 \leqslant x \leqslant 3$.
(i) Write down the x-coordinates of the points at which the line $y = x$ meets the curve $y = x^3 - 2x - 1$.
(ii) Hence, state the solutions of the equation $x^3 - 2x - 1 = x$. Explain your answer.

14. Using a suitable scale, draw the graph of $y = -\frac{x^3}{2} + 3x + 2$ for $-4 \leqslant x \leqslant 4$.
(a) The equation $-\frac{x^3}{2} + 3x = 0$ has three solutions. Explain how this can be seen from your graph.
(b) (i) On the same axes, draw the line $y = 10 - 3x$ for $-4 \leqslant x \leqslant 4$.
(ii) Write down the x-coordinate(s) of the point(s) where this line intersects the curve.
(iii) The value(s) of x in part **(b)(ii)** is/are the solution(s) of the equation $x^3 + Ax + B = 0$. Find the value of A and of B.

15. The variables x and y are connected by the equation $y = x + \frac{1}{2x} - 1$.
The table below shows some values of x and the corresponding values of y, correct to 1 decimal place.

x	0.1	0.5	0.8	1	1.5	2	2.5	3	3.5	4
y	4.1	0.5	0.4	0.5	0.8	1.3	p	2.2	2.6	3.1

(i) Calculate the value of p.
(ii) Using a scale of 4 cm to represent 1 unit on both axes, draw the graph of $y = x + \frac{1}{2x} - 1$ for $0.1 \leqslant x \leqslant 4$.
(iii) Use your graph to find the values of x in the range $0.1 \leqslant x \leqslant 4$ for which $x + \frac{1}{2x} = 1$.

16. Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph $y = \frac{1}{4}x^2 + \frac{8}{x} - 9$ for $0.5 \leqslant x \leqslant 7$.
(i) Use your graph to find the minimum value of y in the given range.
(ii) By drawing suitable straight lines on the same axes, solve each of the following equations, giving your answers correct to 1 decimal place.
(a) $\frac{1}{4}x^2 + \frac{8}{x} = 6$ (b) $\frac{1}{4}x^2 + \frac{8}{x} = x + 4$
(c) $\frac{1}{4}x^2 + 2x = 15 - \frac{8}{x}$

17. A company estimates that the total cost of producing x units of flu vaccine is given by $C = 2600 + 2x + 0.0001x^2$.
(i) Write down an expression for the average cost, $\$A$, of producing 1 unit of flu vaccine.
(ii) By plotting a graph of A against x for $4500 \leqslant x \leqslant 5500$ and $3.01 \leqslant A \leqslant 3.03$, determine the production level at which the average cost is the lowest, and hence estimate the minimum average cost for each vaccine.

6.4 Graphs of exponential functions

The growth of bacteria, the spread of diseases, and the spread of rumours or fake news have something in common: they all start small but can increase rapidly. In this section, we will model such growth phenomena using the exponential function. To begin, we shall attempt to model the spread of fake news.

Assume that the initiator of the fake news can spread news to another five persons in the next hour and each of these five persons can go on to spread the news to another five persons in the next hour. Beginning with one person, the number of new persons who hear about the fake news each hour is given below.

Hour	0	1	2	3	4	5	6	...	N
Number of new persons	1	5	25	125	625	3125	15 625	...	?

Although the number seemed small at first, it soon ballooned to a large number. This rapid growth can be modelled by an **exponential function**, $y = a^N$. Let us investigate the graph of an exponential function.

A. Graphs of $y = a^x$ and $y = ka^x$

Investigation: Graph of $y = a^x$ and $y = ka^x$

1. Using a graphing software, draw each of the following graphs.
   (a) $y = 2^x$ **(b)** $y = 3^x$ **(c)** $y = 4^x$ **(d)** $y = 5^x$
2. For each of the graphs in Question 1, answer the following questions.
   (a) Write down the coordinates of the point where the graph intersects the y-axis.
   (b) As x increases, what happens to the value of y?
   (c) Does the graph intersect the x-axis?
3. **(a)** How does the value of a, where a is a positive integer and $a \neq 1$, affect the shape of the graph of $y = a^x$?
   (b) What will the graph of $y = a^x$ look like if $a = 1$?
4. Using a graphing software, draw each of the following graphs.
   (a) $y = 2^x$ **(b)** $y = 3(2^x)$ **(c)** $y = 5(2^x)$ **(d)** $y = -(2^x)$ **(e)** $y = -4(2^x)$
5. For each of the graphs in Question 4, answer the following questions.
   (a) Write down the coordinates of the point where the graph intersects the y-axis.
   (b) As x increases, what happens to the value of y?
   (c) Does the graph intersect the x-axis?
6. How does the value of k affect the shape of the graph of $y = ka^x$?

From the Investigation on page 182, we observe that for the graph of $y = a^x$, where a is a positive integer and $a \neq 1$,

- the values of y are always positive, i.e. the graph lies entirely above the x-axis,
- the graph intersects the y-axis at $(0, 1)$.

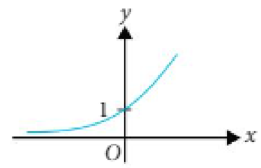
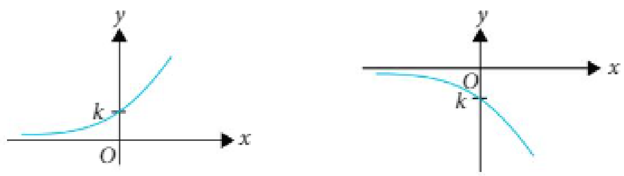


Fig. 6.6

As the positive value of x increases and tends to the right of the graph, the value of y increases very rapidly and approaches infinity. When x is negative and tends to the left of the graph, y becomes smaller as x becomes smaller. The curve gets very close to the x-axis but never touches it.

For the graph of $y = ka^x$, where a is a positive integer and $a \neq 1$,

- if $k > 0$, the values of y are always positive, i.e. the graph lies entirely above the x-axis (see Fig. 6.7**(a)**), if $k < 0$, the values of y are always negative, i.e. the graph lies entirely below the x-axis (see Fig. 6.7**(b)**),
- the graph intersects the y-axis at $(0, k)$.



(a) $y = ka^x$, where $k > 0$ **(b)** $y = ka^x$, where $k < 0$

Fig. 6.7

Journal Writing

A newspaper article states that the number of members of a social network increased **exponentially** in its first year of operation and can be represented by the equation $y = 28^x$, where x is the number of months and y is the number of members.

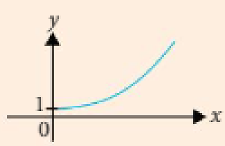


Fig. 6.8

(i) Describe how the number of members of the social network changes with time.

(ii) Search the Internet for more real-life applications of exponential graphs.

> **Big Idea**
>
> **Functions and Models**
>
> Functions such as exponential functions can be used to model real-world situations under certain assumptions. In the previous example of the spread of fake news, the growth of the number of new persons can be modelled using an exponential function based on each person spreading it to five others. This assumption may be an over-simplification and a more accurate model can be formulated if we took other factors into consideration, e.g. the reach of social media. Thus, it is important to be clear about the assumptions and limitations of modelling a real-world situation.

Worked Example 10

Drawing the graph of $y = a^x$

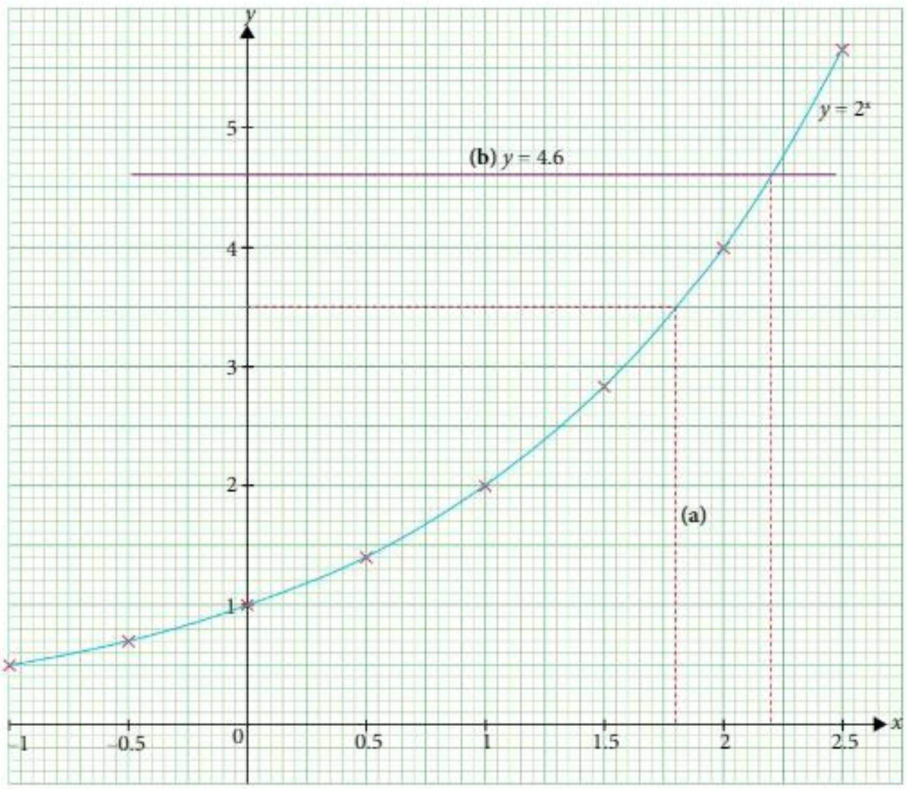
Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 2^x$ for $-1 \leqslant x \leqslant 2.5$. Use your graph to find

(a) the value of y when $x = 1.8$,

(b) the solution of $2^x = 4.6$.

*Solution

x	−1	−0.5	0	0.5	1	1.5	2	2.5
y	0.5	0.71	1	1.41	2	2.83	4	5.66



(a) From the graph, when $x = 1.8$, $y = 3.5$.

(b) From the graph, the line $y = 4.6$ and the curve $y = 2^x$ intersect at (2.2, 4.6).

$\therefore$ the solution is $x = 2.2$.

Practise Now 10

Similar and Further Questions
Exercise 6B
Questions 1, 2, 7, 8, 16, 17

Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 3^x$ for $-2 \leqslant x \leqslant 2$. Use your graph to find

(a) the value of y when $x = -1.25$,

(b) the solution of $3^x + 1 = 1.7$.

B. Sketching graphs of $y = ka^x + b$

In Section 6.4A, we have learnt about the shapes of the graphs of exponential functions $y = ka^x$ and how to plot them. There is one asymptote, $y = 0$ (x-axis). In general, for the graph of an exponential function in the form $y = ka^x + b$, $y = b$ is an asymptote of the graph. How can we sketch them?

Worked Example 11

Sketching graph of an exponential function

Sketch the graph of $y = 3^x - 9$.

Solution

Find asymptote:

$y = -9$ is an asymptote of the graph of $y = 3^x - 9$.

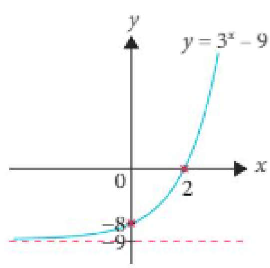
Find y-intercept:

When $x = 0$, $y = 3^x - 9$
$= 3^0 - 9$
$= -8$ y-intercept

Find x-intercept:

When $y = 0$, $3^x - 9 = 0$
$3^x = 9$
$3^x = 3^2$
$x = 2$ x-intercept

Sketch:



Problem-solving Tip

The asymptote of the graph of $y = ka^x + b$ is $y = b$.

Practise Now 11

Similar and Further Questions
Exercise 6B
Questions 3(a)-(d), 9

Sketch each of the following graphs.

(a) $y = 8 - 2^x$

(b) $y = 4.5^x + 2$

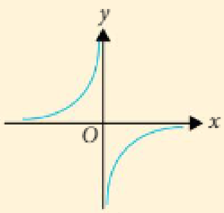
(c) $y = -4^x - 3$

Class Discussion

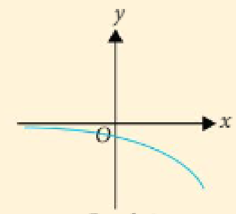
Matching graphs of power functions with the corresponding functions

Match the graphs with their respective functions and justify your answers. If your classmate does not obtain the correct answer, explain to him or her what he or she has done wrongly.

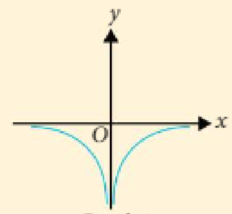
A: $y = 2x^3$	**B:** $y = -\dfrac{6}{x}$	**C:** $y = \dfrac{5}{2x^2}$	**D:** $y = 5^x$
E: $y = -\dfrac{3}{x^2}$	**F:** $y = -2(6^x)$	**G:** $y = \dfrac{1}{2x}$	**H:** $y = -3x^3$



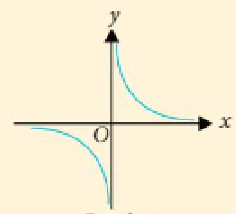
Graph 1



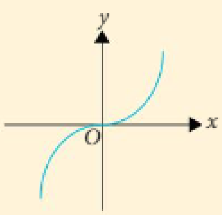
Graph 2



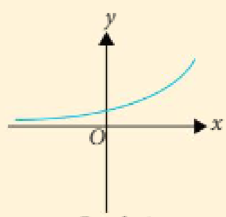
Graph 3



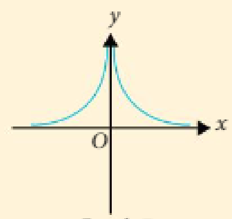
Graph 4



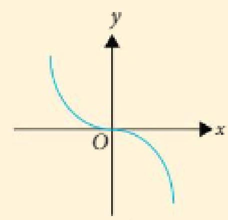
Graph 5



Graph 6



Graph 7



Graph 8

Similar and Further Questions
Exercise 6B
Question 10

Reflection

Given that many different graphs look similar, how can I work out the shape of the graphs of different functions besides committing them to memory?

6.5 Graphs of rational functions

In Section 6.2, we have learnt about the graphs of $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$. They belong to a special case of **rational functions**. Rational functions are of the form $y = \frac{P}{Q}$, where P and Q are **polynomials**.

A polynomial in x is an algebraic expression consisting of non-negative integer powers of x only.

Investigation — Graph of a rational function involving polynomials and its asymptotes

1. Using a graphing software, draw each of the following graphs.

(a) $y = \dfrac{x-1}{(x-2)(x+3)}$ **(b)** $y = \dfrac{(2x+1)(x-1)}{x+2}$

(c) $y = \dfrac{x^2}{(4x-1)(x+3)}$ **(d)** $y = \dfrac{(x+1)(x-3)}{x+1}$

2. For each of the graphs in Question 1, answer the following questions.

(a) Find the value(s) of x for which the denominator of the function is equal to zero.

(b) Is it always true that the value(s) of x in part **(a)** correspond(s) to the vertical asymptote(s) of the graph?

3. Using a graphing software, draw each of the following graphs.

(a) $y = \dfrac{2x-3}{3x^2+x-2}$ **(b)** $y = \dfrac{x^2-2x+1}{4x^3+x}$

(c) $y = \dfrac{2x}{2x^3-x^2+3x-2}$ **(d)** $y = \dfrac{2x+3}{4x}$

(e) $y = \dfrac{x^2+4x-3}{3x^2+5}$ **(f)** $y = \dfrac{4x^3-3x+1}{4x^3+2x}$

(g) $y = \dfrac{x^2-5}{2x+3}$ **(h)** $y = \dfrac{x^3-x^2+x-4}{2x^2+3}$

(i) $y = \dfrac{2x^3+x}{x+6}$

4. For each of the graphs in Question 3, answer the following questions.

(a) The degree of a polynomial is the highest power of its individual terms with non-zero coefficients.
For example in Question 3(**a**), the degree of the polynomial $2x - 3$ is 1, and that of $3x^2 + x - 2$ is 2. Let the degree of the polynomial in the numerator of the expression be m, and that in the denominator be n.
Is $m > n$, $m = n$, or $m < n$?

(b) State the horizontal asymptote of the graph, if any.

From the Investigation on page 187, we observe that we can find the vertical and horizontal asymptotes of the graph of a rational function as follows:

Vertical Asymptote	Horizontal Asymptote
• The vertical asymptote(s) is/are the x-value(s) where the denominator of the rational function = 0 and the numerator ≠ 0. Otherwise, there is no vertical asymptote.	For the graph of a rational function, $y = \frac{ax^m + ...}{bx^n + ...}$, where m and n are the respective degrees of the polynomials, • if $m > n$, there is no horizontal asymptote; • if $m = n$, the horizontal asymptote is $y = \frac{a}{b}$; • if $m < n$, the horizontal asymptote is $y = 0$ (x-axis).

Worked Example 12

Drawing the graph of a rational function

(i) Find the asymptote(s) of the function $f(x) = \frac{2x^2 + 5}{3x^2 + 1}$.

(ii) Using a scale of 1 cm to represent 1 unit on both axes, draw the graph of $y = \frac{2x^2 + 5}{3x^2 + 1}$ and the asymptote(s) in part **(i)**, for $-5 \leqslant x \leqslant 5$.

(iii) Use your graph to

(a) find the value of y when $x = 0.3$,

(b) solve the equation $\frac{2x^2 + 5}{3x^2 + 1} = 2.5$.

Solution

(i) Find vertical asymptote(s):

$3x^2 + 1 = 0$

$3x^2 = -1$

Since $3x^2 \geqslant 0$, there are no real solutions.

∴ the function does not have a vertical asymptote.

Find horizontal asymptote:

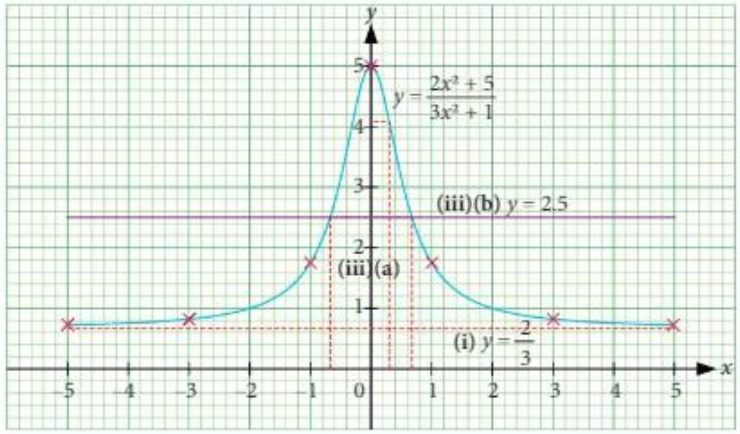
The degree of $2x^2 + 5$ is 2.

The degree of $3x^2 + 1$ is 2.

∴ the horizontal asymptote of the function is $y = \frac{2}{3}$.

(ii)

x	-5	-3	-1	0	1	3	5
y	0.7	0.8	1.8	5	1.8	0.8	0.7



(iii) (a) From the graph, when $x = 0.3$, $y = 4.1$.

(b) From the graph, the line $y = 2.5$ and the curve $y = \frac{2x^2+5}{3x^2+1}$ intersect at $(-0.7, 2.5)$ and $(0.7, 2.5)$.
$\therefore x = -0.7$ or $x = 0.7$

Problem-solving Tip

Based on the scale of the y-axis, we can only plot y-values to the nearest 0.1. Hence, we calculate the y-values correct to 1 d.p. in the table of values, and correct $\frac{2}{3}$ to 1 d.p., which is 0.7, before drawing the line.

Practise Now 12

Similar and Further Questions
Exercise 6B
Questions 4–6, 11, 18

1. **(i)** Find the asymptote(s) of the function $f(x) = \frac{2x-3}{4x^2+2}$.

   (ii) Using a scale of 1 cm to represent 2 units on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = \frac{2x-3}{4x^2+2}$ and the asymptote(s) in part **(i)**, for $-10 \leqslant x \leqslant 10$.

   (iii) Use your graph to find
   (a) the value of y when $x = -5.5$,
   (b) the values of x when $y = -1$.

2. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = \frac{(x+2)(x-3)}{x-5}$.

x	-8	-6	-4	-2	0	2	4	4.4
y	-5.1			0			-6	

   (i) Copy and complete the table, leaving your answers to 1 decimal place where necessary.

   (ii) Find the asymptote(s) of the function $f(x) = \frac{(x+2)(x-3)}{x-5}$.

(iii) Using a scale of 1 cm to represent 1 unit on both axes, draw the graph of $y = \frac{(x+2)(x-3)}{x-5}$ and the asymptote(s) in part **(ii)**, for $-8 \leqslant x \leqslant 4.4$.

(iv) Use your graph to find

(a) the value of y when $x = 4.2$,

(b) the solutions of the equation $(x + 2)(x - 3) = 10 - 2x$.

3. The variables x and y are connected by the equation $y = \frac{x^2 - 2x - 15}{(x+1)(x-3)}$. The table below shows some values of x and the corresponding values of y, correct to 2 decimal places.

x	−7	−5	−3	−2	−1.5	−0.5	0	0.5	1	2.5	3.5	4	6	8
y					−4.33					7.86			0.43	

(i) Copy and complete the table, leaving your answers to 2 decimal places where necessary.

(ii) Find the asymptote(s) of the function $f(x) = \frac{x^2 - 2x - 15}{(x+1)(x-3)}$.

(iii) Using a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = \frac{x^2 - 2x - 15}{(x+1)(x-3)}$ and the asymptote(s) in part **(ii)**, for $-8 \leqslant x \leqslant 10$.

(iv) Use your graph to find

(a) the value of y when $x = 3.7$,

(b) the solutions of the equation $2(x^2 - 2x - 15) = (x + 1)(x - 3)$.

6.6 Gradient of a curve

When a straight line touches a curve at a *single point* A, the line is called the **tangent** to the curve at the point A.

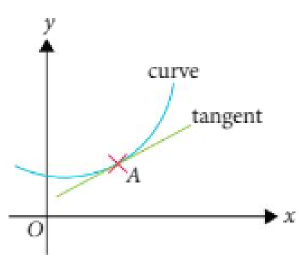


Fig. 6.9

When a line l_1 touches the curve at P, l_1 is called the tangent to the curve at P. Similarly, when a line l_2 touches the curve at Q, l_2 is called the tangent to the curve at Q.

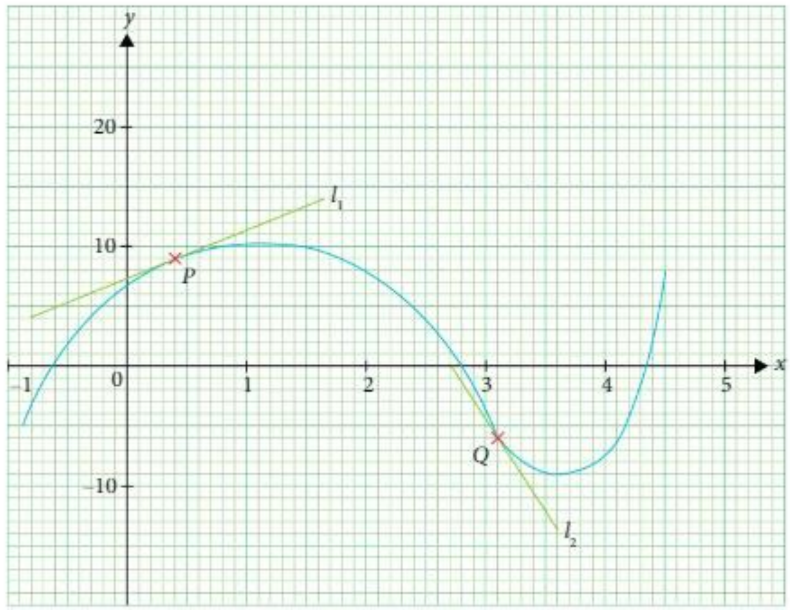


Fig. 6.10

The **gradient of the curve at a point** is defined as the **gradient of the tangent to the curve at that point**. Hence, the gradient of the curve at P in Fig. 6.10 is equal to the gradient of the line l_1 and the gradient of the curve at Q is equal to the gradient of the line l_2.

Why is the gradient important? There are several reasons but the most important one you should understand for now is that the gradient tells us something about the *rate of change* of one variable with respect to the other.

Worked Example 13

Finding gradient of a curve

The variables x and y are connected by the equation $y = \frac{1}{2}(5x - x^2)$.

The table below shows some values of x and the corresponding values of y.

x	$-\frac{1}{2}$	0	1	2	$2\frac{1}{2}$	3	4	5
y	a	0	2	3	b	3	2	0

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = \frac{1}{2}(5x - x^2)$ for $-\frac{1}{2} \leqslant x \leqslant 5$.

(c) By drawing a tangent, find the gradient of the curve at the point $(1, 2)$.

(d) The gradient of the curve at the point (h, k) is zero.

(i) Draw the tangent at the point (h, k).

(ii) Hence, find the value of h and of k.

Solution

(a) When $x = -\frac{1}{2}$,

$$y = \frac{1}{2}\left[5\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2\right]$$
$$= -1\frac{3}{8}$$
$$\therefore a = -1\frac{3}{8}$$
$$= -1.375$$

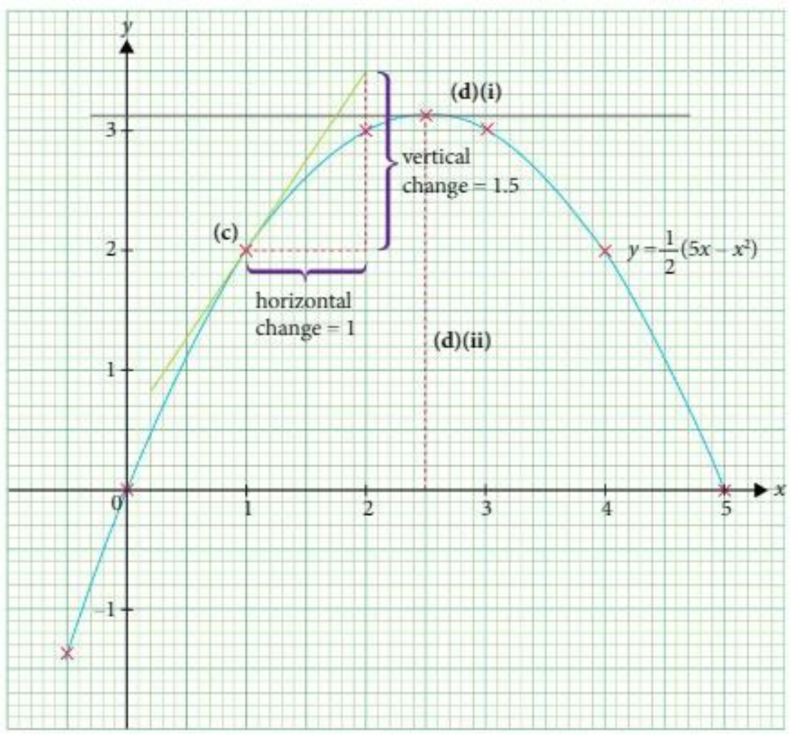
When $x = 2\frac{1}{2}$,

$$y = \frac{1}{2}\left[5\left(2\frac{1}{2}\right) - \left(2\frac{1}{2}\right)^2\right]$$
$$= 3\frac{1}{8}$$
$$\therefore b = 3\frac{1}{8}$$
$$= 3.125$$

Problem-solving Tip

Express the values of a and b in decimals so that it is easier to plot the points.

(b)



(c) A tangent is drawn to the curve at the point (1, 2).
From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{1.5}{1} = 1.5$$

(d) (ii) From the graph and table, $h = 2\frac{1}{2}$, $k = 3\frac{1}{8}$.

Problem-solving Tip

To calculate the gradient of the tangent, choose two points on the tangent that you can read the coordinates of easily. In the graph in **(b)**, the two points chosen are (1, 2) and (2, 3.5). Other than using gradient $= \frac{\text{vertical change}}{\text{horizontal change}}$, you can also use gradient $= \frac{y_2 - y_1}{x_2 - x_1}$.

Practise Now 13

Similar and Further Questions
Exercise 6B
Questions 12–15

The variables x and y are connected by the equation $y = x^2 - 4x$. The table below shows some values of x and the corresponding values of y.

x	−1	0	1	2	3	4	5
y	a	0	−3	−4	b	0	5

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = x^2 - 4x$ for $-1 \leqslant x \leqslant 5$.

(c) By drawing a tangent, find the gradient of the curve at the point where $x = 2.8$.

(d) The gradient of the curve at the point (h, k) is zero.
   (i) Draw the tangent at the point (h, k).
   (ii) Hence, find the value of h and of k.

Recall

A line parallel to the x-axis has a gradient equal to zero.

Advanced | Intermediate | Basic

Exercise 6B

1. The table below shows some values of x and the corresponding values of y, where $y = 4^x$.

x	−1	−0.5	0	0.5	1	1.5	2	2.5
y	0.25		1	2	4			

(i) Copy and complete the table.

(ii) Using a scale of 4 cm to represent 1 unit, draw a horizontal x-axis for $-1 \leqslant x \leqslant 2.5$.
Using a scale of 1 cm to represent 2 units, draw a vertical y-axis for $0 \leqslant y \leqslant 32$.
On your axes, plot the points given in the table and join them with a smooth curve.

(iii) Use your graph to find
   (a) the value of y when $x = 1.8$,
   (b) the value of x when $y = 0.4$.

2. The variables x and y are connected by the equation $y = 3(2^x)$.
The table below shows some values of x and the corresponding values of y, correct to 1 decimal place where necessary.

x	−1	−0.5	0	0.5	1	1.5	2	2.5
y	1.5	2.1			6		12	17.0

(i) Copy and complete the table.

(ii) Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3(2^x)$ for $-1 \leqslant x \leqslant 2.5$.

Basic Intermediate Advanced

Exercise 6B

(iii) Use your graph to find
(a) the values of y when $x = 0.7$ and $x = 2.3$,
(b) the values of x when $y = 2.5$ and $y = 7.4$.

3. Sketch the graph of the following.
(a) $y = 5^x - 125$
(b) $y = -4^x + 16$
(c) $y = 2.5^x + 2$
(d) $y = -2^x - 1.5$

4. **(i)** Find the asymptote(s) of the function $f(x) = \dfrac{x^2 - 1}{2x^2 + 3x + 2}$.
(ii) Using a scale of 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = \dfrac{x^2 - 1}{2x^2 + 3x + 2}$ and the asymptote(s) in part **(i)**, for $-8 \leqslant x \leqslant 6$.
(iii) Use your graph to
(a) find the value of y when $x = 0.3$,
(b) solve the equation $\dfrac{x^2 - 1}{2x^2 + 3x + 2} = -0.5$.

5. The table below shows some values of x and the corresponding values of y, correct to 2 decimal places, where $y = \dfrac{3x - 7}{2x - 5}$.

x	–5	–3	–1	0	1	1.5	2
y			1.43			1.25	

x	2.2	2.8	3	3.5	4	6	8
y	0.67			1.75			

(i) Copy and complete the table, leaving your answers to 2 decimal places where necessary.
(ii) Find the asymptote(s) of the function $f(x) = \dfrac{3x - 7}{2x - 5}$.
(iii) Using a scale of 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = \dfrac{3x - 7}{2x - 5}$ and the asymptote(s) in part **(ii)**, for $-5 \leqslant x \leqslant 8$.

(iv) Use your graph to find
(a) the value of y when $x = 3.3$,
(b) the value of x when $y = -2.2$,

6. The table below shows some values of x and the corresponding values of y, correct to 2 decimal places, where $y = \dfrac{2}{x - 7}$.

x	–4	–2	0	2	4	4.5	5	5.5	6	6.8
y	–0.18							–1.33		

x	7.2	8	8.5	9	9.5	10	12	14	16	18
y			1.33			0.67			0.22	

(i) Copy and complete the table, leaving your answers to 2 decimal places where necessary.
(ii) Find the asymptote(s) of the function $f(x) = \dfrac{2}{x - 7}$.
(iii) Using a scale of 1 cm to represent 2 units on both axes, draw the graph of $y = \dfrac{2}{x - 7}$ for $-4 \leqslant x \leqslant 18$.
(iv) Use your graph to find
(a) the value of y when $x = 12.6$,
(b) the solution of $\dfrac{2}{x - 7} = -3$.

7. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = 2 + 2^x$.

x	–1	–0.5	0	1	1.5	2	2.5	3
y	a	2.7	3	4	4.8	6	b	10

(i) Find the value of a and of b.
(ii) Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 2 + 2^x$ for $-1 \leqslant x \leqslant 3$.
(iii) Use your graph to find
(a) the values of y when $x = -0.7$ and $x = 2.7$,
(b) the solution of $2^x = 5.5$.

Exercise 6B

8. Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3^x$ for $-2 \leqslant x \leqslant 2$.

(a) Use your graph to find the value of x when $y = 5.8$.

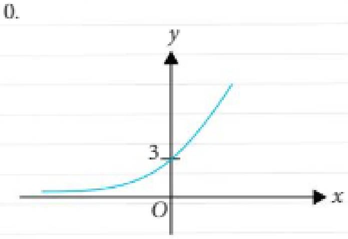
(b) On the same axes, draw the graph of $y = \frac{1}{2}x - \frac{1}{x}$, $x \neq 0$.

(i) Write down the coordinates of the point at which the graph of $y = \frac{1}{2}x - \frac{1}{x}$ meets the curve $y = 3^x$.

(ii) Hence, state the solution of the equation $3^x + \frac{1}{x} - \frac{1}{2}x = 0$.

9. On the same axes, sketch the graphs of $y = 3^x$, $y = 6^x + 1$ and $y = -2^x + 3$.

10. The sketch represents the graph of $y = ka^x$, where $a > 0$.



Write down the value of k.

11. The table below shows some values of x and the corresponding values of y, correct to 2 decimal places, where $y = \frac{2x-5}{x^2-6x+9}$.

x	−3	−2	−1	0	1	1.9	2	2.1
y	−0.31					−0.99		

x	2.6	2.8	3.3	4	4.5	5	6	7	8
y			17.78				0.78		

(i) Copy and complete the table, leaving your answers to 2 decimal places where necessary.

(ii) Find the asymptote(s) of the function $f(x) = \frac{2x-5}{x^2-6x+9}$.

(iii) Using a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = \frac{2x-5}{x^2-6x+9}$ and the asymptote(s) in part **(ii)**, for $-3 \leqslant x \leqslant 8$.

(iv) Use your graph to find

(a) the value of y when $x = 1.5$,

(b) the solutions of the equation $\frac{2x-5}{x^2-6x+9} = x$.

12. The table below shows some values of x and the corresponding values of y, where $y = (x + 2)(4 - x)$.

x	−2	−1	0	1	2	3	4
y	0		8	9			0

(a) Copy and complete the table.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = (x + 2)(4 - x)$ for $-2 \leqslant x \leqslant 4$.

(c) By drawing a tangent, find the gradient of the curve at the point where $x = -1$.

(d) The gradient of the curve at the point (h, k) is zero.

(i) Draw the tangent at the point (h, k).

(ii) Hence, find the value of h and of k.

13. (i) Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = 12 + 10x - 3x^2$ for $-2 \leqslant x \leqslant 5$.

(ii) Find the gradient of the curve when $x = 4$.

(iii) Find the gradient of the curve at the point where the curve intersects the y-axis.

Basic | Intermediate | Advanced

Exercise 6B

14. Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = 2^x + \frac{1}{x^2}$ for $-2 \leqslant x \leqslant 3$.

(a) **(i)** On the same axes, draw the line $y = 1 - x$.

(ii) Hence, solve the equation

$$2^x + \frac{1}{x^2} - 1 + x = 0.$$

(b) Explain why the graph of $y = 2^x + \frac{1}{x^2}$ will not lie below the x-axis for all real values of x.

15. **(i)** Using a suitable scale, draw the graph of $y = 1 + \frac{1}{x}$ for $0.5 \leqslant x \leqslant 3$.

(ii) On the same axes, draw the line $y = -x$.

(iii) Hence, find the coordinates on the graph of $y = 1 + \frac{1}{x}$ at which the gradient of the curve is -1.

16. Albert invested PKR 1500 in a financial product that earns 5% interest compounded yearly.

(i) Show that the amount of money in the bank after x years is given by $1500(1.05)^x$.

(ii) By drawing a graph, determine the number of years needed for his investment to double in value.

17. A radioactive substance decays at a rate of 15% per hour.

(i) Write down an equation for the percentage of substance, S, left after x hours.

(ii) By selecting an appropriate scale for each axis, draw the graph that depicts the percentage of substance left after x hours.

(iii) Given that the substance is safe to handle when at least 25% of it has decayed, use your graph to find the amount of time needed for it to reach this safe threshold.

18. The variables x and y are connected by the equation $y = \frac{x+1}{x^2+4x}$. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place.

x	-8	-6	-5.5	-5	-4.5	-4.2	-3.8	-3.5
y	-0.2		-0.5		-1.6		3.7	

x	-3	-2.5	-2	-1	-0.8	-0.5	-0.2	0.2
y	0.7		0.3		-0.1		-1.1	

x	0.5	0.8	1	3	5
y	0.7		0.4		0.1

(i) Copy and complete the table, leaving your answers to 1 decimal place where necessary.

(ii) Find the asymptote(s) of the function $f(x) = \frac{x+1}{x^2+4x}$.

(iii) Using a scale of 1 cm to represent 1 unit on both axes, draw the graph of $y = \frac{x+1}{x^2+4x}$ and the asymptote(s) in part **(ii)**, for $-8 \leqslant x \leqslant 5$.

(iv) By drawing a suitable line on the same axes, solve the equation $\frac{2x+2}{x} = (x+4)^2$.

6.7 Applications of graphs in real-world contexts

In this section, we will apply our knowledge of coordinate geometry and graphs to analyse and interpret graphs in various real-world contexts, including distance-time and speed-time graphs. Important features of the graphs such as intercepts, gradients, variables and scale of the x- and y-axes will provide information to help us in our analysis.

A. Distance-time graphs

The gradient of a line segment in a **distance-time graph** represents the speed. If the graph is a curve, the gradient of the curve at a point will represent the speed at that instant.

Class Discussion — **Linear distance-time graphs**

Fig. 6.11 shows the graph of a cyclist's journey between 0800 and 1200 hours. The graph can be divided into four sections: 0800 to 0900, 0900 to 0930, 0930 to 1030 and 1030 to 1200.

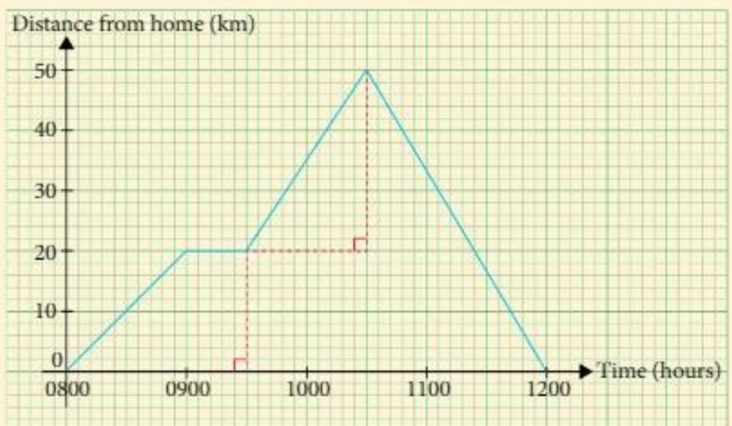


Fig. 6.11

Since the gradient of the graph from 0800 to 0900 hours $= \dfrac{20\text{ km}}{1\text{ h}} = 20$ km/h, the cyclist travels at a constant speed of 20 km/h in the first hour.

1. Consider the section of the graph from 0900 to 0930 hours. Since the graph is a horizontal line, what is its gradient? State clearly what this gradient represents.
2. Find the gradient of the section of the graph from 0930 to 1030 hours. What does this gradient tell you about the motion of the cyclist?
3. Find the gradient of the section of the graph from 1030 to 1200 hours. What does the negative gradient represent? Describe briefly the motion of the cyclist.

Worked Example 14

Distance-time curve

A train left station A and travelled 8 km to station B.

The table below shows the time, in minutes, of the train's departure from station A and the corresponding distance travelled, in km.

Time (in minutes)	1	2	3	4	5	6	7	8
Distance (in km)	0.3	1.1	2.3	4.8	6.8	7.4	7.8	8.0

(i) Using a scale of 2 cm to represent 2 minutes on the horizontal axis and 2 cm to represent 2 km on the vertical axis, plot the points given in the table and join them with a smooth curve.

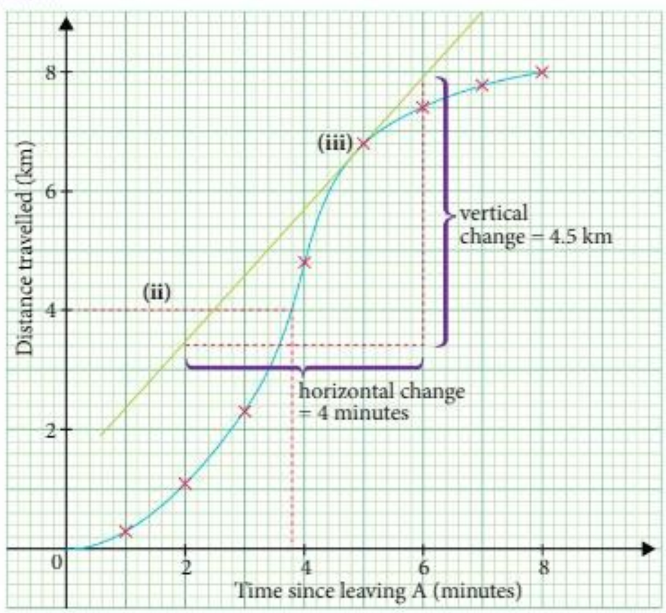
(ii) Use your graph to estimate the time taken for the train to travel the first 4 km of the journey.

(iii) By drawing a tangent, find the approximate speed of the train 5 minutes after leaving station A.

(iv) By considering the gradient of the graph, compare and describe briefly the motion of the train during the first 4 minutes and the last 4 minutes of the journey.

*Solution

(i)



(ii) From the graph, the train takes approximately 3.8 minutes to travel the first 4 km.

(iii) From the graph,

$$\text{Gradient of tangent at 5 min} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{4.5\text{ km}}{4\text{ minutes}}$$
$$= \frac{4.5\text{ km}}{\frac{4}{60}\text{ h}}$$
$$= 67.5\text{ km/h}$$

$\therefore$ the speed of the train 5 minutes after leaving station A is approximately 67.5 km/h.

Problem-solving Tip

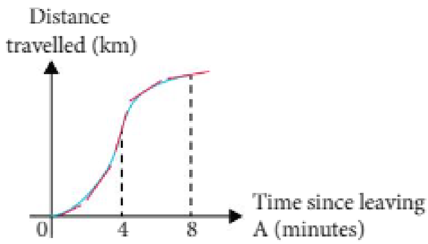
The gradient of the tangent at the point 5 min after leaving station A gives the speed at that particular point. It is called the instantaneous speed.

Attention

The graphical method of finding the gradient of a curve only yields approximate results. Slight changes in the drawing may give very different results.

(iv) During the first 4 minutes, the speed of the train increases as the gradient of the curve increases.

During the last 4 minutes, the speed of the train decreases as the gradient of the curve decreases.



Practise Now 14

Similar and Further Questions
Exercise 6C
Questions 1, 2, 6–9

A train travelled 7.6 km from station P to station Q. The table below shows the time, in minutes, of the train's departure from station P and the corresponding distance travelled, in km.

Time (in minutes)	1	2	3	4	5	6	7	8
Distance (in km)	0.2	0.8	2.6	5.0	6.5	7.2	7.5	7.6

(i) Using a scale of 2 cm to represent 2 minutes on the horizontal axis and 2 cm to represent 2 km on the vertical axis, plot the points given in the table and join them with a smooth curve.

(ii) Use your graph to estimate the time taken for the train to travel the first 4 km of the journey.

(iii) By drawing a tangent, find the approximate speed of the train 6 minutes after leaving station P.

(iv) By considering the gradient of the graph, compare and describe briefly the motion of the train during the first 4 minutes and the last 4 minutes of the journey.

B. Speed-time graphs

How is a **speed-time graph** related to a distance-time graph? On a speed-time graph, the *speed* is represented on the vertical axis and the *time*, on the horizontal axis. Fig. 6.12 is a simple example of a body moving at constant speed.

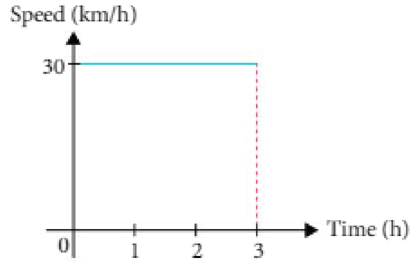


Fig. 6.12

Based on the above example, since the speed of the body is constant at 30 km/h, how far does the body travel in 1 hour, 2 hours and 3 hours respectively? Since distance = speed × time, we get 30 km, 60 km and 90 km. Hence, the area under a speed-time graph represents the distance travelled. Fig. 6.13 shows the distance-time graph of the body.

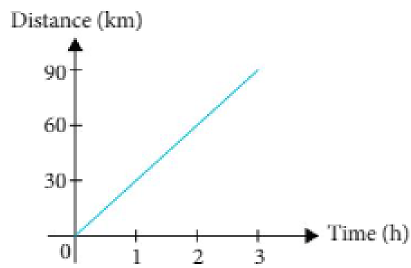


Fig. 6.13

Now, let's look at the speed-time graph with equation of the form $v = at$ where v represents speed and t represents time.

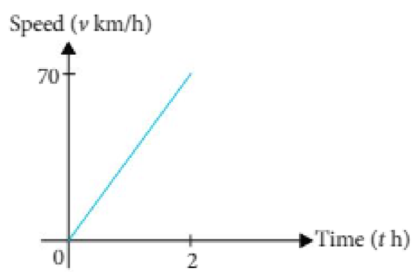


Fig. 6.14

The gradient of the speed-time graph shows the rate of change of speed, which gives the magnitude of the **acceleration, *a*** Acceleration is the measure of how fast the speed is increasing or decreasing over time. Fig. 6.14 shows that the object is moving at a constant acceleration of $\frac{70}{2} = 35$ km/h². When the speed decreases over time, the **acceleration** is negative and is known as **deceleration** or **retardation**.

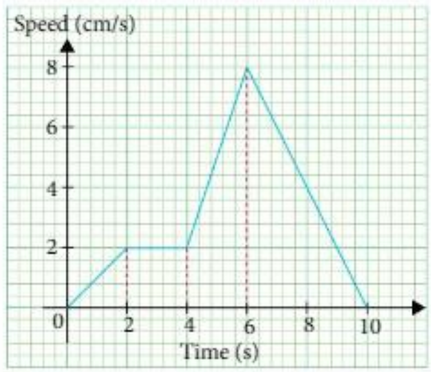
In general,

- the gradient of the distance-time graph is the speed of the object;
- the gradient of the speed-time graph is the acceleration of the object;
- the area under the speed-time graph is the distance travelled by the object.

Worked Example 15

Speed-time graph

The graph shows the speed of an object over a period of 10 seconds.



(i) Find the acceleration of the object in the first 2 seconds.

(ii) Find the total distance travelled by the object for the whole journey.

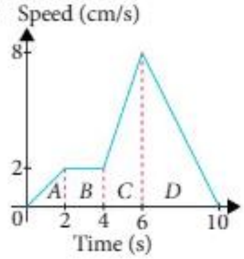
(iii) Find the deceleration of the object in the last 2 seconds.

Solution

(i) Acceleration $= \dfrac{2 \text{ cm/s}}{2 \text{ s}}$

$= 1 \text{ cm/s}^2$

(ii)



Total distance

= area under graph

= area of $(A + B + C + D)$

$= \left(\frac{1}{2}\times 2\times 2\right) + (2\times 2) + \frac{1}{2}(2+8)\times 2 + \left(\frac{1}{2}\times 8\times 4\right)$

$= 2 + 4 + 10 + 16$

$= 32$ cm

Attention

The unit for acceleration is always that of speed per unit time, i.e. if the unit of speed is cm/s, then the unit of acceleration is cm/s².

Problem-solving Tip

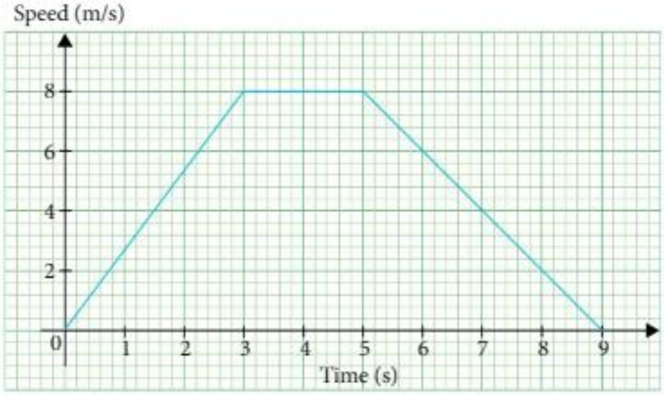
For **(i)**, the gradient can be found by $\dfrac{\text{vertical change}}{\text{horizontal change}}$ or using the formula $\dfrac{y_2 - y_1}{x_2 - x_1}$.

(iii) Deceleration in the last 2 seconds
= deceleration in the last 4 seconds
$= \dfrac{8-0}{10-6}$
$= 2$ cm/s²

Practise Now 15

Similar and Further Questions
Exercise 6C
Questions 3–5, 10–12

The graph shows the speed of an object over a period of 9 seconds.



(i) Find the acceleration of the object in the first 3 seconds.

(ii) What is the total distance travelled by the object during the entire period?

(iii) Find the deceleration of the object in the last 3 seconds.

Worked Example 16

Speed-time curve

A particle moves along a straight line from X to Y so that, t seconds after leaving X, its speed, v m/s, is given by $v = 3t^2 - 15t + 20$.
The table below shows some values of t and the corresponding values of v.

t	0	1	1.5	2	2.5	3	4	5
v	20	8	a	2	b	2	8	20

(i) Find the value of a and of b.

(ii) Using a scale of 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 5 m/s on the vertical axis, draw the graph of $v = 3t^2 - 15t + 20$ for $0 \leqslant t \leqslant 5$.

(iii) Use your graph to estimate
  (a) the value of t when the speed of the particle is 10 m/s,
  (b) the time at which the acceleration of the particle is zero,
  (c) the gradient at $t = 4$, and explain what this value represents,
  (d) the time interval when the speed of the particle is less than 15 m/s.

*Solution

(i) When $t = 1.5$,
$v = 3(1.5)^2 - 15(1.5) + 20$
$= 4.25$
$\therefore a = 4.25$

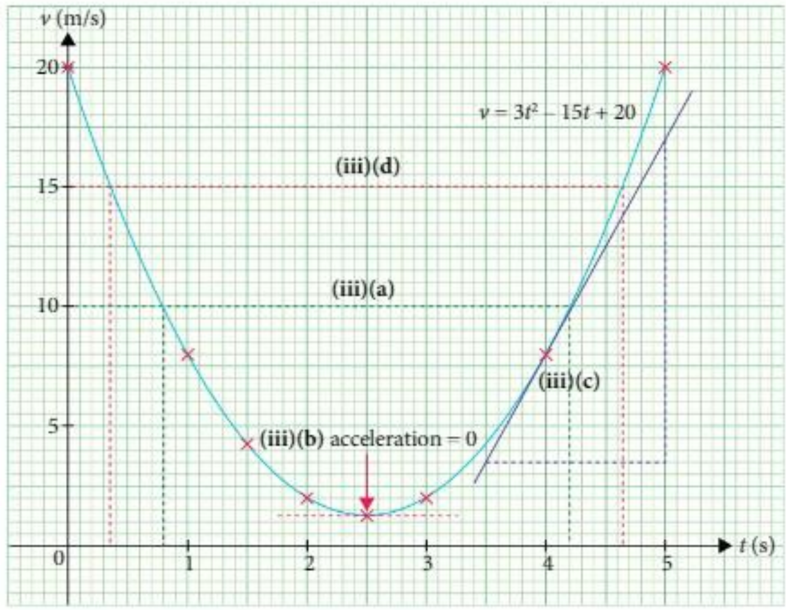
When $t = 2.5$,

$v = 3(2.5)^2 - 15(2.5) + 20$

$= 1.25$

$\therefore b = 1.25$

(ii)



(iii) (a) From the graph, when $v = 10$,
$t = 0.8$ or $t = 4.2$.

(b) The acceleration is zero when the gradient of the curve is zero.
From the graph, the acceleration of the particle is zero at $t = 2.5$.

(c) Gradient of tangent at $t = 4$

$= \frac{\text{vertical change}}{\text{horizontal change}}$

$= \frac{13.5}{1.5}$

$= 9\ \text{m/s}^2$

$\therefore$ the acceleration of the particle at $t = 4$ is $9\ \text{m/s}^2$.

Attention

In **(c)**, the unit for acceleration is $\frac{\text{m/s}}{\text{s}}$, i.e. m/s^2.

(d) From the graph, when $v < 15$, $0.35 < t < 4.65$.

Practise Now 16

Similar and Further Questions
Exercise 6C
Questions 13–15

A particle moves along a straight line from P to Q so that, t seconds after leaving P, its speed, v m/s, is given by $v = 2t^2 - 9t + 12$.

The table below shows some values of t and the corresponding values of v.

t	0	1	2	3	4	5
v	12	5	a	3	8	b

(i) Find the value of a and of b.

(ii) Using a scale of 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 5 m/s on the vertical axis, draw the graph of $v = 2t^2 - 9t + 12$ for $0 \leqslant t \leqslant 5$.

(iii) Use your graph to estimate

(a) the values of t when the speed of the particle is 7 m/s,

(b) the time at which the acceleration of the particle is zero,

(c) the gradient at $t = 4.5$, and explain what this value represents,

(d) the time interval when the speed of the particle is less than 10 m/s.

C. Other graphs

Let us now look at how graphs are used in other real-world contexts.

Worked Example 17

Other types of graphs

The graph below shows the battery level of a smartphone. It had an initial level of 20%, which then increases to 60% in half an hour while connected to the power supply. Waseem then removes the smartphone from the power supply to watch a 20-minute long video clip, before reconnecting the smartphone to the power supply.

(i) Find the battery level of the smartphone when Waseem was exactly halfway through the video clip.

(ii) Find the rate of increase in the battery level of the smartphone when it was later reconnected to the power supply.

Solution

(i) From the graph, the battery level was 50%.

(ii) To find the rate of increase in the battery level, we need to calculate the gradient of the line from the 50th minute to the 90th minute.

$$\text{Gradient} = \frac{100-40}{90-50}$$
$$= 1.5\%/\text{minute}$$

$\therefore$ the rate of increase in the battery level is 1.5%/minute.

Practise Now 17

Similar and Further Questions
Exercise 6C
Question 16

The graph below shows Joyce's heart rate, in beats per minute. She rests at the bench for 10 minutes, then walks briskly for another 10 minutes. She slows down for 5 minutes, before walking briskly for a further 5 minutes. She then jogs at a constant speed for 10 minutes, before gradually slowing down.

(i) Write down her resting heart rate.

(ii) Find the rate of increase in her heart rate during the first brisk walk.

(iii) Find the rate of decrease in her heart rate as she slows down in the last 20 minutes.

Advanced | Intermediate | Basic

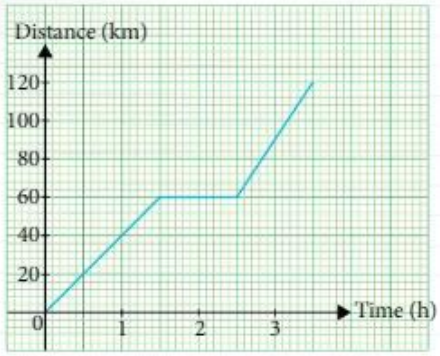
Exercise 6C

1. A cyclist set out at 9 a.m. for a destination 40 km away. He cycled at a constant speed of 15 km/h until 10.30 a.m. He then rested for half an hour before completing his journey at a constant speed. He then reached his destination at 11.53 a.m.
   (i) Draw the distance-time graph to represent his journey.
   (ii) Hence, find the constant speed of the cyclist in the second part of the journey.

2. Imran starts a 30-km journey at 0900 hours. He maintains a constant speed of 20 km/h for the first 45 minutes and then stops for a rest. He then continues his journey at a constant speed of 30 km/h, finally arriving at his destination at 1120 hours.
   (i) Find the distance that he travelled in the first 45 minutes.
   (ii) Draw the distance-time graph to represent his journey.
   (iii) Hence, state the duration of his rest, giving your answer in minutes.

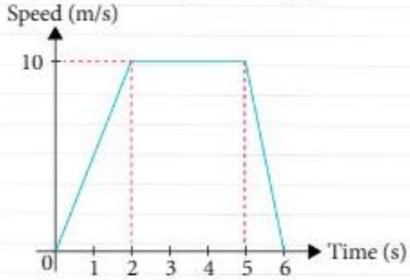
Exercise 6C

3. The figure shows the distance-time graph of a car.



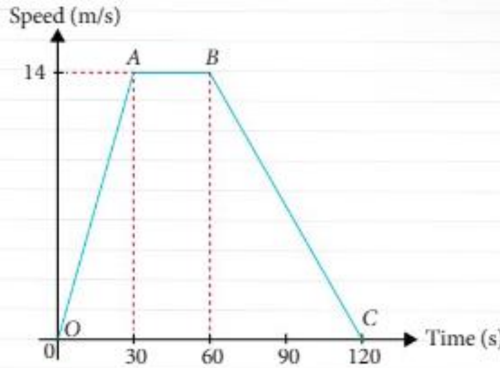
(i) Find the duration during which the car is not moving.
(ii) Find the speed of the car in the first hour of the journey.
(iii) Find the speed of the car for the last part of the journey.
(iv) Sketch the speed-time graph of the car for the whole journey.

4. The graph shows the speed-time graph of a car.



(i) Find the acceleration of the car in the first 2 seconds.
(ii) What is the total distance travelled by the car for the whole journey?

5. The graph shows the speed, v m/s, of a car after t seconds.



(i) State what the gradient of OA represents.
(ii) Find the speed of the car when $t = 15$.

6. A lift moves from ground level to a height of 60 metres in 10 seconds, stops for 10 seconds and then descends to the ground in 10 seconds. Tables **(a)** and **(b)** show the height, h m, of the lift on the upward and downward journeys respectively, t seconds after leaving ground level.

***t* (in seconds)**	0	2	4	6	8	10
***h* (in m)**	0	3	16	44	57	60

Table (a)

***t* (in seconds)**	20	22	24	26	28	30
***h* (in m)**	60	57	44	16	3	0

Table (b)

(i) Using a scale of 2 cm to represent 5 seconds, draw a horizontal t-axis for $0 \leqslant t \leqslant 30$.
Using a scale of 1 cm to represent 5 metres, draw a vertical h-axis for $0 \leqslant h \leqslant 60$.
On your axes, plot the points given in the table and join them with a smooth curve.
(ii) Find the gradient of the graph at $t = 8$ and explain briefly what this gradient represents.

Exercise 6C

A construction worker, waiting at the 40-metre level, starts to walk down at $t = 5$.

(iii) Assuming that he descends at a steady speed of 0.8 m/s, use your graph to find the value of t when the worker and the lift are at the same height, during the downward journey of the lift.

7. An autonomous-vehicle manufacturer conducts a test run on one of its vehicles. It starts from point X and travels to point Y, 3 km away. The table shows the distance, d km, of the vehicle from X, t minutes after leaving X.

Time, t, (in minutes)	0	1	2	3	4	5	6
Distance, d, (in km)	0	0.2	0.7	1.8	2.5	2.9	3.0

(i) Using a scale of 2 cm to represent 1 minute on the horizontal axis and 4 cm to represent 1 km on the vertical axis, plot the points given in the table and join them with a smooth curve.

(ii) Use your graph to find

(a) the approximate time taken for the vehicle to travel the first 1 km,

(b) the gradient of the graph when $t = 1\frac{1}{2}$ and explain briefly what this value represents,

(c) the time taken for the vehicle to travel the last 1 km.

8. Bernard and Yasir start moving towards each other at the same time. The initial distance between them is 32 km.

(i) Given that Bernard is cycling at a constant speed of 20 km/h and Yasir is walking at a constant speed of 7 km/h, draw a distance-time graph to illustrate this information.

(ii) Use your graph to find

(a) the time taken for them to pass each other,

(b) the time(s) at which they are 5 km apart.

9. At 0900 hours, Cheryl travels to meet David, who stays 20 km away. Cheryl travels at a uniform speed of 18 km/h for half an hour. She rests for 20 minutes and then continues her journey at a uniform speed of 8 km/h.

At 0900 hours, David sets off from home on the same road to meet Cheryl and travels at a uniform speed of 7 km/h.

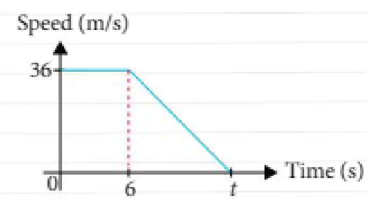
(i) Draw the distance-time graph for the above information.

(ii) Use your graph to find

(a) the time at which Cheryl and David meet,

(b) the distance from David's home at the meeting point.

10. The diagram shows the speed-time graph of an object which travels at a constant speed of 36 m/s and then slows down at a rate of 12 m/s^2, coming to rest at time t seconds.

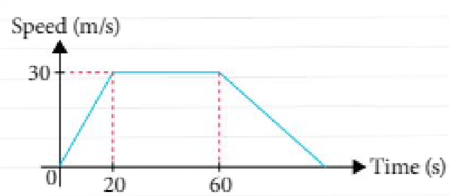


Find

(i) the value of t,

(ii) the total distance travelled by the object for the whole journey.

11. The diagram shows the speed-time graph of a train.



(i) Find the acceleration of the train during the first 20 seconds.

(ii) Given that the train decelerates at a rate of 0.75 m/s^2, find the time taken for the whole journey.

Basic Intermediate Advanced

Exercise 6C

12. A taxi starts from rest and accelerates at a uniform rate for 45 seconds to reach a speed of 30 m/s. It then travels at this constant speed. Sketch the speed-time graph and use it to find the speed of the taxi after 10 seconds.

13. A particle moves along a straight line from X to Y so that, t minutes after leaving X, its speed, v m/min, is given by $v = t^2 - 7t + 16$.

t (minutes)	0	1	2	3	4	5	6
v (m/min)	16	10	6	a	4	6	b

(i) Find the value of a and of b.

(ii) Using a scale of 2 cm to represent 1 minute on the horizontal axis and 1 cm to represent 1 m/min on the vertical axis, draw the graph of $v = t^2 - 7t + 16$ for $0 \leqslant t \leqslant 6$.

(iii) Use your graph to estimate

- **(a)** the value(s) of t when the speed of the particle is 7 m/min,
- **(b)** the time at which the speed of the particle is a minimum,
- **(c)** the gradient at $t = 2$, and explain what this value represents,
- **(d)** the time interval when the speed of the particle is not more than 5 m/min.

14. The speed of a body, v m/s, after time t seconds is given in the table below.

t (s)	0	2	4	6	8	10	12
v (m/s)	0	2	7	12	19	28	42

(i) Using a scale of 1 cm to represent 1 second on the horizontal axis and 1 cm to represent 5 m/s on the vertical axis, plot the graph of v against t for $0 \leqslant t \leqslant 12$.

(ii) Use your graph to estimate the speed of the body when $t = 5$ and when $t = 11$.

(iii) By drawing two tangents, find the acceleration of the body when $t = 4$ and when $t = 10$.

15. Object P moves along a straight line from A to B so that, t hours after leaving A, its speed, v km/h, is given by $v = 3t^2 - 17t + 30$.

t (h)	0	1	2	3	4	5
v (km/h)	30	16	h	6	k	20

(i) Find the value of h and of k.

(ii) Using a scale of 2 cm to represent 1 hour on the horizontal axis and 2 cm to represent 5 km/h on the vertical axis, draw the graph of $v = 3t^2 - 17t + 30$ for $0 \leqslant t \leqslant 5$.

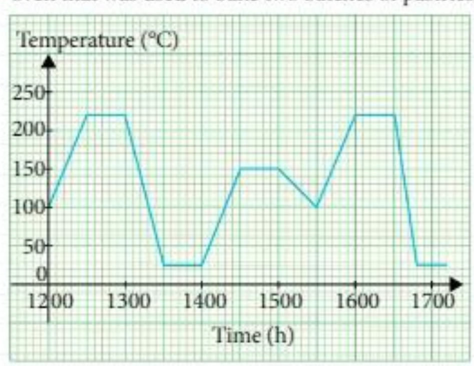
(iii) Use your graph to estimate

- **(a)** the time at which the speed of object P is a minimum,
- **(b)** the gradient at $t = 4.5$, and explain what this value represents,
- **(c)** the time interval when the speed of object P does not exceed 10 km/h.

Object Q moves along a straight line from A to B with a constant speed of 24 km/h.

(iv) Use your graph to determine the value of t at which both objects have the same speed.

16. The graph below shows the temperature inside an oven that was used to bake two batches of pastries.



The oven was switched off after the first batch was taken out and cooled to room temperature.

Advanced Intermediate Basic

Exercise 6C

(i) What was the room temperature?

(ii) Find the rate at which the oven first heated up.

The chef modified the baking technique for the second batch. He baked the pastries at a certain temperature k °C, and then cooled the oven down, before increasing the heat again. The oven was then switched off to stop the baking process.

(iii) State the value of k.

(iv) If the second batch of pastries was only placed in the oven when the temperature was k °C, determine the baking duration for the second batch.

Looking Back

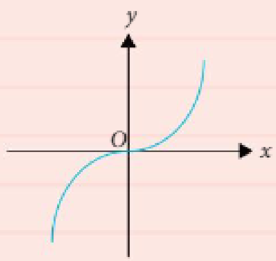
In this chapter, we have seen the power of using mathematical **diagrams**, such as graphs, to solve mathematical problems. In particular, we can use graphs to approximate the solutions of an equation. This method is especially useful when we have yet to develop algebraic methods to solve certain classes of equations, e.g. exponential and reciprocal equations.

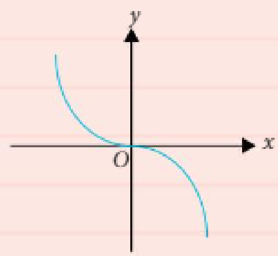
In addition, we have also looked at how **functions** (such as exponential and reciprocal functions) can be used to **model** certain real-world situations under some assumptions. Here, graphs offer us another way to understand real-world phenomena by representing how one variable (input) is related to another variable (output) visually. These visual representations not only allow us to obtain approximate solutions to equations but also give us insights into these situations. For example, a distance-time graph (such as the one in Worked Example 14 on page 198) can show us how the speed of a train changes as it moves from one place to another.

Last but not least, we see that the power of graphical solutions lies in the flexibility to use graphs of different functions to solve an equation, that can be represented in various **equivalent** forms (e.g. see Worked Example 2 on page 162).

Summary

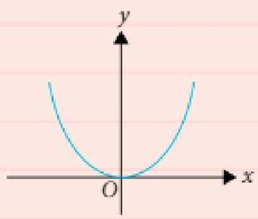
1. Graphs of power functions $y = ax^n$

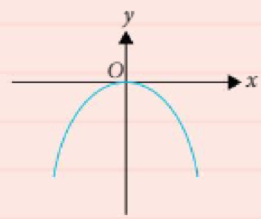




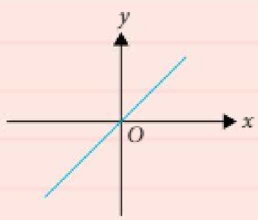
(a) $n = 3, a > 0, y = ax^3$ $\qquad$ $n = 3, a < 0, y = ax^3$

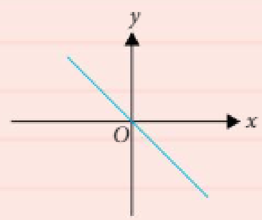
- Explain why the shape of the graph for $y = ax^3$ is as such.



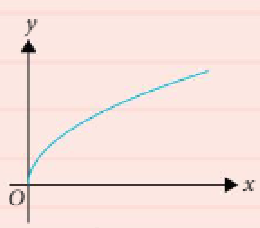


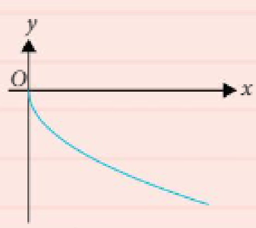
(b) $n = 2, a > 0, y = ax^2$ $\qquad$ $n = 2, a < 0, y = ax^2$





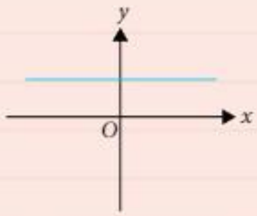
(c) $n = 1, a > 0, y = ax$ $\qquad$ $n = 1, a < 0, y = ax$

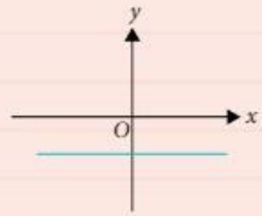




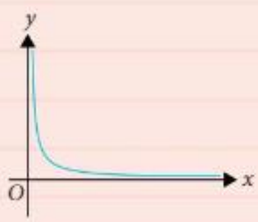
(d) $n = \frac{1}{2}, a > 0, y = a\sqrt{x}$ $\qquad$ $n = \frac{1}{2}, a < 0, y = a\sqrt{x}$

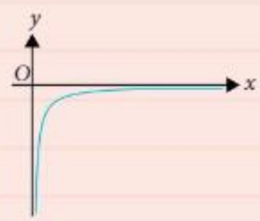
Summary



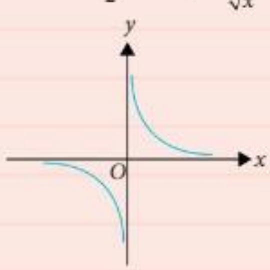


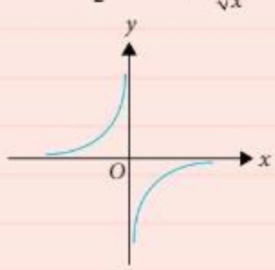
(e) $n = 0, a > 0, y = a$ $\qquad$ $n = 0, a < 0, y = a$





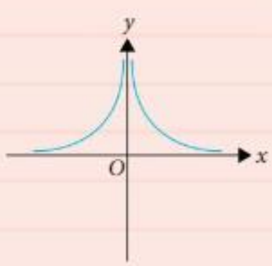
(f) $n = -\frac{1}{2}, a > 0, y = \frac{a}{\sqrt{x}}$ $\qquad$ $n = -\frac{1}{2}, a < 0, y = \frac{a}{\sqrt{x}}$

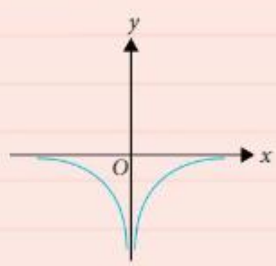




(g) $n = -1, a > 0, y = \frac{a}{x}$ $\qquad$ $n = -1, a < 0, y = \frac{a}{x}$

- Explain why the shape of the graph for $y = ax^{-1}$ is as such.





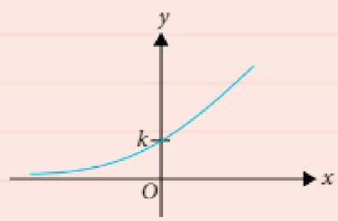
(h) $n = -2, a > 0, y = \frac{a}{x^2}$ $\qquad$ $n = -2, a < 0, y = \frac{a}{x^2}$

- Explain why the shape of the graph for $y = ax^{-2}$ is as such.

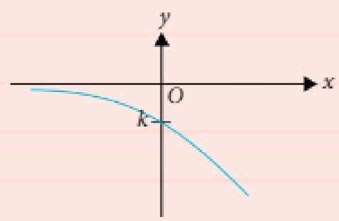
- Give a specific example of a function that corresponds to each of the graphs shown from **(a)** to **(h)**.

Summary

2. Graphs of exponential functions $y = ka^x$, where a is a positive integer and $a \neq 1$



$k > 0, y = ka^x$



$k < 0, y = ka^x$

- What are the similarities and differences between the graphs of $y = \frac{a}{x}$ and $y = ka^x$?

3. Asymptotes of functions

Type	Equation	Vertical Asymptote(s)	Horizontal Asymptote(s)
Linear functions	$y = mx + c$	None	
Quadratic functions	$y = ax^2 + bx + c$	None	
Cubic functions	$y = ax^3 + bx^2 + cx + d$	None	
Reciprocal functions	$y = \frac{a}{x} + b$	$x = 0$	$y = b$
	$y = \frac{a}{x^2}$	$x = 0$	$y = 0$
Functions involving $\sqrt{x}$	$y = a\sqrt{x}$	None	
	$y = \frac{a}{\sqrt{x}}$	$x = 0$	$y = 0$
Exponential functions	$y = ka^x + b$	None	$y = b$
Rational functions	$y = \frac{ax^m + \ldots}{bx^n + \ldots}$, where m and n are the respective degrees of the polynomials	The vertical asymptote(s) is/are the x-value(s) where the denominator of the rational function = 0 and the numerator $\neq$ 0. Otherwise, there is no vertical asymptote.	• If $m > n$: none • If $m = n$: $y = \frac{a}{b}$ • If $m < n$: $y = 0$

4. Gradient of a curve

The gradient of a curve at a point can be obtained by drawing a tangent to the curve at that point and finding the gradient of the tangent.

CHAPTER 7

Volume, Surface Area, and Symmetry of Pyramids, Cones and Spheres

The photo shows some ice cream cones. It is believed that Italo Marchiony invented the first ice cream cone in 1896, after folding waffles into a cone. If you were to unwrap a cone, what will its net look like? How does an ice cream manufacturer determine the volume of ice cream needed to fill each cone completely?

Volume and surface area are two types of **measures**: volume quantifies the space occupied by the solid while surface area quantifies the amount of space enclosed within the boundary of a face of a three-dimensional object.

In this chapter, we will learn how to find the volume and surface area of pyramids, cones and spheres, and their applications in the real world.

Learning Outcomes

What will we learn in this chapter?

- What pyramids, cones and spheres are
- How to find the volume and surface area of pyramids, cones and spheres
- What the symmetrical properties of pyramids and cones are
- How to solve problems involving the volume and surface area of composite solids
- Why volume and surface area of pyramids, cones and spheres have useful applications in real life

Introductory Problem

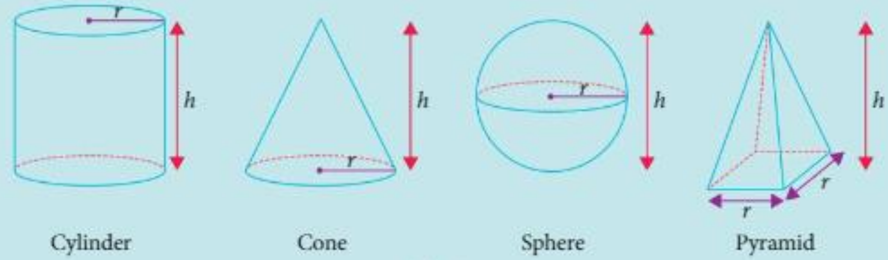


Fig. 7.1

Fig. 7.1 shows four solids with the same height, h cm. The cylinder, cone and sphere have the same radius, r cm. The pyramid has a square base of side r cm. Arrange these solids in ascending order of their volumes.

7.1 Volume, surface area and symmetry of pyramids

A. Pyramids

Class Discussion

What are pyramids?

The photo in Fig. 7.2**(a)** shows the Great Pyramid built by the Egyptians in Giza, Egypt, around the year 2560 BC. The photo in Fig. 7.2**(b)** shows two tetrahedral dice, each with 4 triangular faces. These are real-life examples of pyramids.

(a) The Great Pyramid

(b) Tetrahedral dice

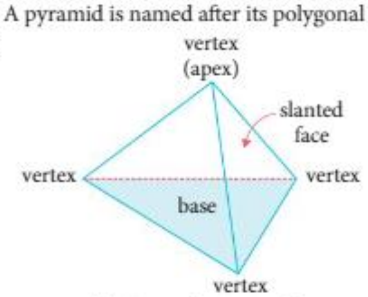
Fig. 7.2

1. What are some common features of these pyramids?
2. What do you notice about
   (a) the *slanted faces*,
   (b) the *bases*,
   of these pyramids?
3. What do you notice about the cross sections of a pyramid? Are they uniform?

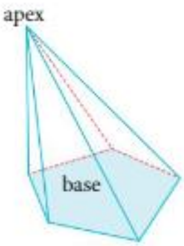
B. Types of pyramids

A pyramid is a solid in which one of the faces is a **polygonal base** and the other **slanted faces** are triangles joined to the edges (or sides) of the base. The corner points of a pyramid are known as **vertices** (singular: vertex) and the vertex where all the slanted faces meet is called the **apex**, which is opposite the base.

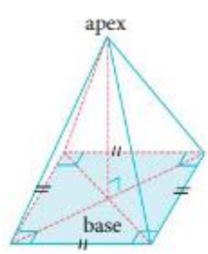
Fig. 7.3 shows some examples of pyramids with different bases. The base of each pyramid is shaded. A pyramid is named after its polygonal base. Can you name the last two pyramids?



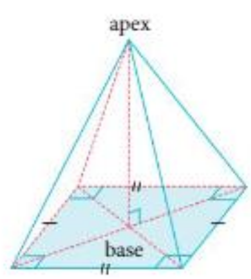
(a) Triangular pyramid



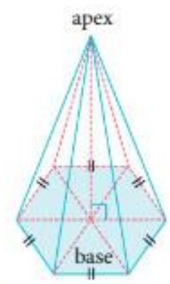
(b) Pentagonal pyramid



(c) Square pyramid



(d) ______ pyramid



(e) ______ pyramid

Fig. 7.3

> **Information**
>
> The object shown is not a pyramid. Do you know why?
>
> 

For the square pyramid in Fig. 7.3(**c**), do you notice that the apex is *vertically above the centre of the base*? Such a pyramid is called a **right pyramid**. Can you identify other right pyramids in Fig. 7.3?

The **perpendicular height** (or simply the **height**) of a pyramid is the perpendicular distance from the apex to the base of the pyramid (see Fig. 7.4). A **slant height** of a pyramid is the distance from the apex to the midpoint of an edge (or side) of the base. The edges that join the apex to the vertices (or corner points) of the base are called the **slant edges**.

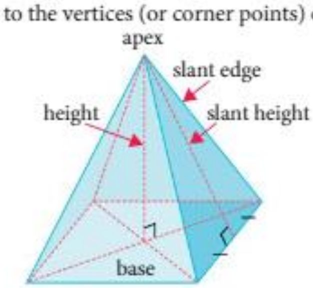
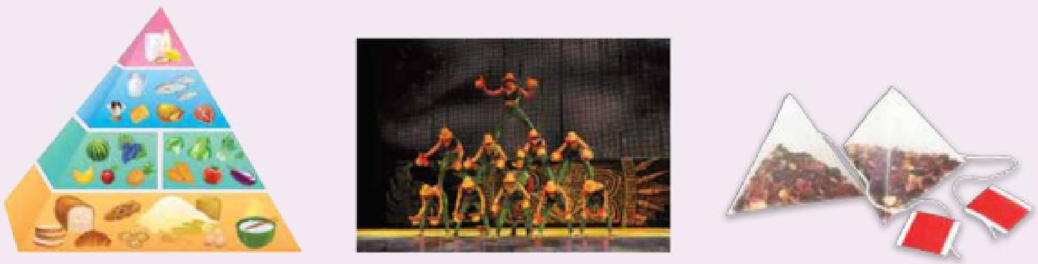


Fig. 7.4

> **Information**
>
> Do you notice that the pyramid in Fig. 7.3(**e**) is a right pyramid with a regular hexagon as its base? Such a right pyramid with a regular polygonal base is known as a **regular pyramid**.
>
> Recall that a *regular polygon* is a polygon with *all sides equal* and *all angles equal*. Since a *regular quadrilateral* has to be a square and not a rectangle, then a rectangular pyramid can never be a regular pyramid. However, are all square pyramids regular pyramids?

Thinking Time

1. What can we say about the slant edges, slant heights and slant faces of regular pyramids?
2. Look at the photos in Fig. 7.5. Are they pyramids? Explain your answer.



(a) Food pyramid (b) Human pyramid (c) Tea bags

Fig. 7.5

In this section, we will study only right pyramids. Therefore, the term 'pyramid' is used to refer to a right pyramid. Also, the term 'height' is used to refer to the perpendicular height unless otherwise stated.

Journal Writing

In Book 2, we learnt about prisms. Write down three differences between a prism and a pyramid. Give three real-life examples of pyramids.

C. Volume of pyramid

In Books 1 and 2, we learnt that the measure of the boundary and the measure of the space enclosed within the boundary are different for a 2D shape and a 3D object (see Table 7.1).

	2D shapes (plane figures)	3D objects
Measure of boundary	Perimeter	Surface area
Measure of space enclosed within the boundary	Area	Volume

Table 7.1

In Book 2, we also learnt that a prism has a uniform polygonal cross section. A cuboid (see Fig. 7.6) is an example of a prism.

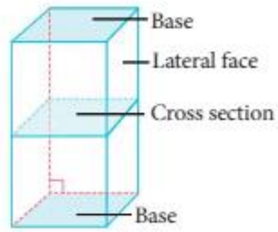


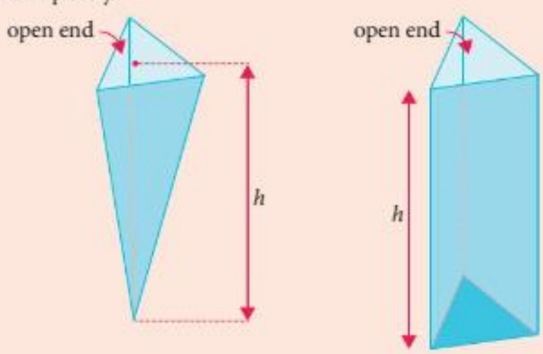
Fig. 7.6

Volume of prism = area of cross section × distance between cross-sectional bases

Let us now find the volume of a pyramid by comparing it with a prism that has the same polygonal base.

Investigation — Volume of pyramid

Consider an open triangular pyramid and an open triangular prism with the *same base* and the *same height* (see Fig. 7.7). If we fill the entire pyramid with sand before pouring it into the prism, how many times will it take to fill the prism completely?



(a) Triangular pyramid **(b)** Triangular prism

Fig. 7.7

Information

An open pyramid refers to a pyramid that is open at its base while an open prism refers to a prism that is open at one end.

Fig. 7.8 shows the **net** of an open triangular pyramid which can be photocopied and pasted on a piece of cardboard before cutting it out and folding along the dotted lines to obtain an open pyramid similar to that shown in Fig. 7.7(**a**).

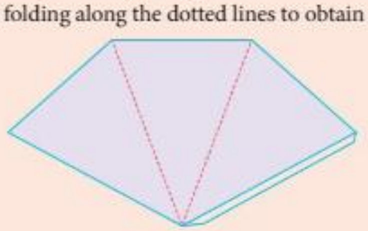


Fig. 7.8

Attention

The tabs are not part of the net as they are for gluing purposes.

Similarly, use the net of an open triangular prism in Fig. 7.9 to make an open prism similar to that shown in Fig. 7.7(**b**).

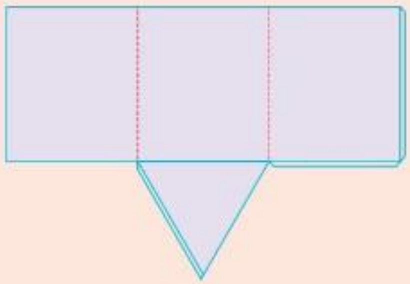


Fig. 7.9

Internet Resources

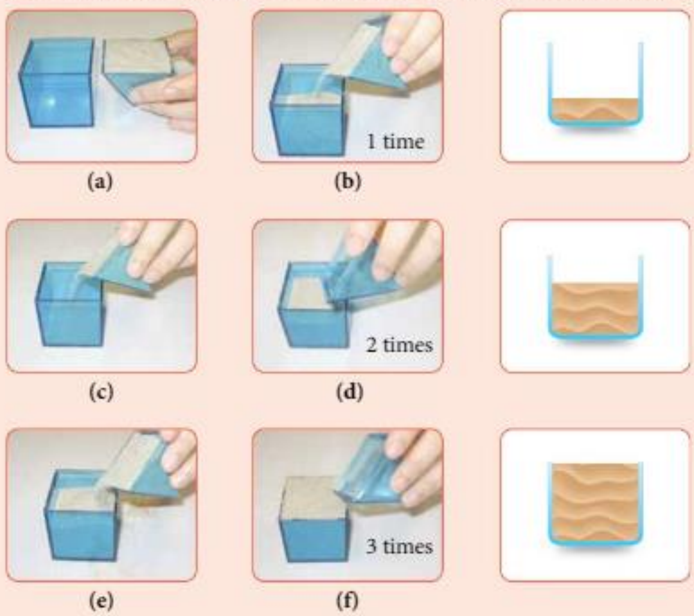
Search the Internet for 'Dynamic Paper - NCTM' for more templates of nets.

Notice that both the pyramid in Fig. 7.8 and the prism in Fig. 7.9 have the *same base* and the *same height*.

Fill the entire pyramid with sand such that the sand is level. Pour the sand into the prism. How many times must you do this until the prism is completely filled? Do you get the same result as your classmates? This suggests that:

Volume of pyramid = ____________ × volume of corresponding prism

You may want to repeat the experiment in the Investigation for different pyramids and their *corresponding prisms* (i.e. prisms with the same base and height). Fig. 7.10 shows a series of pictures where sand is poured from a square pyramid into a *square prism* with the *same base* and the *same height*. The process is repeated until the prism is completely full. It shows that it takes 3 times the volume of a pyramid to fill the prism completely.



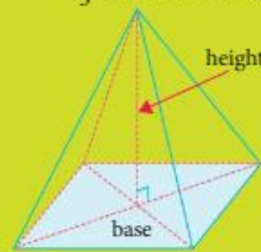
(a) (b) (c) (d) (e) (f)

Fig. 7.10

This suggests that:

Volume of pyramid $= \frac{1}{3} \times$ volume of *corresponding* prism

$= \frac{1}{3} \times$ base area $\times$ height



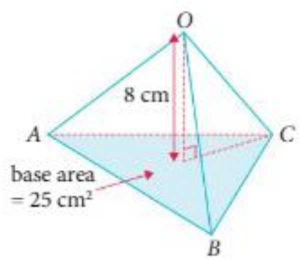
Attention

The *corresponding* prism of a pyramid has the same base and the same height as the pyramid.

Worked Example 1

Finding volume of pyramid, given base area and height

OABC is a triangular pyramid with a base area of 25 cm² and a height of 8 cm. Find the volume of the triangular pyramid.



Information

Another name for a triangular pyramid is 'tetrahedron'. If all the edges are of the same length, it is called a regular tetrahedron.

Solution

Volume of triangular pyramid $= \frac{1}{3} \times$ base area $\times$ height

$= \frac{1}{3} \times 25 \times 8$

$= 66\frac{2}{3}$ cm³

Practise Now 1

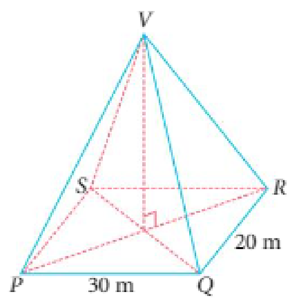
Similar and Further Questions
Exercise 7A
Questions 1–3, 9, 10, 17

1. A triangular pyramid has a base area of 36 cm² and a height of 7 cm. Find the volume of the triangular pyramid.
2. The height of the Great Pyramid of Egypt is 146 m and its base is a square of sides 229 m. Find the volume of the pyramid, leaving your answer correct to 3 significant figures.

Worked Example 2

Finding height of pyramid, given volume and dimensions of base

VPQRS is a rectangular pyramid where $PQ = 30$ m and $QR = 20$ m. Given that the volume of the pyramid is 7000 m^3, find its height.



Solution

$$\text{Volume of pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

$$7000 = \frac{1}{3} \times (30 \times 20) \times \text{height}$$

$$7000 = 200 \times \text{height}$$

$\therefore$ height = 35 m

Practise Now 2

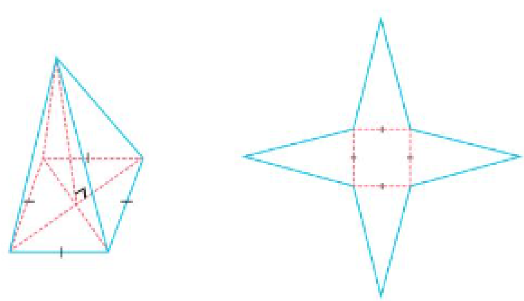
Similar and Further Questions
Exercise 7A
Questions 4, 5, 11, 12

The volume of a pyramid with a square base of length 5 m is 75 m^3. Find its height.

D. Surface area of pyramid

In Book 2, we learnt that the total surface area of a solid is a **measure** of the total area occupied by the surface of the solid. This is equal to the area of all the faces of the net. We shall now use this concept to find the total surface area of a pyramid. Let us consider a square pyramid as shown in Fig. 7.11(**a**).

A net of the pyramid is shown in Fig. 7.11(**b**).



(a) **(b)**

Fig. 7.11

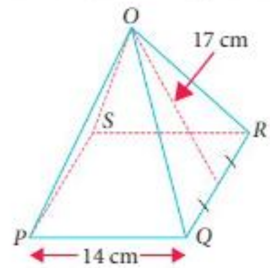
From Fig. 7.11(**b**), it can be seen that:

Total surface area of pyramid = total area of all faces
= area of base + area of all lateral faces

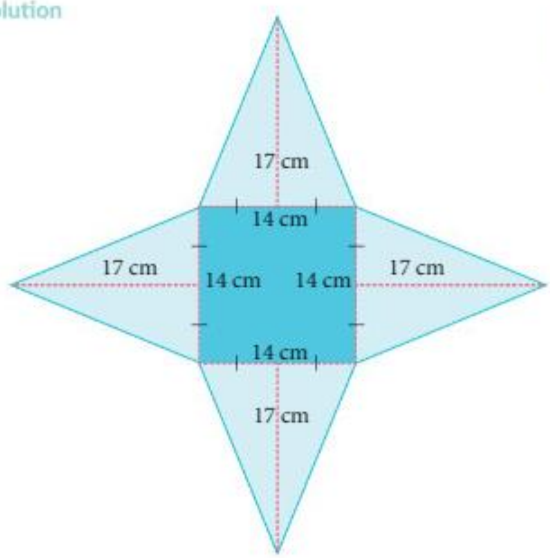
Worked Example 3

Finding total surface area of pyramid, given slant height and dimensions of base

OPQRS is a pyramid with a square base of length 14 cm. Given that the slant height of the pyramid is 17 cm, draw its net and hence, find its total surface area.



*Solution



Big Idea

Diagrams
Diagrams help us visualise the given information so that we can think of how to solve the problem.

Area of each triangular face $= \frac{1}{2} \times 14 \times 17$
$= 119 \text{ cm}^2$

Area of square base $= 14 \times 14$
$= 196 \text{ cm}^2$

$\therefore$ total surface area $= 4 \times$ area of each triangular face + area of square base of pyramid
$= 4 \times 119 + 196$
$= 672 \text{ cm}^2$

Practise Now 3

Similar and Further Questions
Exercise 7A
Questions 6, 13, 14

A pyramid has a square base of length 12 m. Given that the slant height of the pyramid is 15 m, draw its net and hence, find its total surface area.

Worked Example 4

Finding volume of pyramid, given total surface area and dimensions of base

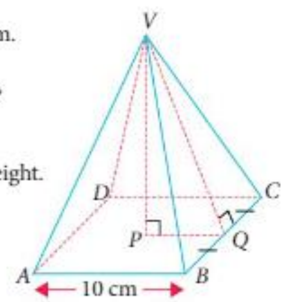
A pyramid has a square base of length 10 cm and a total surface area of 272 cm². Find the volume of the pyramid.

*Solution

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

Stage 1: **Understand the problem**

We need to sketch a diagram to help us understand the problem.
Label the base as $ABCD$ and the apex as V.
What other information is given? What are we supposed to find?



Stage 2: **Think of a plan**

To find the volume of the pyramid, we need its base area and height.
Since the length of the square base and the total surface area are given, how can we calculate the base area and height of the pyramid?

Observe that there is a right-angled triangle in the figure.
Do you recall a formula that can be applied to the lengths of right-angled triangles?

Stage 3: **Carry out the plan**

Attention

In the figure, VP is the height of the pyramid and VQ is the height of the lateral face $\triangle VBC$.

Total surface area of pyramid
= 4 × area of each triangular face + area of square base

Area of each triangular face

$$= \frac{\text{total surface area of pyramid} - \text{area of square base}}{4}$$

$$= \frac{272 - 10 \times 10}{4}$$

$$= \frac{172}{4}$$

$= 43$ cm²

$$\text{Area of } \triangle VBC = \frac{1}{2} \times 10 \times VQ = 43$$

$$5 \times VQ = 43$$

$$VQ = 8.6 \text{ cm}$$

$$PQ = \frac{1}{2} \times AB$$

$$= \frac{1}{2} \times 10$$

$$= 5 \text{ cm}$$

In $\triangle VPQ$, $\angle P = 90°$.

Using Pythagoras' Theorem,

$VQ^2 = VP^2 + PQ^2$

$8.6^2 = VP^2 + 5^2$

$VP^2 = 8.6^2 - 5^2$

$= 73.96 - 25$

$= 48.96$

$\therefore VP = \sqrt{48.96}$ (since $VP > 0$)

$= 6.9971$ cm (to 5 s.f.)

$\therefore$ volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$= \frac{1}{3} \times 10 \times 10 \times 6.9971$

$= 233\text{ cm}^3$ (to 3 s.f.)

Recall

In order for the final answer to be accurate to 3 s.f., any intermediate working must be correct to 4 or 5 s.f. Alternatively, use height $= \sqrt{48.96}$ in the calculation of the volume.

Stage 4: **Look back**

Is the answer reasonable?

Since we observed that there are two right angles in the figure, then finding VQ allows us to apply Pythagoras' Theorem in $\triangle VPQ$ to find VP, the height of the pyramid.

Are the dimensions of $\triangle VPQ$ such that $VP + PQ < VQ$?

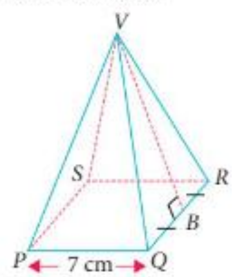
Practise Now 4

Similar and Further Questions
Exercise 7A
Questions 15, 18, 19

VPQRS is a pyramid where the length of each side of its square base is 7 cm. Given that the total surface area of the pyramid is 161 cm², find its

(i) slant height VB,

(ii) volume.



E. Symmetry of pyramid

In Book 1, we learnt about the line and rotational symmetries of a plane figure, and in Book 2, the plane and rotational symmetries of a prism and a cylinder. Using what we learnt about symmetry in Books 1 and 2, we shall now explore the plane and rotational symmetries of a regular pyramid.

Investigation Symmetry in pyramids

Fig. 7.12(**a**) and (**b**) show a square pyramid cut by a plane symmetrically in two different ways. Fig. 7.12(**c**) shows the square pyramid rotating about the axis XY, which passes through the centre of the base of the pyramid.

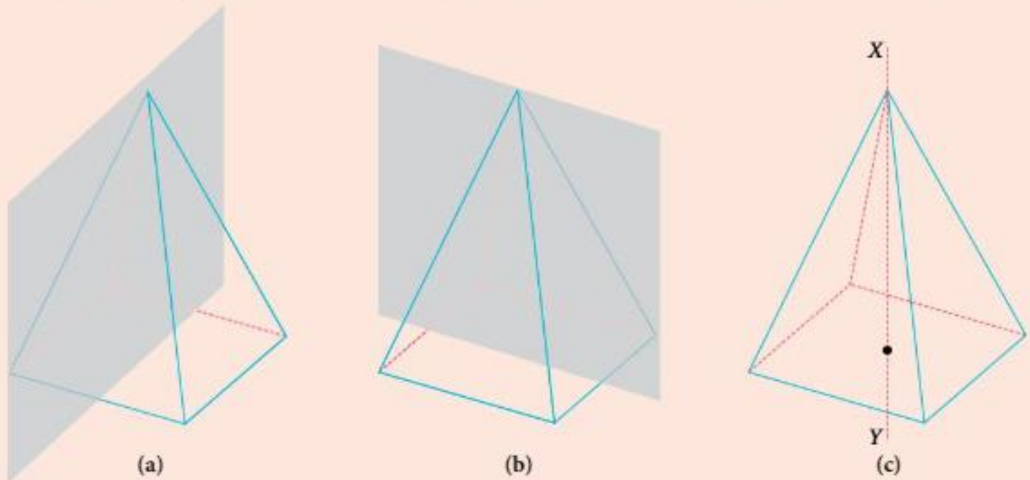


Fig. 7.12

1. For the square pyramid, find
   (**a**) the total number of planes of symmetry,
   (**b**) the order of rotational symmetry when rotated about the axis XY.

Fig. 7.13(**a**) and (**b**) show a pentagonal pyramid cut by a plane symmetrically in two different ways. Fig. 7.13(**c**) shows the pentagonal pyramid rotating about the axis XY, which passes through the centre of the base of the pyramid.

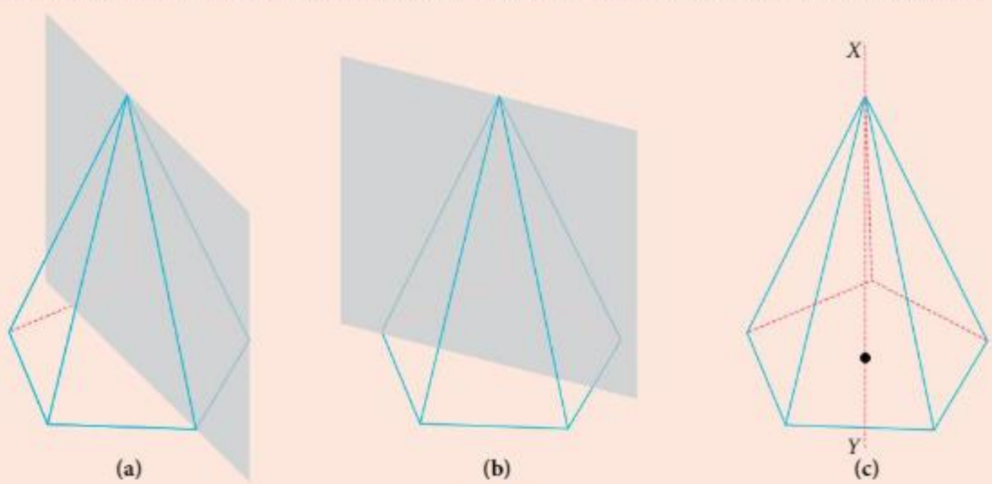


Fig. 7.13

2. For the pentagonal pyramid, find
   (a) the total number of planes of symmetry,
   (b) the order of rotational symmetry when rotated about the axis XY.
3. Repeat the steps in Questions 1 and 2 to complete Table 7.2 for the other pyramids.

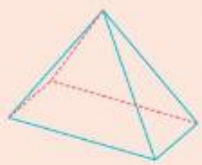
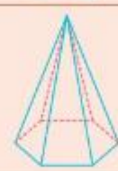
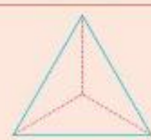
Name	Figure	Number of planes of symmetry	Order of rotational symmetry
Square pyramid			
Rectangular pyramid			
Regular pentagonal pyramid			
Regular hexagonal pyramid			
Regular tetrahedron			

Table 7.2

Similar and Further Questions
Exercise 7A
Questions 7(a), (b), 8(a), (b), 16(a), (b), 20

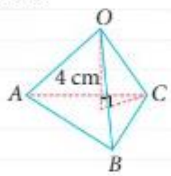
Reflection

1. What do I already know about the volume, surface area and symmetry of prisms that could guide my learning of the volume, surface area and symmetry of pyramids?
2. What have I learnt in this section that I am still unclear of?

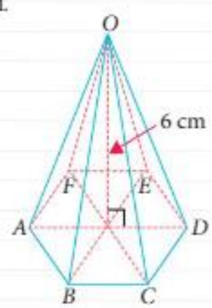
Basic | Intermediate | Advanced

Exercise 7A

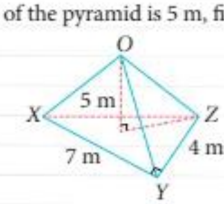
1. $OABC$ is a triangular pyramid with a base area of 15 cm² and a height of 4 cm. Find the volume of the triangular pyramid.



2. $OABCDEF$ is a hexagonal pyramid with a base area of 23 cm² and a height of 6 cm. Find the volume of the pyramid.



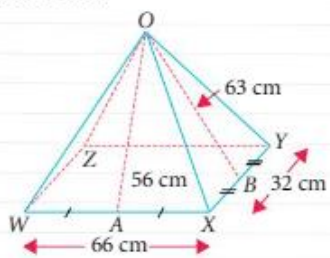
3. $OXYZ$ is a pyramid whose base is a right-angled triangle where $XY = 7$ m and $YZ = 4$ m. Given that the height of the pyramid is 5 m, find its volume.



4. A pyramid with a triangular base has a volume of 50 cm³. If the base and the height of the triangular base are 5 cm and 8 cm respectively, find the height of the pyramid.

5. The volume of a square pyramid with a height of 12 m is 100 m³. Find the length of its square base.

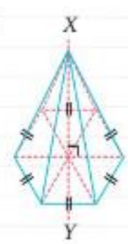
6. $OWXYZ$ is a rectangular pyramid where $WX = 66$ cm and $XY = 32$ cm. Given that the slant heights OA and OB of the pyramid are 56 cm and 63 cm respectively, draw its net and hence, find its total surface area.



7. Find the order of rotational symmetry of each of the following solids.

(a)

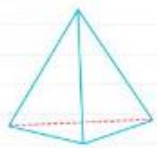
(b)



Exercise 7A

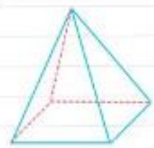
8. For the following solids, state the number of
(i) planes of symmetry,
(ii) axes of rotational symmetry.

(a)



A regular tetrahedron

(b)

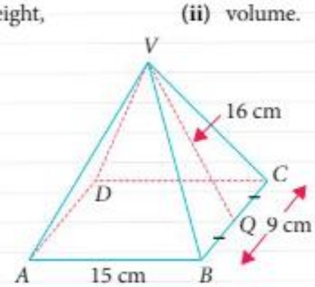


A square pyramid

9. A glass paperweight is in the shape of a solid pyramid with a square base of length 6 cm and a height of 7 cm. Given that the density of the glass is 3.1 g/cm^3, find the mass of four identical paperweights.

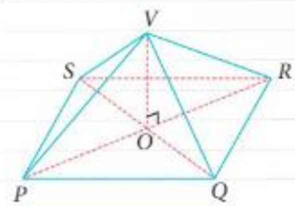
Hint: Density = $\dfrac{\text{mass}}{\text{volume}}$

10. *VABCD* is a pyramid with a rectangular base of sides 15 cm by 9 cm. Given that the slant height *VQ* of the pyramid is 16 cm, find its
(i) height, **(ii)** volume.

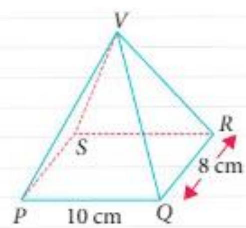


11. A solid pentagonal pyramid has a mass of 500 g. It is made of a material with a density of 6 g/cm^3. Given that the base area of the pyramid is 30 cm^2, find its height.

12. OPEN *VPQRS* is a rectangular pyramid with a volume of 100 cm^3. Suggest a possible height of the pyramid and the corresponding dimensions of the rectangular base.



13. *VPQRS* is a pyramid with a rectangular base of sides 10 cm by 8 cm. Given that the volume of the pyramid is 180 cm^3, find its
(i) height, **(ii)** total surface area.



14. A pyramid has a rectangular base of sides 16 m by 14 m. Given that the volume of the pyramid is 700 m^3, find its
(i) height, **(ii)** total surface area.

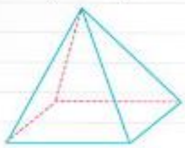
15. A pyramid has a square base of length 4.8 m and a total surface area of 85 m^2. Find the volume of the pyramid.

Basic Intermediate Advanced

Exercise 7A

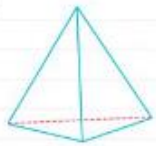
16. Copy and draw the following figures. For each figure, draw two planes of symmetry and one axis of rotational symmetry.

(a)



A right pyramid with a rectagular base

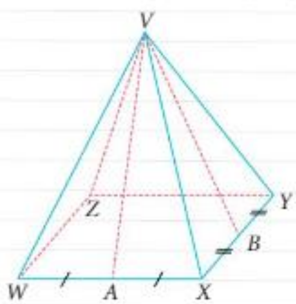
(b)



A regular tetrahedron

17. A solid pyramid has a rectangular base of sides 15 cm by 10 cm and a height of 20 cm. It is placed inside an open cubical tank of sides 30 cm. The tank is then completely filled with water. If the pyramid is removed, what will be the depth of the remaining water in the tank?

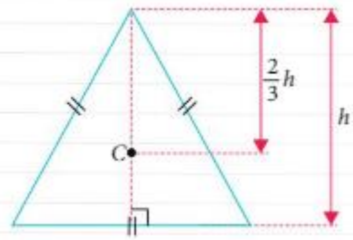
18. $VWXYZ$ is a rectangular pyramid where WX is longer than XY. Is the slant height VA longer or shorter than the slant height VB? Explain your answer.



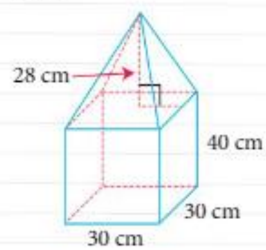
19. The length of each edge of a regular tetrahedron, whose faces are identical equilateral triangles, is 8 cm. Find its

(i) slant height, **(ii)** volume.

Hint: All sides are equal. The centre C of an equilateral triangle is $\frac{2}{3}$ of its height h.



20. A structure is made up of a pyramid attached to a cuboid with a square base of sides 30 cm and a height of 40 cm.



Find

(i) the number of planes of symmetry,
(ii) the number of axes of rotational symmetry,
(iii) the order of rotational symmetry,
of the structure.

7.2 Volume, surface area and symmetry of cones

A. Cones

Class Discussion **What are cones?**

Fig. 7.14 shows some real-life examples of cones.

(a) Ice-cream cone **(b)** Party hat

Fig. 7.14

1. What are some common features of these cones?
2. What are some differences between
   (a) a cone and a cylinder?
   (b) a cone and a pyramid?
   (c) a cone and a triangular prism?

Information

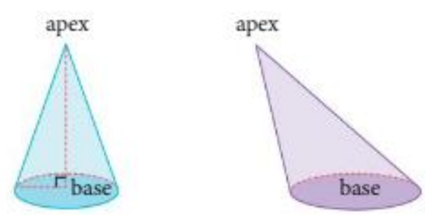
The tetrahedral dice is not a cone. Do you know why?

B. Types of cones

A cone is a solid in which the base is bounded by a **simple closed curve** and the curved surface tapers into a point called the **apex**, which is opposite the base. If the base is circular, the cone is known as a **circular cone**. Fig. 7.15 shows some examples of cones with different bases.

Information

A simple closed curve refers to a closed curve that does not intersect itself.



(a) Circular base **(b)** Elliptical base

Fig. 7.15

In Fig. 7.15**(a)**, the apex is *vertically above the centre of the circular base*. We call such a cone a **right circular cone**.

The **perpendicular height** (or simply the **height**) of a cone is the perpendicular distance from the apex to the base of the cone (see Fig. 7.16). The **slant height** of a right circular cone is the distance from the apex to the circumference of the base.

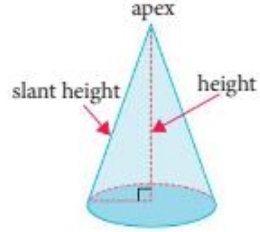


Fig. 7.16

In this section, we will study only right circular cones. Therefore, the term 'cone' is used to refer to a right circular cone. Also, the term 'height' is used to refer to the perpendicular height unless otherwise stated. The radius of the circular base is also known as the **base radius**.

Thinking Time

1. Does Fig. 7.17 show a right circular cone? Explain.
2. Is a pyramid also a cone? Explain.
3. Give three real-life examples of cones.


Fig. 7.17

C. Volume of cone

In Book 2, we learnt:

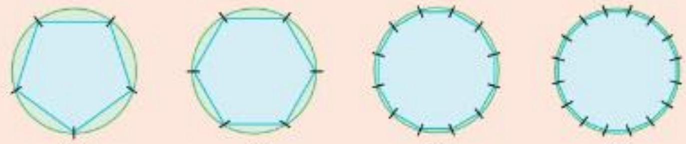
Volume of cylinder = area of cross section × distance between cross-sectional bases
= base area × height
= $\pi r^2 h$,
where r = base radius and h = height of the cylinder.

Let us now find the volume of a cone by comparing it with a pyramid that has a regular polygonal base.

Investigation

Comparison between a cone and a pyramid

1. Fig. 7.18 shows **(a)** a regular pentagon, **(b)** a regular hexagon, **(c)** a regular 12-gon, and **(d)** a regular 16-gon inside a circle respectively.

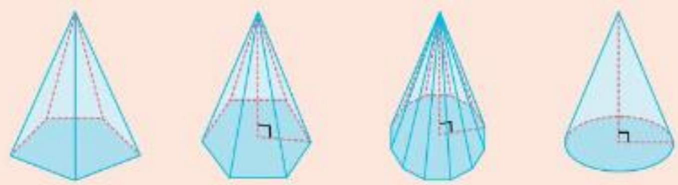


(a) (b) (c) (d)

Fig. 7.18

If the number of sides of a regular polygon is increased indefinitely, what will the polygon start to look like?

2. Fig. 7.19 shows a sequence of regular pyramids, i.e. right pyramids with regular polygonal bases.



(a) (b) (c) (d)

Fig. 7.19

If the number of sides of the regular polygonal base of a pyramid is increased indefinitely, what will the pyramid start to look like?

In many ways, a cone is *like* a pyramid. However, a cone is *not* a pyramid because the base of a pyramid must be a polygon.

Since a cone is like a pyramid (see Fig. 7.19), by analogy, the formula for the volume of a cone should be the same as the formula for the volume of a pyramid. We have:

Volume of cone $= \frac{1}{3} \times \text{base area} \times \text{height}$

$= \frac{1}{3}\pi r^2 h,$

where r is the base radius and h is the height of the cone.

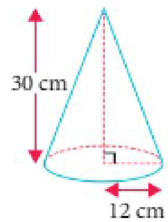
Thinking Time

If a cone and a cylinder have the *same base* and the *same height*, what is the relationship between the volume of the cone and the volume of this corresponding cylinder?

Worked Example 5

Finding volume of cone, given base radius and height

A cone has a circular base of radius 12 cm and a height of 30 cm. Find the volume of the cone.



Solution

$$\begin{aligned}\text{Volume of cone} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\times\pi\times 12^2\times 30 \\ &= 1440\pi \\ &= 4520 \text{ cm}^3 \text{ (to 3 s.f.)}\end{aligned}$$

Recall

If the question does not specify the value of π, we use the value of π stored in the calculator.

Problem-solving Tip

Leave intermediate working in terms of π for better accuracy.

Practise Now 5

Similar and Further Questions
Exercise 7B
Questions 1(a)–(d), 2–4, 11, 12

1. A cone has a circular base of radius 8 cm and a height of 17 cm. Find the volume of the cone.
2. The volume of a cone with a circular base of radius 6 m is 84π m^3. Find its height.

D. Surface area of cone

As a cone has a curved surface, it is difficult to use the flat slant faces of a pyramid as an analogy to find a formula for the curved surface area of a cone. Instead, we will use the same approach as finding the formula for the area of a circle.

Investigation — Curved surface area of cone

Consider the cone as shown in Fig. 7.20(**a**).

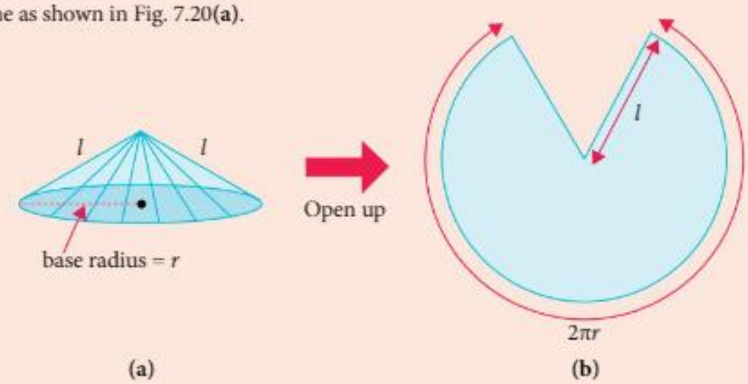


Fig. 7.20

Unfolding the curved surface of the cone will result in the sector shown in Fig. 7.20(**b**). The sector can be divided into 44 smaller sectors and these can be rearranged to form the shape as shown in Fig. 7.21(**b**).

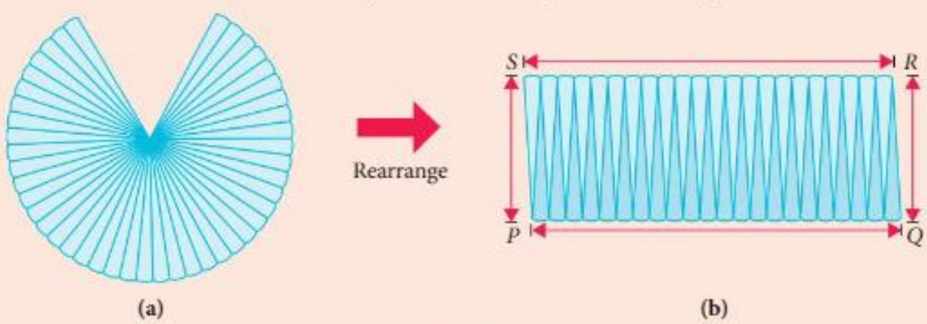


Fig. 7.21

If the number of sectors is increased indefinitely, then the shape in Fig. 7.21(**b**) will look like a ________ *PQRS*.

Since $PQ + RS$ = circumference of the base circle in Fig. 7.20(**a**), then the length of the rectangle, PQ = ________.

Since PS = slant height of the cone in Fig. 7.20(**a**), then the breadth of the rectangle, PS = ________.

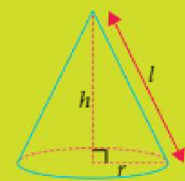
$\therefore$ curved surface area of cone = area of rectangle

= ________ × ________

= ________

From the Investigation on page 229, we have:

Curved surface area of cone = πrl,
where r is the base radius and l is the slant height of the cone.



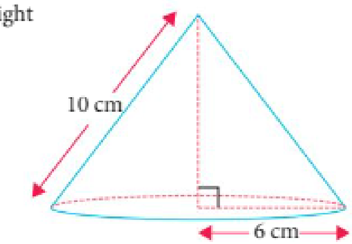
Thinking Time

What is the total surface area of a *solid* cone?

Worked Example 6

Finding total surface area of cone, given base radius and slant height

A cone has a circular base of radius 6 cm and a slant height of 10 cm. Find the total surface area of the cone.



Solution

$$\begin{aligned}\text{Total surface area of cone} &= \pi rl + \pi r^2 \\ &= \pi \times 6 \times 10 + \pi \times 6^2 \\ &= 60\pi + 36\pi \\ &= 96\pi \\ &= 302 \text{ cm}^2 \text{ (to 3 s.f.)}\end{aligned}$$

Practise Now 6

Similar and Further Questions
Exercise 7B
Questions 5(a)–(c), 6–8, 13, 14

1. A cone has a circular base of radius 2 cm and a slant height of 5 cm. Find the total surface area of the cone.
2. A cone has a circular base of radius 8 m. Given that the total surface area of the cone is 350 m^2, find its slant height. (Take π to be 3.142.)

Worked Example 7

Finding curved surface area of cone, given base radius and height

A cone has a circular base of radius 3 m and a height of 4 m. Find the curved surface area of the cone.

Solution

Let the slant height of the cone be l m.

Using Pythagoras' Theorem, $l = \sqrt{3^2 + 4^2}$

$= 5$

$\therefore$ curved surface area of cone $= \pi rl$

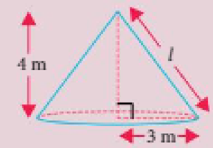
$= \pi \times 3 \times 5$

$= 15\pi$

$= 47.1 \text{ m}^2$ (to 3 s.f.)

Problem-solving Tip

A sketch is helpful.



To find the curved surface area of the cone, we need to know its slant height, but it is not given. However, we notice that there is a right-angled triangle in the figure and so we can make use of Pythagoras' Theorem.

Practise Now 7

Similar and Further Questions
Exercise 7B
Questions 15–17, 19, 20

1. A cone has a circular base of radius 8 m and a height of 15 m. Find the curved surface area of the cone.
2. A cone has a circular base of radius 7 cm and a slant height of 12 cm. Find the volume of the cone.

E. Symmetry of cone

In the following investigation, we shall explore the plane and rotational symmetries of a right cone.

Investigation Symmetry in cones

Fig. 7.22(**a**) shows a cone cut symmetrically by one plane and Fig. 7.22(**b**) shows the cone rotating about the axis XY, which passes through the centre of the base of the cone.

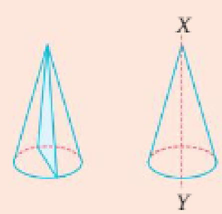


Fig. 7.22

For the cone, find

(**a**) the total number of planes of symmetry,

(**b**) the order of rotational symmetry when rotated about the axis XY.

Similar and Further Questions
Exercise 7B
Questions 9, 10, 18

From the Investigation on page 235, we observe that for a right cone, it has:

- infinite planes of symmetry which pass through the centre of the circular base of the cone and its vertex,
- infinite order of rotational symmetry, about the axis which passes through the centre of the circular base of the cone and its vertex.

Reflection

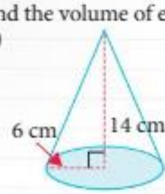
1. What do I already know about the volume and surface area of cylinders that could guide my learning of the volume and surface area of cones?
2. What have I learnt in this section that I am still unclear of?

Basic | Intermediate | Advanced

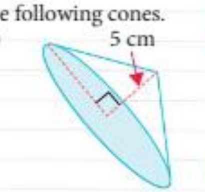
Exercise 7B

1. Find the volume of each of the following cones.

(a)

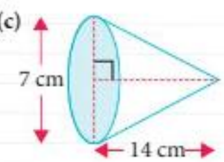


(b)

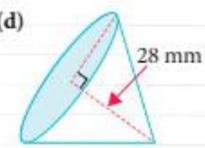


Area of base = 154 cm²

(c)



(d)



Circumference of base = 132 mm

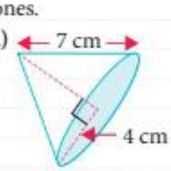
2. A cone has a base area of 20 m² and a volume of 160 m³. Find the height of the cone.

3. OPEN The volume of a cone is 320π cm³ and its base area is πr^2 cm², where r is a prime number. Suggest a value of r and find the corresponding height of the cone.

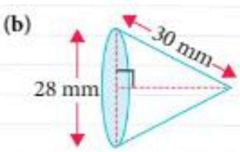
4. A cone has a height of 14 cm and a volume of 132 cm³. Find the radius of the circular base. (Take π to be $\frac{22}{7}$.)

5. Find the total surface area of each of the following cones.

(a)



(b)



(c)

Circumference of base = 132 cm

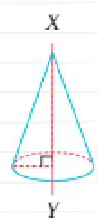
Exercise 7B

6. A cone has a circular base of radius 6 mm. Given that the curved surface area of the cone is 84π mm², find its slant height.

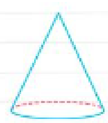
7. A cone has a total surface area of 1000 cm². Find a possible slant height and radius of the circular base. (Take π to be 3.142.)

8. An open cone has a slant height of 5 m and a curved surface area of 251 m². Find the radius of the circular base.

9. Find the order of rotational symmetry of the following cone.



10. For a right circular cone, state
(i) the number of planes of symmetry,
(ii) the number of axes of rotational symmetry.

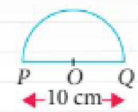


11. A conical funnel of diameter 23.2 cm and depth 42 cm contains water filled to the brim. The water is poured into a cylindrical tin of diameter 16.2 cm. If the tin must contain all the water, find its least possible height, giving your answer to the nearest integer.

12. A conical block of silver has a height of 16 cm and a base radius of 12 cm. The silver is melted to form coins $\frac{1}{6}$ cm thick and $1\frac{1}{2}$ cm in diameter. Find the number of coins that can be made.

13. An open cone has a circular base of radius 10 cm and a slant height of 20 cm. Draw the net of the cone and label its dimensions.

14. The semicircle shown is folded to form a right circular cone so that the arc PQ becomes the circumference of the base. Given that the diameter of the semicircle, PQ, is 10 cm, find
(i) the diameter of the base,
(ii) the curved surface area,
of the cone.

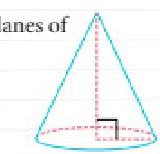


15. A cone has a circular base of radius 5 cm and a height of 12 cm. Find the curved surface area of the cone.

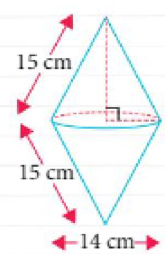
16. A cone has a circular base of radius 8 cm and a slant height of 20 cm. Find the volume of the cone.

17. A circular cone has a height of 17 mm and a slant height of 21 mm. Find
(i) the volume,
(ii) the total surface area,
of the cone.

18. For the following cone, draw two planes of symmetry and one axis of rotational symmetry.



19. A solid is made up of two identical cones, each with a base diameter of 14 cm and a slant height of 15 cm. Find
(i) the total surface area,
(ii) the volume,
of the solid.



20. A cone has a circular base of radius 13.5 m. Given that the total surface area of the cone is 1240 m², find its volume.

7.3 Volume and surface area of spheres

Fig. 7.23 shows some real-life examples of spheres.


(**a**) Soccer ball


(**b**) Crystal ball


(**c**) Globe

Fig. 7.23

Internet Resources

Map projection is a method of representing the surface of a *sphere* on a *plane*. Is it possible to draw the map of all the countries in the world on a plane accurately without distorting the shape or the area? Search the Internet to find out more about conformal (or area-preserving) map projections.

What do you think is an important feature of spheres?

Thinking Time

1. What is a **hemisphere**? What are some real-life examples of hemispheres?
2. Is a rugby ball a sphere? How about a chicken's egg? Explain your answer.

A. Volume of sphere

Class Discussion — **Is the King's crown made of pure gold?**

Archimedes is one of the greatest mathematicians of all time. He lived from 287 BC to 212 BC in Greece. One day, the King asked Archimedes to find out whether his crown was made of pure gold. Archimedes thought about how to find the volume of the crown for a long time. He then decided to take a break by taking a bath. As he stepped into the bathtub, the water overflowed. Archimedes was so excited by this discovery that he dashed out into the street, unclothed, shouting 'Eureka!' which means 'I have found it!'.

Archimedes realised that a sinking solid displaces an amount of water that is equal to the volume of the solid. To find the volume of the crown, we can fill up the Eureka can (named as a result of the above story) as shown in Fig. 7.24(**a**) with water until it overflows before putting in the crown. The volume of water displaced into the container as shown in Fig. 7.24(**b**) is equal to the volume of the crown.

(a) Before

(b) After

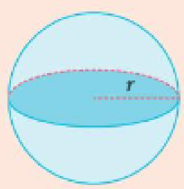
Fig. 7.24

Suppose Archimedes found out that the volume of water displaced was 714 cm^3 and the mass of the crown was 11.6 kg. Given that the density of gold is 19.3 g/cm^3, determine whether the crown was made of pure gold. Do you arrive at the same conclusion as your classmates?

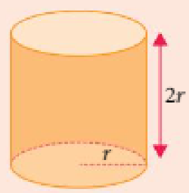
Investigation — Volume of sphere

Part 1: Archimedes' discovery of the volume of sphere

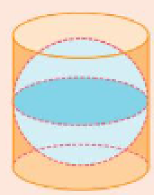
Using the displacement method mentioned in the above Class Discussion, Archimedes discovered a formula to find the volume of a sphere. Fig. 7.25(**a**) shows a sphere of radius r and Fig. 7.25(**b**) shows its related circular cylinder of base radius r and height $2r$. Archimedes filled the cylinder with water and placed the sphere inside it as shown in Fig. 7.25(**c**).



(**a**) Sphere



(**b**) Related cylinder



(**c**) Sphere inside related cylinder

Fig. 7.25

Archimedes found out that the volume of water displaced was equal to $\frac{2}{3}$ of the volume of the cylinder. This was one of his greatest discoveries.

Part 2: Work out the volume of sphere

Volume of cylinder $= \pi r^2 h$

$= \pi \times r^2 \times 2r$

$=$ ______

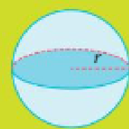
Volume of sphere $= \frac{2}{3} \times$ volume of cylinder

$= \frac{2}{3} \times$ ______

$=$ ______

From the Investigation on page 239, we have:

Volume of sphere $= \frac{4}{3}\pi r^3$,
where r is the radius of the sphere.



Big Idea

Proportionality
In the Investigation on page 239, we learnt that the volume of a sphere is $\frac{2}{3}$ of the volume of a related cylinder. Earlier, we learnt that the volume of a cone is $\frac{1}{3}$ of the volume of a cylinder with the same base and height. These are examples of how we use the idea of proportionality, whereby the volumes of these two solids are in the same ratio, to determine the formulae for the volumes of the various solids.

Worked Example 8

Finding volume of sphere, given radius

A ball bearing (which is spherical in shape) has a radius of 0.3 cm.
Find
(i) the volume of the ball bearing,
(ii) the mass of 6000 identical ball bearings if they are made of steel of density 7.85 g/cm³.

Hint: Density $= \frac{\text{mass}}{\text{volume}}$

Solution

(i) Volume of ball bearing $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \pi \times 0.3^3$$
$$= 0.036\pi$$
$$= 0.113 \text{ cm}^3 \text{ (to 3 s.f.)}$$

(ii) Mass of 6000 ball bearings = volume of 6000 ball bearings × density

$$= 6000 \times 0.036\pi \times 7.85$$
$$= 5330 \text{ g (to 3 s.f.)}$$

Practise Now 8

Similar and Further Questions
Exercise 7C
Questions 1(a)–(c), 2(a)–(f), 7–11, 14

1. A lead ball bearing has a diameter of 0.4 cm. Find the mass of 5000 identical lead ball bearings if the mass of 1 cm³ of lead is 11.3 g.
2. A basketball has a volume of 5600 cm³. Find its radius.

Introductory Problem Revisited

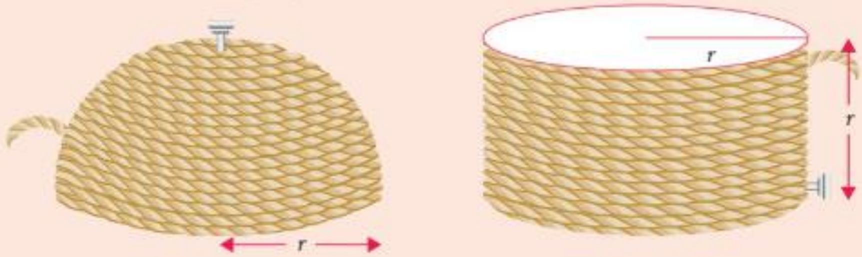
In the Introductory Problem, you may not have known how to compare the volumes of a cone with a radius r cm and height h cm, a pyramid with a square base of side r cm and height h cm, as well as a sphere of radius r cm. After learning the formulae for the volumes of pyramids, cones and spheres, do you know how to arrange them in ascending order of their volumes? Discuss your answers with your classmates.

B. Surface area of sphere

Investigation — Surface area of sphere

Part 1: Archimedes' discovery of the surface area of sphere

Archimedes also discovered a formula to find the surface area of a sphere. Fig. 7.26(**a**) shows a hemisphere of radius r. One end of a piece of twine is stuck in the centre of the curved surface of the hemisphere by a pin before the twine is coiled around the curved surface completely. Fig. 7.26(**b**) shows a circular cylinder of base radius r and height r. One end of a piece of twine is stuck at the bottom of the curved surface of the cylinder by a pin before the twine is coiled around the curved surface completely.



(**a**) Hemisphere (**b**) Cylinder

Fig. 7.26

Archimedes found out that the two pieces of twine were of the same length. Since the two pieces of twine cover the curved surface of the hemisphere and the curved surface of the cylinder in Fig. 7.26 respectively, this means that:

Curved surface area of hemisphere = curved surface area of cylinder

$= 2\pi rh$

$= 2\pi \times r \times r$

= ________

$\therefore$ surface area of sphere = $2 \times$ curved surface area of hemisphere

$= 2 \times$ ________

= ________

Part 2: Simple experiment to find the surface area of sphere

1. Take an orange and cut it into halves. Place one half flat on a sheet of paper and draw a circle around it. The area of the circle is πr^2, where r is the radius of the orange, assuming that the orange is spherical.

2. Repeat step 1 and draw a few more circles.
3. Peel the skin of both halves of the orange, tear into small pieces and cover as many circles as completely as possible with the skin.
4. How many circles are covered completely with the orange skin?
5. What do you think the surface area of the orange (spherical in shape) is?

In Part 2, about 4 circles are completely covered by the orange skin.
From the above Investigation, we have:

Surface area of sphere $= 4\pi r^2$,
where r is the radius of the sphere.

Thinking Time

What is the total surface area of a solid hemisphere?

Journal Writing

Think about various items that you have come across in your daily lives. Which of these are pyramids, or cones, or spheres? Are you able to make sketches of them? Can you think of any reasons why they are shaped as pyramids or cones or spheres?

Worked Example 9

Finding surface area of sphere, given radius

A solid sphere has a radius of 7 cm. Find its surface area.

*Solution

Surface area of sphere $= 4\pi r^2$

$= 4 \times \pi \times 7^2$

$= 196\pi$

$= 616\text{ cm}^2$ (to 3 s.f.)

Practise Now 9

Similar and Further Questions
Exercise 7C
Questions 3(a)–(c), 4, 12, 15

1. A solid sphere has a radius of 0.6 m. Find its surface area.
2. A solid sphere has a diameter of 25 cm. Find its surface area.

Worked Example 10

Finding radius of hemisphere, given curved surface area

A solid hemisphere has a curved surface area of 175 cm². Find its radius.

*Solution

Curved surface area of solid hemisphere $= \frac{1}{2} \times 4\pi r^2 = 2\pi r^2 = 175$

$$r^2 = \frac{175}{2\pi}$$

$$\therefore r = \sqrt{\frac{175}{2\pi}} \quad (\text{since } r > 0)$$

$$= 5.28 \text{ cm (to 3 s.f.)}$$

Practise Now 10

Similar and Further Questions
Exercise 7C
Questions 5(a)–(f), 6, 13

A hemisphere has a curved surface area of 200 cm². Find its radius.

Advanced | Intermediate | Basic

Exercise 7C

1. Find the volume of each of the following spheres with the given radius.
(a) 8 cm (b) 14 mm
(c) 4 m

2. Find the radius of each of the following spheres with the given volume.
(a) 1416 cm³ (b) 12 345 mm³
(c) 780 m³ (d) 972π cm³
(e) 498π mm³ (f) $15\frac{3}{16}\pi$ m³

3. Find the surface area of each of the following spheres.
(a) radius = 12 cm (b) radius = 9 mm
(c) diameter = 6 m

4. Find the total surface area of a solid hemisphere of radius 7 cm. (Take π to be 3.142.)

5. Find the radius of each of the following spheres with the given surface area.
(a) 210 cm² (b) 7230 mm²
(c) 3163 m² (d) 64π cm²
(e) 911π mm² (f) 49π m²

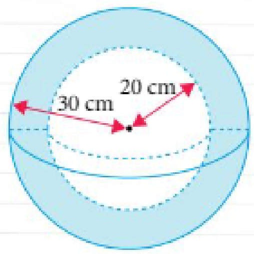
6. A hemisphere has a curved surface area of 364.5π cm². Find its radius.

7. Find the number of steel ball bearings, each of diameter 0.7 cm, which can be made from 1 kg of steel, given that 1 cm³ of steel has a mass of 7.85 g.

Basic Intermediate Advanced

Exercise 7C

8. A hollow aluminium sphere has an internal radius of 20 cm and an external radius of 30 cm. Given that the density of aluminium is 2.7 g/cm³, find the mass of the sphere in kg.



9. 54 solid hemispheres, each of diameter 2 cm, are melted to form a single sphere. Find the radius of the sphere.

10. A sphere of diameter 26.4 cm is half-filled with acid. The acid is drained into a cylindrical beaker of diameter 16 cm. Find the depth of the acid in the beaker.

11. A cylindrical tin has an internal diameter of 18 cm. It contains water to a depth of 13.2 cm. A heavy spherical ball bearing of diameter 9.3 cm is dropped into the tin. Find the new height of water in the tin, leaving your answer correct to 2 decimal places.

12. A sphere has a volume of 850 m³. Find its surface area.

13. A basketball has a surface area of 1810 cm². Find its volume.

14. A cylindrical can has a base radius of 3.4 cm. It contains a certain amount of water such that when a sphere is placed inside the can, the water just covers the sphere. If the sphere fits exactly inside the can, find
   (i) the surface area of the can that is in contact with the water when the sphere is inside the can,
   (ii) the depth of water in the can before the sphere was placed inside the can.

15. There are 20 identical solid hemispheres. The curved surface and the flat surface of each hemisphere are to be painted red and yellow respectively. Find the ratio of the amount of red paint needed to the amount of yellow paint needed.

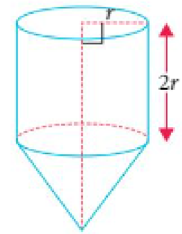
7.4 Volume and surface area of composite solids

In this section, we shall learn how to solve problems involving the volume and surface area of composite solids.

Worked Example 11

Solving problem involving a cylinder and a cone

A container is made up of a hollow cone with an internal base radius of r cm and a hollow cylinder with the same base radius and an internal height of $2r$ cm. Given that the height of the cone is two-thirds of the height of the cylinder and 5 litres of water are needed to fill the conical part of the container completely, find the amount of water needed to fill the container completely, giving your answer in litres.



Solution

$$\text{Height of cone} = \frac{2}{3} \times \text{height of cylinder}$$
$$= \frac{2}{3} \times 2r$$
$$= \frac{4}{3}r$$
$$\text{Volume of cone} = \frac{1}{3}\pi r^2\left(\frac{4}{3}r\right)$$
$$= \frac{4}{9}\pi r^3$$

Method 1:

Since volume of cone $= 5\ l = 5000\ \text{cm}^3$,

$$\text{then } \frac{4}{9}\pi r^3 = 5000$$
$$\pi r^3 = \frac{5000 \times 9}{4}$$
$$= 11\ 250$$
$$\text{Volume of cylinder} = \pi r^2(2r)$$
$$= 2\pi r^3$$
$$= 2 \times 11\ 250$$
$$= 22\ 500\ \text{cm}^3$$
$$= 22.5\ l$$

$\therefore$ amount of water needed to fill container completely $= 22.5 + 5$
$= 27.5\ l$

Recall

$1\ l = 1000\ \text{cm}^3$

Problem-solving Tip

Alternatively, we can solve for r and substitute this value to find the volume of cylinder. Since this is an intermediate step, the value of r used must be correct to at least 4 s.f.

Method 2:

Since volume of cone $= \frac{4}{9}\pi r^3 = 5\ l$,

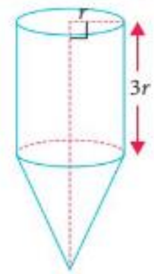
$$\text{then volume of cylinder} = 2\pi r^3$$
$$= \frac{2}{\frac{4}{9}} \times \text{volume of cone}$$
$$= \frac{2}{\frac{4}{9}} \times 5$$
$$= 22.5\ l$$

$\therefore$ amount of water needed to fill container completely $= 22.5 + 5$
$= 27.5\ l$

Practise Now 11

Similar and Further Questions
Exercise 7D
Questions 1, 2, 5, 6

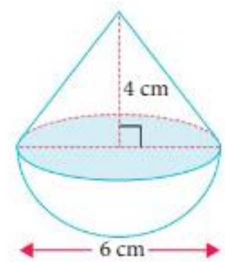
A container is made up of a hollow cone with an internal base radius of r cm and a hollow cylinder with the same base radius and an internal height of $3r$ cm. Given that the height of the cone is three-quarters of the height of the cylinder and 10 litres of water are needed to fill the conical part of the container completely, find the amount of water needed to fill the container completely, giving your answer in litres.



Worked Example 12

Solving problem involving a cone and a hemisphere

A solid consists of a cone and a hemisphere which share a common base. The cone has a height of 4 cm and a base diameter of 6 cm.



(a) Find

(i) the volume, **(ii)** the total surface area,

of the solid.

(b) The solid is melted and recast to form a solid cylinder with a height of 4 cm. Find the radius of the cylinder.

(c) If 1000 identical cylinders are to be painted and each tin of paint is enough to paint an area of 5 m^2, find the number of tins of paint needed.

Solution

(a) Radius of cone = radius of hemisphere

$= 6 \div 2$

$= 3$ cm

(i) Volume of solid = volume of cone + volume of hemisphere

$$= \frac{1}{3} \times \pi \times 3^2 \times 4 + \frac{1}{2} \times \frac{4}{3} \times \pi \times 3^3$$
$$= 12\pi + 18\pi$$
$$= 30\pi$$
$$= 94.2 \text{ cm}^3 \text{ (to 3 s.f.)}$$

(ii) Using Pythagoras' Theorem,

Slant height of cone $= \sqrt{3^2 + 4^2}$

$= 5$ cm

Total surface area of solid

= curved surface area of cone + curved surface area of hemisphere

$= \pi \times 3 \times 5 + 2 \times \pi \times 3^2$

$= 15\pi + 18\pi$

$= 33\pi$

$= 104$ cm^2 (to 3 s.f.)

(b) Volume of cylinder $= \pi r^2(4) = 30\pi$

$$4\pi r^2 = 30\pi$$

$$r^2 = \frac{30}{4}$$

$$= 7.5$$

$$\therefore r = \sqrt{7.5} \text{ (since } r > 0)$$

$$= 2.74 \text{ cm (to 3 s.f.)}$$

(c) Surface area of one cylinder $= 2 \times \pi \times \left(\sqrt{7.5}\right)^2 + 2 \times \pi \times \sqrt{7.5} \times 4$

$$= 115.95 \text{ cm}^2 \text{ (to 5 s.f.)}$$

Surface area of 1000 cylinders $= 1000 \times 115.95$

$$= 115\,950 \text{ cm}^2$$

$$= 11.595 \text{ m}^2$$

Since $\frac{11.595}{5} = 2.319$, then 3 tins of paint are needed to paint 1000 cylinders.

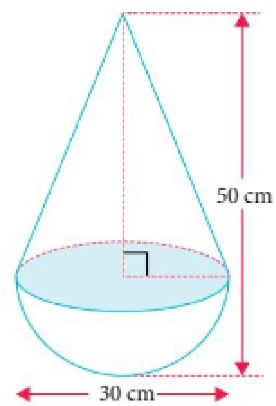
Attention

We cannot round 2.319 off to the nearest whole number because 2 tins of paint are not sufficient to paint 1000 cylinders.

Practise Now 12

Similar and Further Questions
Exercise 7D
Questions 3, 4, 7–10

A solid consists of a cone and a hemisphere which share a common base. The solid has a height of 50 cm and the hemisphere has a diameter of 30 cm.



(a) Find
  (i) the volume,
  (ii) the total surface area,
of the solid.

(b) The solid is melted and recast to form a solid cylinder with a radius of 12.5 cm. Find
  (i) the height of the cylinder,
  (ii) the surface area of the cylinder, leaving your answer in terms of π.

Reflection

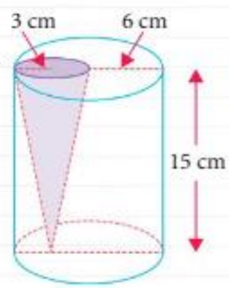
1. **(a)** What do I have to do to find the volume of a composite solid?
   (b) What do I have to do to find the surface area of a composite solid?
   (c) Are my approaches to parts **(a)** and **(b)** the same?
2. What have I learnt in this section or chapter that I am still unclear of?
3. How can I consolidate all that I have learnt about finding the volume and the surface area of 3D solids?

Basic | Intermediate | Advanced

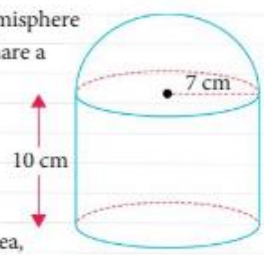
Exercise 7D

1. A rocket is in the shape of a cone attached to a cylinder with the same base radius. The cone has a slant height of 15 m. The cylinder has a base diameter of 12 m and a height of 42 m. Find the total surface area of the rocket.

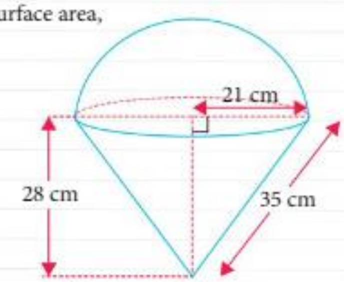
2. A cylinder has a radius of 6 cm and a height of 15 cm. A hole in the shape of a cone is bored into one of its ends. If the cone has a radius of 3 cm, find the volume of the remaining solid.



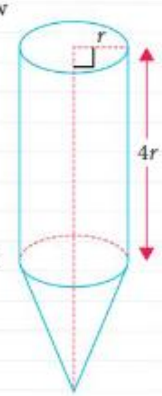
3. A solid consists of a hemisphere and a cylinder which share a common base. The cylinder has a base radius of 7 cm and a height of 10 cm. Find
(i) the volume,
(ii) the total surface area,
of the solid.



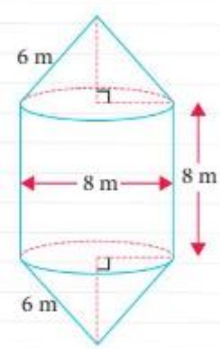
4. A solid consists of a hemisphere and a cone which share a common base. The cone has a base radius of 21 cm, a height of 28 cm and a slant height of 35 cm. Find
(i) the volume,
(ii) the total surface area,
of the solid.



5. A container is made up of a hollow cone with an internal base radius of r cm and a hollow cylinder with the same base radius and an internal height of $4r$ cm. Given that the height of the cone is three-fifths of the height of the cylinder and 7 litres of water are needed to fill the conical part of the container completely, find the amount of water needed to fill the container completely, giving your answer in litres.



6. Find
(i) the total surface area,
(ii) the volume,
of the solid cylinder with conical ends as shown.

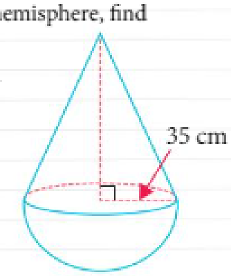


Exercise 7D

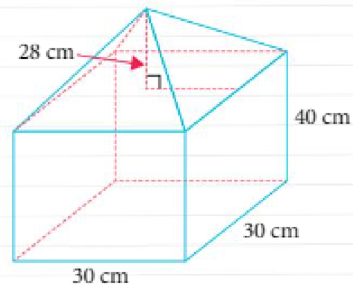
7. A storage tank consists of a hemisphere and a cylinder which share a common base. The tank has a height of 16.5 m and the cylinder has a base diameter of 4.7 m. Find the capacity of the tank.

8. A solid metal ball of radius 3 cm is melted and recast to form a solid circular cone of radius 4 cm. Find the height of the cone.

9. A solid consists of a cone and a hemisphere which share a common base. The cone has a base radius of 35 cm. Given that the volume of the cone is equal to $1\frac{1}{5}$ of the volume of the hemisphere, find
   (i) the height of the cone,
   (ii) the total surface area of the solid, leaving your answer in terms of π.



10. A solid consists of a pyramid of height 28 cm attached to a cuboid with a square base of sides 30 cm and a height of 40 cm. Find
   (i) the volume,
   (ii) the total surface area,
   of the solid.

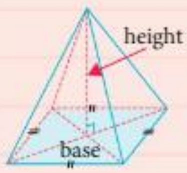


Looking Back

In this chapter, we see how knowing the relationships between solids can help us use what we know about the volumes of prisms and cylinders to find the volumes of pyramids, cones and spheres. For example, one of our class Investigations earlier showed that the volume of a cone is $\frac{1}{3}$ the volume of a cylinder with the same base and height. This relationship is an example of the idea of **proportionality**. We can use this important idea to solve real-world problems involving solids such as pyramids, cones, and spheres. These solids are all around us — in tea bags, party hats, and soccer balls, just to name a few. Associating these solids with real-world objects allows us to model them using ideas of proportionality in similar figures and solids, so as to apply the relationships to solve relevant real-world problems. This will be covered in Book 4.

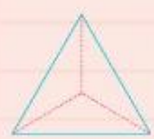
Summary

1. Volume and total surface area of solids

Name	Figure	Volume	Total surface area
Pyramid		For a pyramid and its *corresponding* prism with the *same base* and the *same height*, volume of pyramid $= \frac{1}{3} \times$ volume of corresponding prism $= \frac{1}{3} \times$ base area $\times$ height	Total area of all faces = area of base + area of all lateral faces
Cone		For a cone and its *corresponding* cylinder with the *same base radius r* and the *same height h*, volume of cone $= \frac{1}{3} \times$ volume of *corresponding* cylinder $= \frac{1}{3} \times$ base area $\times$ height $= \frac{1}{3}\pi r^2 h$	Total area of all faces $= \pi r^2 + \pi r l$
Sphere		$\frac{4}{3}\pi r^3$	$4\pi r^2$

- Think of real-world objects that you can use these formulae to find the volumes and surface areas of.

2. Plane and rotational symmetries of right pyramids and cones

Name	Square pyramid	Rectangular pyramid	Regular pentagonal pyramid	Regular hexagonal pyramid	Regular tetrahedron	Cone
Figure						
Number of planes of symmetry	4	2	5	6	6	Infinite
Order of rotational symmetry	4	2	5	6	3 for each axis of rotational symmetry	Infinite

CHAPTER 8

Averages of Statistical Data

Each year, the Department of Statistics, Singapore, publishes key statistics about Singapore. For example, in 2018, it was found that:

- the average life expectancy at birth was 81.0 years for males and 85.4 years for females;
- the mean years of schooling amongst residents aged 25 years and above was 11.1;
- the median age of residents was 40.8 years.

Statistics is important for us to understand the world around us so that we can make informed decisions. In Book 2, we have studied how numerical data can be summarised and analysed with the help of histograms. Here, we will extend our study of the interpretation of numerical data to the idea of a measure of central tendency, i.e. where the data points are centred about. This centre represents a data set with a single value.

Different scenarios may require the use of different **measures** of central tendency to help us interpret the data in a meaningful and useful way. In this chapter, we will learn when to use the three types of measures of central tendency: mean, median and mode.

Learning Outcomes

What will we learn in this chapter?

- What the types of average (mean, median and mode) of a data set are
- How to calculate the mean, median and mode of a data set or a frequency distribution
- How to estimate the mean of a set of grouped data
- Which type of average to use as a measure of the central tendency

Introductory Problem

In primary school, we learnt how to find the **average** of a set of data:

$$\text{Average} = \frac{\text{total number or amount}}{\text{number of data values}} \text{ or } \frac{\text{sum of data values}}{\text{number of data values}}$$

For example, consider the monthly salaries (in dollars) of nine employees in a company:

2000, 2100, 2700, 2400, 2200, 2000, 10 000, 2800, 2600.

1. Find the average monthly salary of the employees.
2. How many employees earn less than the average found in Question 1?
3. Does the average found in Question 1 present a fitting picture of their general monthly salaries? Why or why not?
4. Is there another way of calculating the average monthly salary that will present a more fitting picture? If yes, describe the method of calculation.

In this chapter, we will recap the type of average learnt in primary school and learn two other types of average of a statistical data set. In addition, we will also learn how to estimate the average of a set of grouped data.

8.1 Mean

A. Mean of raw data

Raw data are data that have not been organised into a frequency table. For example, the data in the **Introductory Problem** are raw data.

In primary school, we learnt how to find the average of raw data. This average is called the **mean**:

Mean of raw data

$$\text{Mean} = \frac{\text{sum of data values}}{\text{number of data values}}$$

Information

The mean in this chapter refers to the arithmetic mean. There are other types of mean, such as the geometric mean, the harmonic mean and the weighted mean. You can search the Internet to find out more about these other types of mean.

Worked Example 1

Calculating mean of raw data

The numbers of books borrowed by five students from a library are 3, 4, 4, 9 and 10. Calculate the mean number of books borrowed by the five students.

Solution

$$\text{Mean number of books} = \frac{\text{sum of data values}}{\text{number of data values}}$$
$$= \frac{3 + 4 + 4 + 9 + 10}{5}$$
$$= \frac{30}{5}$$
$$= 6$$

Problem-solving Tip

It is important to distinguish between the data values and the number of data values. In this example, the data values are the number of books borrowed, and there are 5 data values.

Attention

An idea of the mean is the notion of '**even out**'. In this example, if we put all the books together and distribute them ***evenly*** among the 5 students, each of them will get 6 books, which is the average number of books.

Practise Now 1

Similar and Further Questions
Exercise 8A
Questions 1, 2

The heights of seven students are 1.54 m, 1.67 m, 1.49 m, 1.54 m, 1.76 m, 1.69 m and 1.58 m.
Calculate the mean height of the students.

Worked Example 2

Finding unknown given mean

The mean of \$10, \$15, \$12, \$20 and \$$x$ is \$13. Find the value of x.

Solution

$$\text{Mean} = \frac{10 + 15 + 12 + 20 + x}{5} = 13$$

$$\therefore 10 + 15 + 12 + 20 + x = 13 \times 5$$
$$57 + x = 65$$
$$x = 8$$

Big Idea

Measures

Another idea of the mean is that it is a **measure of the central tendency** (or the centre) of a data set. In this example, do you realise that the mean (which is 6) is somewhere in the centre of the data set: 3, 4, 4, 9, 10?

Attention

What does $10 + 15 + 12 + 20 + x = 13 \times 5$ in Worked Example 2 signify? The LHS of the equation is the sum of the data values while the RHS is the mean multiplied by the number of data values. In other words, mean × number of data values = sum of data values.

Practise Now 2

Similar and Further Questions
Exercise 8A
Question 3

The mean of 44, 47, y, 58 and 55 is 52. Find the value of y.

Worked Example 3

Solving problem involving mean

The mean of six numbers is 17. Four of the numbers are 15, 17, 20 and 22. The remaining two numbers are equal. Calculate

(i) the sum of the six numbers,
(ii) the remaining two numbers.

Solution

(i) $\text{Mean} = \frac{\text{sum of data values}}{\text{number of data values}}$,

$\therefore$ sum of data values $=$ mean $\times$ number of data values
$= 17 \times 6$
$= 102$

$\therefore$ sum of the six numbers $= 102$

Attention

In Worked Example 2, we have learnt that sum of data values = mean × number of data values. So, we can calculate the sum of the data values using the mean, without knowing each individual value.

(ii) Let each of the two remaining numbers be x.

Then sum of data values $= 15 + 17 + 20 + 22 + x + x = 102$

$$74 + 2x = 102$$
$$2x = 28$$
$$x = 14$$

$\therefore$ the remaining two numbers are 14 and 14.

Practise Now 3

Similar and Further Questions
Exercise 8A
Questions 4, 5, 11, 12

1. The mean of seven numbers is 11. Five of the numbers are 3, 17, 20, 4 and 15. The remaining two numbers are equal. Find
   (i) the sum of the seven numbers, **(ii)** the remaining two numbers.
2. The mean height of 20 boys and 14 girls is 161 cm. If the mean height of the 14 girls is 151 cm, find the mean height of the 20 boys.
3. Given that the mean of 16, w, 17, 9, x, 2, y, 7 and z is 11, find the mean of w, x, y and z.

Worked Example 4

Can you take the average of two means?

Class A has 30 students and Class B has 40 students. The two classes organise a bake sale. The mean number of cookies baked by the students from Class A is 73 while that of Class B is 67. Find the mean number of cookies baked by the students from both classes combined.

Solution

Total number of cookies baked by Class A
$=$ mean $\times$ number of students
$= 73 \times 30$
$= 2190$

Total number of cookies baked by Class B
$=$ mean $\times$ number of students
$= 67 \times 40$
$= 2680$

Total number of cookies baked by the two classes $= 2190 + 2680$
$= 4870$

Total number of students in the two classes $= 30 + 40$
$= 70$

$\therefore$ mean number of cookies baked by both classes combined $= \dfrac{4870}{70}$
$= 69.6$ (to 3 s.f.)

Reflection

If we take the average of the two means (or find the mean of the two means), we will get $\frac{73+67}{2} = 70$. Why is this different from the mean for both classes combined?

Hint: Recall the issue of taking the average of two percentages in Chapter 8, or the average of two speeds in Chapter 9, of Book 1.

Practise Now 4

Similar and Further Questions
Exercise 8A
Questions 6, 7, 13, 20

In March, the mean number of toys a manufacturer sold per day was 8. In April, the mean number of toys he sold per day was 11. Find the mean number of toys he sold per day in the two months.
(There are 31 days in March and 30 days in April.)

From Worked Example 4 and Practise Now 4, we learn that:

In general, we should *not* take the average of means because the *bases* may be different.

Recall

In Book 1, we learnt that we should not take the average of percentages or the average of speeds (or rates) because the bases may be different.

In Worked Example 4, the base is the number of students in each class, which is different for the two classes.
In Practise Now 4 and similar questions in Exercise 8A, what is the base for each question?

Class Discussion — Understanding the concept of mean

All the students in a class brought a different number of sweets one day. Vasi brought the most sweets, which was 7. Those who brought more gave some to those who brought less until every student had the same number of sweets, x.

1. Is it possible for x to be 8? Explain.

2. If $x = 5$, does this necessarily mean that at least one of the students brought 5 sweets to the class? Why or why not?

3. Is the number of sweets given by those who brought more equal to the number of sweets received by those who brought less? Explain.

The students in another class brought 10 basketballs to school. They want to divide the balls equally among the 40 students in the class.

4. How many balls should each student get? Does it make sense to you? Explain.

In 2018, the average household size in Singapore was approximately 3.24 persons.

5. Does "3.24 persons" make sense to you? Explain.

6. How should we interpret "average household size"?

B. Mean of frequency distribution

In the previous section, the data are *raw data*.
In this section, the raw data have been organised into a frequency table, forming what we call a **frequency distribution**.
How do we find the mean of a frequency distribution from the frequency table?

Attention

Data organised and presented in statistical diagrams such as a frequency table or a dot diagram might not be grouped data. In Book 2, we have learnt that grouped data are data that are grouped into class intervals. We will learn how to estimate the mean of grouped data in Section 8.1C.

Worked Example 5

Calculating mean from frequency table

The weekly salaries of the employees in a company are recorded.

Weekly salary ($)	1000	1100	1200	2100	2500
Number of employees	20	8	10	7	5

Calculate

(i) the total number of employees in the company,
(ii) the total salary paid to the employees in a week,
(iii) the mean weekly salary of the employees.

Solution

(i) Total number of employees $= 20 + 8 + 10 + 7 + 5$
$= 50$

(ii) Total salary paid to the employees in a week
$= 20 \times \$1000 + 8 \times \$1100 + 10 \times \$1200 + 7 \times \$2100 + 5 \times \$2500$
$= \$68\ 000$

(iii) Mean weekly salary $= \dfrac{\text{total salary}}{\text{total number of employees}}$
$= \dfrac{\$68\ 000}{50}$
$= \$1360$

Problem-solving Tip

(iii) Mean $= \dfrac{\text{sum of data values}}{\text{number of data values}}$

In Worked Example 5, the sum of data values is the total salary, and the number of data values is the total number of employees.

Practise Now 5

Similar and Further Questions
Exercise 8A
Questions 8, 9, 14

The amount of money spent by the visitors at a carnival was recorded.

Amount spent ($)	40	60	80	100	160	200
Number of visitors	12	32	54	68	18	16

Find

(i) the total number of visitors who were at the carnival,
(ii) the total amount of money spent by the visitors at the carnival,
(iii) the mean amount of money spent by the visitors at the carnival.

From Worked Example 5 and Practise Now 5, we have learnt how to find the mean of a frequency distribution. In general, a set of data $x_1, x_2, x_3, \ldots, x_n$, with the corresponding frequencies $f_1, f_2, f_3, \ldots, f_n$, is usually displayed in the form of a frequency table as shown in Table 8.1.

x	x_1	x_2	x_3	...	x_n
f	f_1	f_2	f_3	...	f_n

Table 8.1

Then the mean of the **frequency distribution** is given by:

$$\text{Mean} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \ldots + f_nx_n}{f_1 + f_2 + f_3 + \ldots + f_n}$$

If we write the mean as $\bar{x}$, the sum of fx as Σfx, and the sum of f as Σf, then we have

Mean of frequency distribution

$$\bar{x} = \frac{\Sigma fx}{\Sigma f},$$

where x is a data value and f is the frequency of the data value.

Big Idea

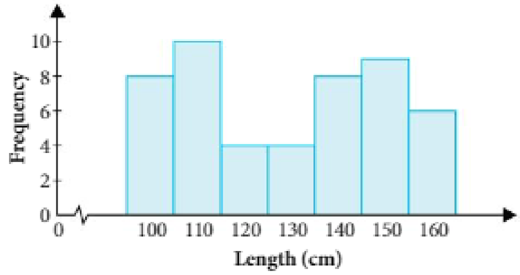
Notations

Σ (pronounced as sigma) is the uppercase of the 18th letter of the Greek alphabet. In Mathematics, the symbol Σ is used as a notation for the summation operator, which represents the sum of a series of numbers in a concise and precise manner, without writing out each individual number. So Σf is used to concisely represent $f_1 + f_2 + f_3 + \ldots + f_n$ and Σfx is used to concisely represent $f_1x_1 + f_2x_2 + f_3x_3 + \ldots + f_nx_n$.

Worked Example 6

Calculating mean from histogram

The histogram shows the lengths, in centimetres, of some rods.



Find the mean length of the rods.

Solution

$$\text{Mean length} = \frac{\sum fx}{\sum f}$$

$$= \frac{8 \times 100 + 10 \times 110 + 4 \times 120 + 4 \times 130 + 8 \times 140 + 9 \times 150 + 6 \times 160}{8 + 10 + 4 + 4 + 8 + 9 + 6}$$

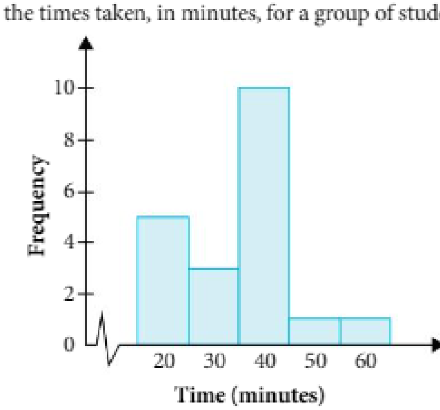
$$= \frac{6330}{49}$$

$$= 129 \text{ cm (to 3 s.f.)}$$

Practise Now 6

Similar and Further Questions
Exercise 8A
Questions 10, 15

The histogram shows the times taken, in minutes, for a group of students to pack for a camp.



Find the mean time taken by the group of students to pack for the camp.

C. Mean of grouped data

In the previous sections, we have learnt how to *calculate* the mean of a set of data that has not been grouped (whether in the form of raw data or a frequency distribution). In this section, we will learn how to **estimate** the mean of a set of **grouped data**. It is only an estimate because we are unable to calculate the actual mean without the individual data values.

Worked Example 7

Estimating mean of grouped data

The times taken, in minutes, for a group of teenagers to cycle between two parks are recorded.

Time taken (x minutes)	$15 < x \leqslant 20$	$20 < x \leqslant 25$	$25 < x \leqslant 30$	$30 < x \leqslant 35$	$35 < x \leqslant 40$
Number of teenagers	5	10	20	15	18

Find an estimate of the mean time taken by the teenagers to cycle between the two parks.

Solution

Time (x minutes)	Frequency (f)	Mid-value (x)	fx
$15 < x \leqslant 20$	5	17.5	87.5
$20 < x \leqslant 25$	10	22.5	225
$25 < x \leqslant 30$	20	27.5	550
$30 < x \leqslant 35$	15	32.5	487.5
$35 < x \leqslant 40$	18	37.5	675
	$\Sigma f = 68$		$\Sigma fx = 2025$

$$\text{Estimated mean time taken} = \frac{\Sigma fx}{\Sigma f} = \frac{2025}{68} = 29.8 \text{ minutes (to 3 s.f.)}$$

Attention

We do not know how much time each teenager took, so we can only *estimate* their mean time. To do so, we represent the data in each class interval by the **mid-value** of the class interval, e.g. the mid-value of the class interval $15 < x \leqslant 20$ is $\frac{15+20}{2} = 17.5$. We apply the same formula $\frac{\Sigma fx}{\Sigma f}$, where x is now the mid-value of a class interval and not a data value.

Practise Now 7

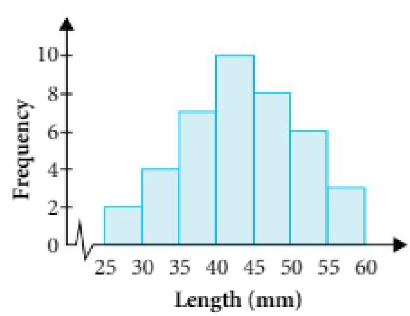
Similar and Further Questions
Exercise 8A
Questions 16–19, 21

1. The ages, in years, of the employees of an advertising firm are recorded.

Age (x years old)	$20 < x \leqslant 30$	$30 < x \leqslant 40$	$40 < x \leqslant 50$	$50 < x \leqslant 60$	$60 < x \leqslant 70$
Number of employees	12	10	20	15	18

Calculate an estimate of the mean age of the employees.

2. The histogram shows the approximate lengths, in mm, of 40 leaves from different plants.



Problem-solving Tip

If x represents the length of the leaves, then the first class interval is either $25 \leqslant x < 30$ or $25 < x \leqslant 30$. The mid-value of the class interval is the same for both cases.

Find an estimate of the mean length of the leaves.

From Worked Example 7, we learn that:

Estimated mean of grouped data

$$\bar{x} = \frac{\sum fx}{\sum f},$$

where x is the mid-value and f is the frequency of the class interval.

Reflection

1. What do I already know about the average learnt in primary school that could help me understand how to find the mean of a frequency distribution or grouped data?
2. What have I learnt in this section that I am still unclear of?

Advanced | Intermediate | Basic

Exercise 8A

1. The number of passengers on coaches travelling along 12 popular scenic routes are 29, 42, 45, 39, 36, 41, 38, 37, 43, 35, 32 and 40.
Find the mean number of passengers on the coaches.

2. Consider the prices, in \$, of various computing books at a bookstore:
19.90, 24.45, 34.65, 26.50, 44.05,
38.95, 56.40, 48.75, 29.30, 35.65.
Find the mean price of the books.

3. The mean of 7 cm, 15 cm, 12 cm, 5 cm, h cm and 13 cm is 10 cm. Find the value of h.

4. The mean mass of five boys is 62 kg. When the mass of a boy is excluded, the mean mass of the remaining four boys becomes 64 kg. Find the mass of the boy that has been excluded.

5. The mean of eight numbers is 12. Five of the numbers are 6, 8, 5, 10 and 28. The remaining three numbers are each equal to k. Find
(i) the sum of the eight numbers,
(ii) the value of k.

6. There are two buildings in a condominium. Building X has 60 units while Building Y has 80 units. The mean number of people living in each unit of Buildings X and Y are 3 and 4 respectively. Find the mean number of people living in each unit of the condominium.

7. The mean mass of 28 students in Class A is 50.5 kg. The mean mass of 35 students in Class B is 52.6 kg. Find the mean mass of the students in the two classes.

8. The number of goals scored per match by a team during a soccer league season was recorded.

Number of goals scored per match	0	1	2	3	4	5	6
Number of matches	6	8	5	6	2	2	1

Find
(i) the total number of matches played,
(ii) the total number of goals scored,
(iii) the mean number of goals scored per match, by the team.

Basic Intermediate Advanced

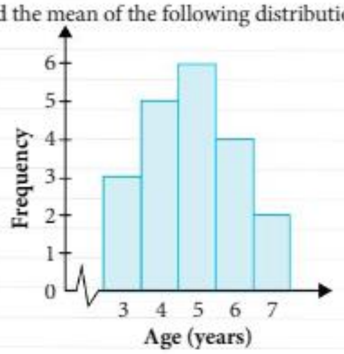
Exercise 8A

9. The number of days the students in a class were absent from school in a term was recorded.

Number of days absent	0	1	2	3	4	5	6	9
Number of students	23	4	5	2	2	1	2	1

Find the mean number of days of absence.

10. Find the mean of the following distribution.



11. The mean of 10 numbers is 14. Three of the numbers have a mean of 4. The remaining seven numbers are 15, 18, 21, 5, m, 34 and 14. Find

(i) the sum of the remaining seven numbers,

(ii) the value of m.

12. The heights of three plants A, B and C in a garden are in the ratio 2 : 3 : 5. Their mean height is 30 cm.

(i) Find the height of Plant B.

(ii) If another Plant D is added to the garden such that the mean height of the four plants is now 33 cm, find the height of Plant D.

13. The mean monthly wage of seven experienced and five inexperienced workers is \$1000. If the mean monthly wage of the five inexperienced workers is \$846, find the mean monthly wage of the seven experienced workers.

14. The marks obtained in an English and a Mathematics test by 40 students were recorded. The total score of each test is 10.

Marks obtained	Number of students (English)	Number of students (Mathematics)
0	0	0
1	1	4
2	6	1
3	14	6
4	4	5
5	8	10
6	2	3
7	4	5
8	0	3
9	1	1
10	0	2

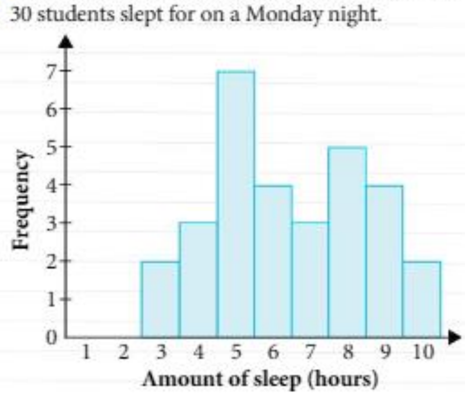
(a) Find the mean mark of the students for each subject.

(b) Given that the percentage passing mark for each subject was 50%, find the percentage of students who

(i) passed English,

(ii) did not pass Mathematics.

15. The histogram shows the duration, in hours, that 30 students slept for on a Monday night.



(i) Find the mean amount of time that the 30 students slept for.

Exercise 8A

(ii) The mean amount of time that 20 other students slept for was 7.5 hours. Find the mean amount of time that the 50 students slept for.

16. The heights, measured to the nearest cm, of the plants in a nursery are recorded.

Height (x cm)	Number of plants
$0 < x \leqslant 10$	4
$10 < x \leqslant 20$	6
$20 < x \leqslant 30$	14
$30 < x \leqslant 40$	6
$40 < x \leqslant 50$	10

(i) Calculate an estimate of the mean height of the plants.
(ii) A plant is selected at random. Find the probability that the plant is not taller than 40 cm.

17. One hundred lorries are required to transport raw materials from a quarry to a construction site. The time taken, in minutes, for the lorries to travel from the quarry to the construction site using a fixed route was recorded.

Time taken (t minutes)	Number of lorries
$116 \leqslant t < 118$	1
$118 \leqslant t < 120$	6
$120 \leqslant t < 122$	23
$122 \leqslant t < 124$	28
$124 \leqslant t < 126$	27
$126 \leqslant t < 128$	9
$128 \leqslant t < 130$	5
$130 \leqslant t < 132$	1

(i) Calculate an estimate of the mean travelling time of the lorries.
(ii) Find the fraction of lorries which took less than 124 minutes to travel from the quarry to the construction site.

18. The speeds, in km/h, of 100 vehicles are recorded.

Speed (x km/h)	Number of vehicles
$30 < x \leqslant 40$	16
$40 < x \leqslant 50$	25
$50 < x \leqslant 60$	35
$60 < x \leqslant 70$	14
$70 < x \leqslant 80$	10

(i) Calculate an estimate for the mean speed of the vehicles.
(ii) Find the ratio of the number of vehicles which travel at a speed of not more than 40 km/h to the number of vehicles which travel at a speed of more than 60 km/h.

19. The table shows the mean distances, d million km, of 20 moons from Jupiter.

21.03	23.55	22.93	23.12	23.22
23.03	21.11	23.22	23.28	21.31
21.28	23.58	21.27	23.40	21.17
21.15	23.62	23.18	22.25	23.36

(i) Complete the frequency table for the data.

Mean distance (d million km)	Frequency
$21.0 \leqslant d < 21.5$	
$21.5 \leqslant d < 22.0$	
$22.0 \leqslant d < 22.5$	
$22.5 \leqslant d < 23.0$	
$23.0 \leqslant d < 23.5$	
$23.5 \leqslant d < 24.0$	

(ii) Using the frequency table in part **(i)**, calculate an estimate for the mean of the mean distances of the moons from Jupiter.

20. The mean of three numbers x, y and z is 6 and the mean of five numbers x, y, z, a and b is 8. Find the mean of a and b.

Basic Intermediate Advanced

Exercise 8A

21. The table shows the lifespans, x hours, of 30 light bulbs.

167	171	179	167	171	165	175	179	169	168
171	177	169	171	177	173	165	175	167	174
177	172	164	175	179	179	174	174	168	171

(i) Find the mean lifespan of the light bulbs by dividing the sum of the lifespans of the light bulbs by 30.

(ii) Using the class intervals $164 \leqslant x < 167$, $167 \leqslant x < 170$ and so on, construct a frequency table for the data.

(iii) Using the frequency table in part **(ii)**, calculate an estimate for the mean lifespan of the light bulbs.

(iv) Comment on your answers to parts **(i)** and **(iii)**.

8.2 Median

In this section, we will learn another type of average called the median. Why is the median an average and when do we use the median (instead of the mean) as a measure of the central tendency (or the centre) of a data set?

A. Median of raw data

Introductory Problem Revisited

In the **Introductory Problem**, the following monthly salaries (in dollars) of nine employees in a company were shown:

2000, 2100, 2700, 2400, 2200, 2000, 10 000, 2800, 2600.

In Question 1, you should have calculated that the average (or *mean*) monthly salary is \$3200.
In Question 2, you should have observed that 8 out of 9 employees earn less than the mean salary of \$3200.
In Question 3, you should have realised that the mean salary does not present a fitting picture of the amount of money the employees in the company earn in general because most of them earn less than this mean salary of \$3200. The **extreme value** (or **outlier**) of \$10 000 has caused the mean salary to be much higher than what most of the employees earn.

To answer Question 4 on another way of calculating the average monthly salary of the employees, we first arrange the monthly salaries in ascending order (i.e. from the smallest to the largest):

2000, 2000, 2100, 2200, 2400, 2600, 2700, 2800, 10 000.

1. Which salary is the *middle value* in the above list?
This salary is called the **median** salary and it lies in the middle of the list of salaries that have been arranged in *ascending order*.

2. What if we arrange the salaries in *descending order*? Will the salary that lies in the middle of this list be equal to the salary in Question 1?
3. Does the salary in Question 1 present a fitting picture of the general average monthly salary? Why or why not?
4. Why is the median salary not affected by the extreme value of \$10 000?

Consider the monthly salaries of the employees in Company B that have been arranged in ascending order:

800, 5500, 5900, 6300, 10 500.

5. Which are the extreme values? How many are there?
6. Calculate the mean monthly salary of the employees in Company B.
7. Find the median monthly salary of the employees in Company B.
8. Are the mean and median monthly salaries close to each other?
9. Why is the mean monthly salary not adversely affected by the two extreme values of \$800 and \$10 500?
10. Do both the mean and median monthly salaries present a fitting picture of the average monthly salary of the employees in Company B in general? Why or why not?

Big Idea

Measures

The median is another **measure of the central tendency** (or the centre) of a data set. In this example, do you realise that the median (which is 2400) is somewhere in the centre of the data set? The median is the middle value in the data set, if the number of data values is odd (see Worked Example 9 for an even number of data values).

From the **Introductory Problem Revisited**, we observe that the **median** is another average of a data set. The median is a more useful average than the mean when the mean is affected by extreme values. However, we also observe that the mean is *not always* adversely affected by extreme values, e.g. when there are both high and low extreme values within the data set. Another example is when there is only one extreme value in a very large data set such that the extreme value does not adversely affect the mean of the data set.

Both data sets in the previous discussion have an odd number of data values. How do we find the median of a data set with an even number of data values or if the data are presented in a frequency table as shown in Table 8.2?

Height (cm)	152	154	156	158	160
Number of students	2	2	5	1	8

Table 8.2

First, we need to learn about the *position* of the median in a data set that has been arranged in ascending order.

Worked Example 8

Finding median when number of data values is odd

Find the median of the following set of data.

20, 25, 21, 24, 22, 26, 20

Solution

Total number of data values, $n = 7$

$$\text{Position of median} = \frac{n+1}{2} = \frac{7+1}{2} = 4$$

Attention

The number of data values in Worked Example 8 is small enough for us to easily observe that the median position is the 4th position. But we need a way to find the median position when the data set is large. The *position of the median* is given by $\frac{n+1}{2}$. Here, $n = 7$, so the median position is $\frac{7+1}{2} = 4$, i.e. the median is the 4th data value when the data set is arranged in ascending order.

Rearranging the data in ascending order:

20, 20, 21, 22, 24, 25, 26

4^{th} position

$\therefore$ median = 22

Practise Now 8

Similar and Further Questions
Exercise 8B
Questions 1(a), (b)

Find the median of each of the following sets of data.

(a) 20, 16, 9, 3, 18, 11, 15

(b) 11.2, 15.6, 30.2, 17.3, 18.2

Worked Example 9

Finding median when number of data values is even

Find the median of the following set of data.

12, 8, 19, 30, 14, 21, 9, 5

Solution

Total number of data values = 8

$$\text{Position of median} = \frac{n+1}{2}$$
$$= \frac{8+1}{2}$$
$$= 4.5$$

Rearranging the data in ascending order:

5, 8, 9, 12, 14, 19, 21, 30

4.5^{th} position

4^{th} value = 12

5^{th} value = 14

$\therefore$ median = mean of 4^{th} value and 5^{th} value

$$= \frac{12+14}{2}$$
$$= 13$$

Problem-solving Tip

The 4.5^{th} position means that the median lies between the 4^{th} value and the 5^{th} value. Therefore, median = mean of 4^{th} value and 5^{th} value.

Reflection

Is the median always one of the data values? If not, what does it depend on?

Hint: Compare Worked Examples 8 and 9.

Practise Now 9

Similar and Further Questions
Exercise 8B
Questions 1(c), (d), 7

1. Find the median of each of the following sets of data.

   (a) 32, 15, 20, 15, 25, 12

   (b) 8, 7.3, 8.9, 6.8, 8.8, 8.9, 10, 6.7

2. The median of 9, 3, 6, 7, 5 and x is 6.5.

   OPEN Q Find a possible value of x.

In general, to obtain the median of a set of raw data, we arrange the data in *ascending* (*or descending*) *order* first.

Median

If the total number of data values is *odd*, the median is the *middle value* of the arranged data.

If the total number of data is *even*, the median is the *mean of the two middle values* of the arranged data.

Class Discussion **Understanding the concept of median**

1. Albert said, "The median is always the middle value of a data set that has been arranged in ascending order." Do you agree? Explain.
2. Li Ting said, "The median of a data set is 10. This means that 50% of the data values are greater than 10 and the other 50% are less than 10." Do you agree? Explain.

B. Median of frequency distribution

In the previous section, the data are *raw data*.

How can we find the median when the raw data have been organised into a frequency table, forming a **frequency distribution**, without listing out each data value? Let us learn to find the median from a frequency table or a histogram (for ungrouped data).

Worked Example 10

Finding median of data set displayed in frequency table

The heights, in cm, of 18 students are recorded.

Height (cm)	152	154	156	158	160
Number of students	2	2	5	1	8

Find the median height of the students.

Solution

Total number of data, $n = 18$

$$\begin{aligned}\text{Position of median} &= \frac{n+1}{2}\\ &= \frac{18+1}{2}\\ &= 9.5\end{aligned}$$

$$\begin{aligned}\therefore \text{ median height} &= \text{mean of } 9^{\text{th}} \text{ value and } 10^{\text{th}} \text{ value}\\ &= \frac{156+158}{2}\\ &= 157 \text{ cm}\end{aligned}$$

Attention

Do *not* take the value in the middle column (i.e. 156 cm) as the median height just because the 3rd column is in the middle of the 5 columns.

Instead, interpret the frequency table as follows. The 1st column shows that there are 2 students with height 152 cm. So the 2 smallest data values are 152 cm and 152 cm.

1st column, 3rd column, 4th column

Height (cm)	152	154	156	158	160
Number of students	2	2	5	1	8

$\underbrace{2 + 2 + 5}_{} = 9$ 10th value

Then add the number of students from the left until the 9th value, i.e. $2 + 2 + 5 = 9$. So the 9th value lies in the 3rd column, i.e. 9th value = 156 cm. Therefore, the 10th value lies in the 4th column, i.e. 10th value = 158 cm.

Practise Now 10

Similar and Further Questions
Exercise 8B
Questions 2(a)–(d)

1. The times taken, in minutes, for a group of children to complete a puzzle are recorded.

Time taken (minutes)	5	6	7	8	9
Number of children	8	4	3	10	3

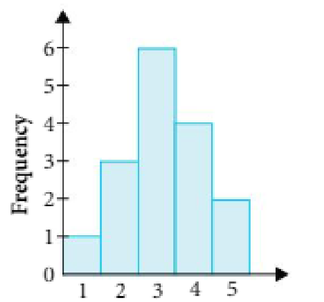
Find the median time taken by the group of children to complete the puzzle.

2. The table shows the prices, in dollars, of various items sold in a shop.

Price ($\$x$)	Number of items
$0 < x \leqslant 5$	11
$5 < x \leqslant 10$	23
$10 < x \leqslant 15$	10
$15 < x \leqslant 20$	7
$20 < x \leqslant 25$	4

State the class interval which contains the median price.

3. The histogram shows the number of concert tickets sold by the members of the Guitar Club in a school in a morning. Find the median of the distribution.



Problem-solving Tip

How do you interpret the histogram? What is the smallest data value (i.e. fewest concert tickets sold)? What is the 2nd smallest data value? How many data values are there?

Reflection

1. Referring to the grouped data in Worked Example 7, do you think you can find the class interval in which the median lies, or estimate the median of the grouped data? Explain.
2. When there are extreme values, do we always use the median as a measure of the central tendency? Why or why not?
3. How can I explain when we should use median (and not mean) as a measure of the central tendency of a data set?

8.3 Mode

When we analysed statistical diagrams in Book 1, one of the comparisons we made was to find out the most popular category or the category that contains the most data.

In statistics, we call the category or the data value that occurs the *most frequently* in a set of data the **mode**. In fact, the mode is another type of average. But why is the mode an average? And when do we use the mode (instead of the mean or median) as a measure of the central tendency (or the centre) of a data set?

A. Mode of raw data

Class Discussion — **Mode as an average**

The owner of a shop kept a record of the sizes of blouses sold on a particular day. The record showed:

8 8 10 8 10 12 10 8 8 12

The mode of the blouse sizes is the size that occurs the most frequently.

1. Find the **(a)** mean, **(b)** median, and **(c)** mode of the sizes of blouses sold on that day.
2. Does the mean, median or mode present the most fitting picture of the size of blouses sold on that day? Explain.

From the above Class Discussion, we realise that the mean of 9.4 and the median of 9 are not very meaningful because there are no such sizes for the blouses. Knowing the popular size(s) will help the owner prepare the size(s) to stock up on. Therefore, we use the mode as a measure of the average when we are interested in the data value that occurs the most often.

Big Idea

Measures

The mode is another **measure of the central tendency** (or the centre) of a data set. It tells us the data value that occurs most often, i.e. the most common data value.

In general, to obtain the mode of a set of data or frequency distribution, we look out for the data value which occurs most frequently:

Mode = data value(s) with the highest frequency

It is possible for a data set to have more than one mode (see Worked Example 11), or to have no mode at all (see Exercise 8B Question 3(c)).

Worked Example 11

Finding mode of raw data

The numbers of books 10 students read in a year were recorded as 8, 9, 10, 10, 3, 5, 6, 10, 6 and 1.

(i) State the modal number of books.

(ii) If two more students' data of 6 and 8 are added, what will the new modal number(s) of books be?

*Solution

(i) Modal number = 10 — since it occurs most frequently, i.e. 3 times

(ii) New modal numbers = 6 and 10 — since they occur most frequently, i.e. 3 times each

Information

If a data set has two modes, we say that the data set is **bimodal**. If all the data values have the same frequency, we say that the data set has *no mode*.

Practise Now 11

Similar and Further Questions
Exercise 8B
Questions 3(a)–(c), 4

The lengths of 10 ribbons are 100 cm, 110 cm, 95 cm, 60 cm, 20 cm, 60 cm, 110 cm, 88 cm, 102 cm and 120 cm.

(i) State the modal length(s) of the ribbons.

(ii) If a ribbon of length 110 cm is removed, what will the new modal length of the ribbons be?

Class Discussion — Understanding the concept of mode

All the students in a class brought a different number of sweets one day. Vasi brought the most, which was 7.

1. Was the mode of the number of sweets brought to class equal to 7? Why or why not?
2. If the mode was 4, does this mean that some students must have brought 4 sweets to class? Explain.

B. Mode of frequency distribution

Worked Example 12

Finding mode of data set displayed in frequency table

State the mode of the frequency distribution.

Commission ($)	1000	1200	1500	2000
Number of salesmen	2	5	3	1

*Solution

Mode = $1200 — since it has the highest frequency, i.e. 5 salesmen

Problem-solving Tip

The *mode* is not the highest frequency, but the *data value* with the highest frequency. Therefore, it is important to interpret which numbers in the frequency table are data values, and which numbers are frequencies.

Practise Now 12

Similar and Further Questions
Exercise 8B
Questions 5(a)–(c), 6(a)–(d)

For each of the following, state the mode(s) or modal class of the distribution.

(a)

Monthly salary (PKR)	2000	3000	3500	4500	5000
Number of employees	10	56	42	25	17

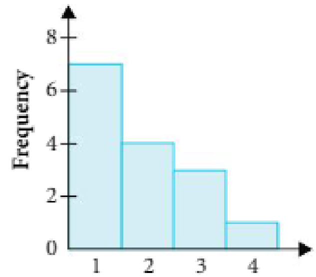
(b)

Length (x cm)	Frequency
$0 < x \leqslant 15$	11
$15 < x \leqslant 30$	20
$30 < x \leqslant 45$	18
$45 < x \leqslant 60$	24
$60 < x \leqslant 75$	15

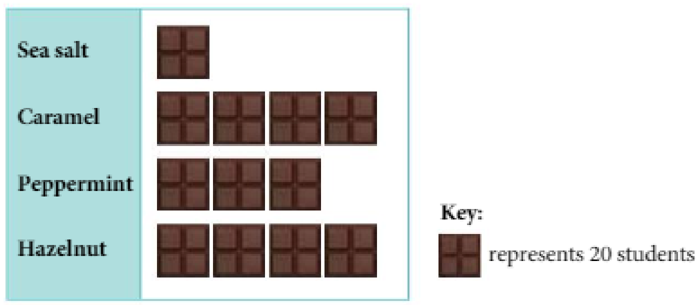
Attention

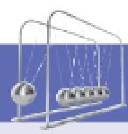
When data is presented in the form of a frequency table or a histogram involving class intervals, the **modal class** is the class interval with the highest frequency.

(c)



(d) **Students' favourite chocolate flavour**





Reflection

1. Can we find the mean or median of a set of categorical data? For example, in the pictogram in Practise Now 12(**c**), we can find the mode, but can we find the mean or the median of the distribution? Why or why not?
2. Referring to the grouped data in Worked Example 7, can I find the mode of the grouped data? Explain.
3. How can I explain when we should use mode (and not mean or median) as a measure of the central tendency of a data set?

8.4 Measures of central tendency

In this chapter, we have learnt that an average is a measure of the central tendency (or the centre) of a data set or a frequency distribution.

We have also learnt three types of average (mean, median and mode) and when to use which average in each of the above three sections. Let us now consolidate what we have learnt.

Class Discussion

Comparing different measures of central tendency

1. Based on what we have learnt in the previous three sections, discuss when we should use the
   (i) mean, **(ii)** median, **(iii)** mode,
   as a measure of the central tendency of a data set or a frequency distribution.

2. The number of people who watched a movie at three cinemas over five days is shown in Table 8.3.

	Cinema X	Cinema Y	Cinema Z
Day 1	87	60	75
Day 2	16	62	54
Day 3	92	98	98
Day 4	88	61	76
Day 5	98	98	75
Mean	76.2	75.8	75.6
Median	88	62	75
Mode	No mode	98	75

Table 8.3

For each of the three cinemas, which average (mean, median or mode) would you use to describe the typical daily number of moviegoers over the five days? Justify your answer.

Similar and Further Questions
Exercise 8B
Questions 8, 9

From the above Class Discussion and the previous sections, we observe the following:

Appropriate use of average

- The *mean* is preferred when there are no extreme values because the mean takes into account *all* the data values in its computation.
- The *median* is preferred when there are extreme values that affect the mean.
- The *mode* is preferred when we are interested in the data value that is the most common or popular.

Worked Example 13

Solving problem involving mean, median and mode

The number of spelling errors a group of students made in an essay is recorded.

Number of spelling errors	0	1	2	3	4	5
Number of students	4	8	x	6	5	4

(a) If the mean number of spelling errors the students made is 2.4, calculate the value of x.

(b) If the modal number of spelling errors the students made is 2, state the smallest possible value of x.

(c) If the median of the distribution is 3, find the possible values of x.

Solution

(a) $$\frac{4\times0 + 8\times1 + 2x + 6\times3 + 5\times4 + 4\times5}{4 + 8 + x + 6 + 5 + 4} = 2.4$$

$$\frac{2x + 66}{x + 27} = 2.4$$

$$2x + 66 = 2.4(x + 27)$$
$$= 2.4x + 64.8$$
$$0.4x = 1.2$$
$$x = 3$$

(b) Smallest possible value of $x = 9$ since $x > 8$

(c) **Method 1:**

We write the data as follows:

$$\underbrace{0, \ldots, 0}_{4}, \underbrace{1, \ldots, 1}_{8}, \underbrace{2, \ldots, 2}_{x}, \underbrace{3, \ldots, 3}_{6}, \underbrace{4, \ldots, 4}_{5}, \underbrace{5, \ldots, 5}_{4}$$

The greatest value of x occurs when the median is here.

$\therefore 4 + 8 + x = 5 + 5 + 4$
$12 + x = 14$
$x = 2$

Hence, greatest value of $x = 2$

The smallest value of x occurs when the median is here (if possible).

$\therefore 4 + 8 + x + 5 = 5 + 4$
$17 + x = 9$
$x = -8$ (not possible since $x > 0$)

Hence, smallest value of $x = 0$

$\therefore$ the possible values of x are 0, 1 and 2.

Method 2:

Position of median $= \frac{n+1}{2}$

$$= \frac{(4 + 8 + x + 6 + 5 + 4) + 1}{2}$$
$$= \frac{x + 28}{2}$$

For greatest value of x, position of median $= 4 + 8 + x + 1$
$= x + 13$

$$\therefore \frac{x + 28}{2} = x + 13$$
$$x + 28 = 2(x + 13)$$
$$= 2x + 26$$
$$x = 28 - 26$$
$$= 2$$

For smallest value of x, position of median $= 4 + 8 + x + 6$
$= x + 18$

$$\therefore \frac{x + 28}{2} = x + 18$$
$$x + 28 = 2(x + 18)$$
$$= 2x + 36$$
$$x = 28 - 36$$
$$= -8 \text{ (not possible since } x \geqslant 0)$$

Hence, smallest value of $x = 0$

$\therefore$ the possible values of x are 0, 1 and 2.

Problem-solving Tip

(iii) For **Method 2**, you can observe the position of the median when x is the greatest (or least) by looking either at the list of data in **Method 1**, or better still, from the frequency table directly.

Reflection

(iii) Which method do you prefer? Why?

Practise Now 13

Similar and Further Questions
Exercise 8B
Questions 10(a), (b), 11(a), (b), 16, 17(a)–(c), 18

The number of books a group of students borrowed from the school library is recorded.

Number of books	0	1	2	3	4
Number of students	2	x	3	4	1

(a) If the mean number of books the students borrowed is 1.8, find the value of x.
(b) If the modal number of books the students borrowed is 3, state the greatest possible value of x.
(c) If the median of the distribution is 2, find the possible values of x.

Worked Example 14

Finding data set given averages and other information

Cheryl has written down five numbers.
The mean of these numbers is 9, the median is 11 and the mode is 12.
The largest number is four times the smallest number.
Find the five numbers.

Solution

Since the median is 11, then the third number of the data set that has been arranged in ascending order is 11:

___ , ___ , 11 , ___ , ___

Since the mode is 12, this number must occur at least twice. As there are only two numbers larger than the median of 11, then the two largest numbers must be 12:

___ , ___ , 11 , 12 , 12

Since the largest number is four times the smallest number, then the smallest number is $12 \div 4 = 3$:

3 , ___ , 11 , 12 , 12

$$\begin{aligned}\text{Sum of the five numbers} &= \text{mean} \times 5\\ &= 9 \times 5\\ &= 45\end{aligned}$$

So the last number is $45 - 3 - 11 - 12 \times 2 = 7$.
$\therefore$ the five numbers are 3, 7, 11, 12 and 12.

Practise Now 14

Similar and Further Questions
Exercise 8B
Questions 12, 13

David has written down six numbers.
The mean of these numbers is 2, the median is 1.5 and the mode is 1.
Half of the numbers are prime and the smallest number has an infinite number of factors.
Find the six numbers.

Worked Example 15

How is average affected by systemic measurement error?

Sara weighed seven oranges.
The modal mass of the oranges was 148 grams.
The median mass of the oranges was 153 grams.
The mean mass of the oranges was 153.7 grams.
The weighing machine used by Sara was found to be inaccurate.
The correct mass of each orange was 18 grams more than what Sara has recorded.
Write down the correct values of the modal, median and mean masses of the oranges.

Solution

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

Stage 1: Understand the problem

What information is given and what information do we need?

- We are given the number of oranges, the incorrect modal, median and mean masses, and that the correct mass of each orange was 18 g more than what was recorded.
- We are not given the mass of each of the seven oranges.

What are we supposed to find and what is not necessary?

- We are supposed to find the correct modal, median and mean masses of the oranges.
- We do not need to find the incorrect or correct mass of each orange.

Stage 2: Think of a plan

Without the recorded mass of each orange, we use a histogram to visualise a random distribution of the data as shown in Fig. 8.1.
Since the correct mass of each orange was 18 g more than what was recorded, each of the rectangle in the left histogram will shift by the same amount of 18 g.

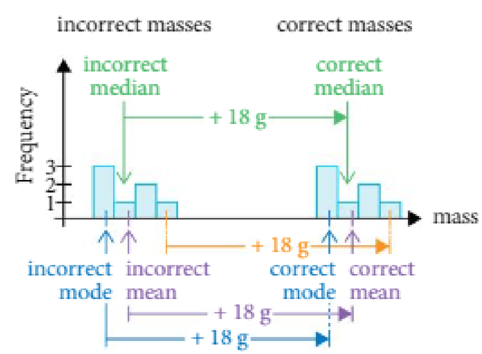


Fig. 8.1

What can we observe about the distribution of the correct masses?

The distribution of the masses remains unchanged, i.e. the distribution of the correct masses looks exactly the same as the distribution of the incorrect masses.

From Fig. 8.1, what is the difference between the incorrect and the correct
- modal masses?
- median masses?

We may not know the exact position of the incorrect mean mass or the correct mean mass.

However, since the distribution of the masses remains unchanged, the *relative* position of the correct mean mass inside the right histogram will be the same as the *relative* position of the incorrect mean mass inside the left histogram.
What is the difference between the incorrect and the correct mean masses?

Stage 3: Carry out the plan

Correct modal mass = incorrect modal mass + 18
= 148 + 18
= 166 g

Correct median mass = incorrect median mass + 18
= 153 + 18
= 171 g

Method 1:

Correct mean mass = incorrect mean mass + 18
= 153.7 + 18
= 171.7 g

Method 2:

Total incorrect mass of 7 oranges = incorrect mean mass × 7
= 153.7 × 7
= 1075.9 g

Total correct mass of 7 oranges = 1075.9 + 18 × 7
= 1201.9 g

Correct mean mass = 1201.9 ÷ 7
= 171.7 g

Reflection

Which method do you prefer? Why?

Stage 4: Look back

What have we learnt from this problem?
We realise that if every data value is shifted by the same amount (i.e. a constant value is added or subtracted), then each of the three types of average (mean, median and mode) also shifts by the same amount.

Practise Now 15

Similar and Further Questions
Exercise 8B
Questions 14, 15

The masses of 30 students in a class were measured and recorded.
The modal mass of the students was 60 kg.
The median mass of the students was 62 kg.
The mean mass of the students was 65.3 kg.
The weighing machine used was found to be faulty.
The correct mass of each student should be 8 kg less than what was recorded.
Write down the correct values of the modal, median and mean masses of the students.

Journal Writing

Create as many sets of data as possible that satisfy all the following conditions:

- Each set of data consists of 7 values.
- The difference between the minimum and the maximum value is 20.
- The mean of the data is greater than its median.
- The mode of the data is less than its median.

Are your data sets the same as those obtained by your classmates?

Reflection

1. Do I know when to use the mean, the median or the mode as a measure of the central tendency of a data set? If yes, elaborate.
2. What have I learnt in this section or chapter that I am still unclear of?

Advanced | Intermediate | Basic

Exercise 8B

1. Find the median of each of the following sets of numbers.

(a) 5, 6, 1, 5, 3, 5, 6

(b) 1.2, 1.1, 4.1, 3.2, 4.1, 1.6, 2.8

(c) 30, 33, 37, 28, 29, 25

(d) 39.6, 12, 13.5, 22.6, 31.3, 8.4, 5.5, 4.7

2. For each of the following, find the median of the distribution or the class interval which contains the median.

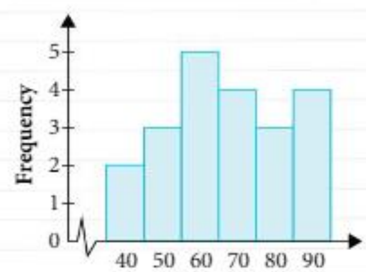
(a)

h	100	200	300	400	500	600
Frequency	2	4	1	5	1	2

(b)

Age (x years)	Frequency
$0 < x \leq 5$	5
$5 < x \leq 10$	6
$10 < x \leq 15$	10
$15 < x \leq 20$	8
$20 < x \leq 25$	7
$25 < x \leq 30$	5
$30 < x \leq 35$	3

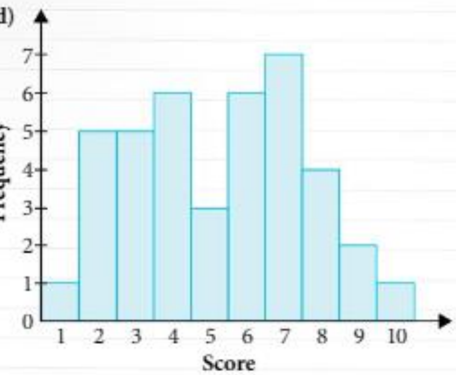
(c)



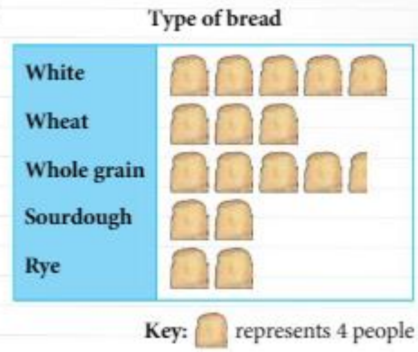
Basic Intermediate Advanced

Exercise 8B

(d)



(c)



3. Find the mode(s) of each of the following sets of numbers.

(a) 2, 5, 8, 3, 7, 5, 3, 9, 7, 3

(b) 8.1, 7.7, 7.8, 9.3, 6.4, 7.7, 9.3, 8.7

(c) 11.3, 10, 21, 14.5, 26, 11, 20.1

4. The temperatures, taken at midnight, of 6 consecutive nights in Singapore are given as follows:

22 °C, 27 °C, 26 °C, 28 °C, 27 °C, 23 °C.

(i) State the modal temperature.

The temperature, taken at midnight, of the 7th day in Singapore was 22 °C.

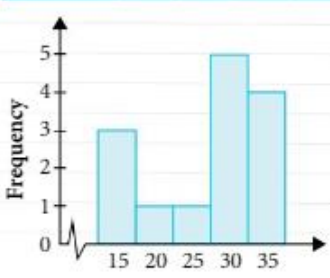
(ii) If 22 °C is added to the above set of data, what will the new modal temperature(s) be?

5. For each of the following, state the mode(s) of the distribution.

(a)

x	6	7	8	9	10
Frequency	1	1	1	1	1

(b)



6. For each of the following, state the mode(s) or modal class of the distribution.

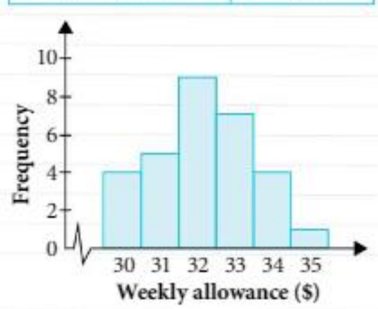
(a)

Colour	Frequency
Red	28
Yellow	15
Green	28
Blue	35
Orange	19
Purple	35
Pink	28
Brown	20
Grey	11

(b)

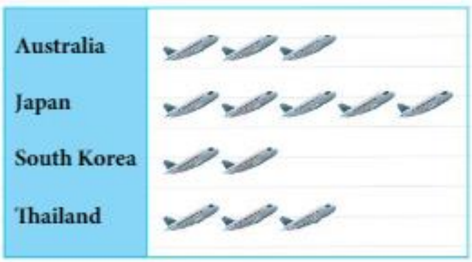
Temperature (x °C)	Frequency
$0 \leqslant x < 10$	114
$10 \leqslant x < 20$	41
$20 \leqslant x < 30$	44
$30 \leqslant x < 40$	56
$40 \leqslant x < 50$	110

(c)



Exercise 8B

(d) **Holiday destinations**



Key: represents 5 students

7. The median of eight numbers is 4.5. Given that seven of the numbers are 9, 2, 3, 4, 12, 13 and 1, find the eighth number.

8. Albert and Bernard were playing golf. Their scores on the first nine holes are shown in the table. In golf, the lower the score, the better it is.

Hole	1	2	3	4	5	6	7	8	9	**Total**
Albert	3	2	5	7	3	2	2	4	17	45
Bernard	4	4	6	8	3	3	2	6	6	42

On the ninth hole, Albert hit his golf ball into a sand trap and lost the game.

(i) Find the mean score on the nine holes for each player.

(ii) Which player scored better on most of the holes? Do the mean scores indicate this?

(iii) Find the median score for each player.

(iv) State the modal score of each player.

(v) Which average, i.e. the mean, the mode or the median, gives the best comparison of the abilities of Albert and Bernard? Explain your answer.

9. Two classes, each with 21 students, took a physical fitness test. The number of pull-ups done by each student in 30 seconds was recorded.

Class A	**Number of pull-ups**	$\leqslant 5$	6	7	8	9	$\geqslant 10$
	Number of students	3	7	4	4	2	1

Class B	**Number of pull-ups**	$\leqslant 5$	6	7	8	9	$\geqslant 10$
	Number of students	3	4	4	7	2	1

(i) Explain why we are unable to calculate the mean number of pull-ups done by the students in each class.

(ii) Find the median number of pull-ups done by the students in each class.

(iii) State the modal number of pull-ups done by the students in each class.

(iv) Does the median or the mode give a better comparison of the number of pull-ups done by the students in the two classes? Explain your answer.

10. The number of pets 40 students own is recorded.

Number of pets	2	4	6	8	10
Number of students	x	2	y	6	14

(a) **(i)** Show that $x + y = 18$.

(ii) If the mean of the distribution is 6.4, show that $x + 3y = 30$.

(iii) Hence, find the value of x and of y.

(b) Using your answers to part **(a)(iii)**, find

(i) the median,

(ii) the mode,

of the distribution.

Basic Intermediate Advanced

Exercise 8B

11. The number of magazines read by a group of women in a week is recorded.

Number of magazines	0	1	2	3
Number of women	5	2	1	x

(a) If the mean of the distribution is 2, find the value of x.

OPEN Q **(b)** If the median of the distribution is 1, find a possible value of x.

12. There are seven positive numbers in a data set. The mean of these numbers is 7, the median is 8 and the modes are 2 and 10. The largest number is a perfect square. Find the seven numbers.

13. Imran has written down six numbers less than 15.
OPEN Q The mean of these numbers is 7, the median and the mode are both 9. Half of the numbers are the same, and one of them is a single digit negative number. Two of the positive numbers are primes while the rest are multiples of 3. Find a possible set of the six numbers.

14. The times taken for 12 students to solve a puzzle were recorded, to the nearest minute. The modal, median and mean timings were 13 min, 13 min and 15 min respectively. The students were given 3 min to read an insert before solving the puzzle, which was not included in the timings recorded. If the 3 min had been included, state the values of the modal, median and mean timings that would have been recorded.

15. Raju has six money boxes. The modal, median and mean amounts of money in the money boxes are \$50, \$40 and \$35 respectively. If he takes out \$10 from each money box, find the new values of the modal, median and mean amounts of money in the money boxes.

16. The number of major hurricanes, which strike the Atlantic coast each year over a period of 50 years, was recorded.

Number of major hurricanes	0	1	2	3	4	5	6
Number of years	5	13	15	x	1	y	2

(i) If the mean of the distribution is 2.18, find the value of x and of y.

(ii) Find
  (a) the median,
  (b) the mode,
of the distribution.

(iii) There are at most p major hurricanes striking the Atlantic coast each year in 36% of the years. Find the value of p.

17. The number of social networking accounts a group of students own is recorded.

Number of social networking accounts	0	1	2	3	4	5
Number of students	4	6	3	x	3	2

(a) If the mean number of social networking accounts the students own is 2.2, find the value of x.

(b) If the median of the distribution is 2, find the greatest possible value of x.

(c) If the modal number of social networking accounts the students own is 3, state the smallest possible value of x.

Advanced | Intermediate | Basic

Exercise 8B

18. The number of books read by a group of students in a week is recorded.

Number of books	0	1	2	3	4	5	6
Number of students	$x+1$	$x-2$	$x+2$	x	$x-2$	$x-4$	$x-3$

(i) Given that the mean and the mode of the distribution are equal, find the value of x.

(ii) Using your answer to part **(i)**, find the median of the distribution.

Looking Back

In this chapter, we have learnt about the ideas related to the three **measures** of central tendency of a given data set — mean, median, and mode. These measures enable us to use a single value to represent the entire data set and provide us an idea of the centre of the data set. However, it is also important to consider which measure to use because using different measures can give rise to different interpretations and conclusions.

The measure of central tendency is a good example of how mankind has created new measures to quantify important attributes of physical objects or phenomena. The use of measures allows us to apply mathematical ideas to make decisions or solve problems in the real world. We can now consider an important question in statistics: Is the measure of central tendency sufficient to represent a given data set? Why or why not? We will explore this further in Book 4.

Summary

1. Mean

Mean of raw data, $\bar{x} = \dfrac{\text{sum of data values}}{\text{number of data values}}$

Mean of frequency distribution, $\bar{x} = \dfrac{\sum fx}{\sum f}$, where x is a data value and f is the frequency of the data value x

Estimated mean of grouped data, $\bar{x} = \dfrac{\sum fx}{\sum f}$, where x is the mid-value and f is the frequency of the class interval

- Are the above three formulae for the mean essentially the same? Explain.
- In general, can we take the average of two means? Elaborate.

Summary

2. Median

If the total number of data values is *odd*, the median is the *middle value* of the arranged data.

If the total number of data is *even*, the median is the *mean of the two middle values* of the arranged data.

- How do we find the median of a frequency distribution displayed in a frequency table or a histogram (for ungrouped data)?

3. Mode

Mode = data value(s) with the highest frequency

- Can a data set have more than one mode or no mode? If yes, give an example for each case.

4. Appropriate use of average

The *mean* is preferred when there are no extreme values because the mean takes into account *all* the data values in its computation.

The *median* is preferred when there are extreme values that affect the mean.

The *mode* is preferred when we are interested in the data value that is the most common or popular.

- Give an example of a data set where the mean is preferred, an example of another data set where the median is preferred, and an example of a third data set where the mode is preferred as a measure of the central tendency.

Answer Keys

Chapter 1 Algebraic Fractions and Formulae

Practise Now 1

(a) $\frac{2x^2}{3y^3}$ (b) $\frac{x^2(x-y)^2}{3y^3}$

Practise Now 2

(a) $\frac{h+7k}{5k}$ (b) $\frac{3}{2p-1}$

(c) $\frac{z}{4}$

Practise Now 3

1. (a) $\frac{3v}{v+3}$ (b) $\frac{p-3q}{5p}$

   (c) $\frac{x+2z}{y-2z}$

2. $\frac{n^2-2}{n^2+3}$

Practise Now 4

1. (a) $\frac{3}{4a^2c^2}$ (b) $\frac{7p^4}{10r^3}$

   (c) $\frac{14(x-3)}{15}$ (d) $\frac{h-3}{h}$

2. $-\frac{1}{2}$

Practise Now 5

1. (a) $\frac{63}{40a}$ (b) $\frac{5}{3(2b+3c)}$

   (c) $\frac{4h}{2-3k}$

2. (a) $\frac{6n^2+5mn-m^2}{6mn}$

   (b) $\frac{8q-11p}{12(p-q)}$

   (c) $\frac{10x+7y}{2(4x-3y)}$

Practise Now 6

(a) $\frac{x-13}{(x+1)(2x-5)}$

(b) $\frac{2-y^2-3y}{(y+3)(y-3)}$

(c) $\frac{2}{z+5}$

(d) $\frac{4w+5}{(w+5)(w+3)}$

Practise Now 7

Yes; $\frac{3x+3}{(x-3)(x+3)}$

Practise Now 8

(a) $\frac{79}{11}$ (b) -27

Practise Now 9

1. (i) $a=\frac{v-u}{t}$

   (ii) 10

2. (i) $T=\frac{100I}{PR}$

   (ii) 4 years

Practise Now 10

1. (i) $x=\frac{7y+5}{3y-2}$

   (ii) $\frac{16}{11}$

2. (i) $k=\frac{bx^2}{3(p-a)}$

   (ii) -27

Practise Now 11

1. (i) $x=\frac{b^2-9y^2}{4a}$

   (ii) 1 (iii) 0

2. (i) $x=\pm\sqrt{\frac{3k(p-a)}{b}}$

   (ii) ± 3 (iii) -1

Practise Now 12

1. (a) 2 (b) $\frac{13}{5}$

   (c) 2

2. 13

3. (a) $\frac{36}{7}$ (b) No

Introductory Problem Revisited

(i) $\frac{1000}{49}$

(ii) $\frac{750}{37}$, $\frac{1200}{119}$, $\frac{60}{13}$, $\frac{1120}{153}$

Exercise 1A

1. (a) $\frac{1}{3x}$ (b) $\frac{2b^2}{3a^2}$

   (c) $\frac{q^2}{3r^2s}$ (d) $\frac{n}{6m^2p^3}$

   (e) $\frac{c^2}{5ab^4}$ (f) $\frac{1}{4xyz^2}$

2. (a) $\frac{y}{4}$ (b) $\frac{4}{c}$

   (c) $\frac{a+2b}{6}$ (d) $\frac{c}{c-d}$

   (e) $\frac{m-n}{m}$ (f) $\frac{q}{3-2q}$

3. (a) $\frac{1}{2a-b}$ (b) $\frac{c-3d}{4c}$

   (c) $\frac{3}{a+3}$ (d) $\frac{x+7}{x}$

   (e) $\frac{k+3}{k-4}$ (f) $\frac{k}{m-4}$

4. (a) $\frac{3}{2b^4}$ (b) $\frac{3}{4}$

   (c) $\frac{3}{8}$ (d) $2c$

5. (a) $\frac{1}{3x^2(a-b)}$

   (b) $\frac{a(a-3b)^2}{3b}$

   (c) $\frac{b^2(2a+3b)}{4a}$

   (d) $\frac{n^2}{12a(b+c)}$

   (e) $\frac{y+3}{y+2}$ (f) $\frac{-m-4}{2m+1}$

   (g) $\frac{-y-3x}{y+x}$ (h) $\frac{3x-y}{4x-y}$

   (i) $\frac{-a-b}{2a+3b}$ (j) $\frac{y+1}{2y-3}$

   (k) $\frac{3}{a-1}$ (l) $\frac{a-b}{a+b}$

   (m) $\frac{a-n}{a+n}$

6. (a) $\frac{5a^2c^5}{144b^6}$ (b) $\frac{27}{64d^3f^3}$

   (c) $\frac{16}{xy^2}$ (d) $\frac{6p^2s^4}{r^2q^3}$

   (e) $-\frac{9}{5w^2}$ (f) $-\frac{9x}{4y}$

   (g) $\frac{h}{h+3}$ (h) $\frac{d(c+d)^2}{c-d}$

   (i) $\frac{m+2}{m}$ (j) $-\frac{z}{z+2}$

   (k) $b(a-2b)$ (l) $\frac{y-2}{3(1-3y)}$

7. $\frac{x+y-z}{x-y-z}$

Exercise 1B

1. (a) $\frac{29}{18a}$ (b) $\frac{1}{b}$

   (c) $\frac{d-c}{3cd}$ (d) $\frac{2f-17h}{24k}$

   (e) $\frac{5a}{x-3y}$ (f) $\frac{1}{z}$

2. (a) $\frac{8a+20}{a(a+4)}$ (b) $\frac{c-5b}{2b(b+c)}$

   (c) $\frac{10d+2}{(d-5)(2d+3)}$

   (d) $\frac{-f-17}{(f+5)(f-1)}$

   (e) $\frac{52-49h}{(3h-7)(6-5h)}$

   (f) $\frac{2k+5}{(k+1)(k-1)}$

   (g) $\frac{8-10m}{(2m+1)(2m-1)}$

   (h) $\frac{2n-1}{(n-2)^2}$

3. (a) 3 (b) 3

   (c) 1 (d) 6

   (e) $\frac{4}{3}$ (f) $\frac{11}{42}$

   (g) $-\frac{1}{2}$

4. (a) $\frac{7}{6(a-b)}$

   (b) $\frac{2c-1}{2(3c-7)}$

   (c) $\frac{12f+10d}{15(2f-d)}$

   (d) $\frac{5u+7}{6(u-4)}$

   (e) $\frac{7m-1}{6(3n-2)}$

   (f) $\frac{5h+7k}{8(p-q)}$

   (g) $\frac{3x^2}{2(x-y)}$

   (h) $\frac{10-21x}{6(z-2y)}$

5. (a) $\frac{24a^2-23a}{(3a-5)(4a-1)}$

   (b) $\frac{8b+5}{(2b+1)^2}$

   (c) $\frac{5-2h}{h(h-6)}$

   (d) $\frac{6m^2-25m+12}{m(m-4)(m-3)}$

   (e) $\frac{x^2+6xy-6y^2}{(x-y)(x+y)}$

(f) $\frac{3}{2z+3}$

6. (a) $\frac{2a+5}{(a+3)(a+1)}$

(b) $\frac{1-b^2-b}{(b+1)(b-6)}$

(c) $\frac{4p^2+4p+1}{2(p+1)(p-5)}$

(d) $\frac{4-2x^2-xy}{(x+y)(x+2y)}$

7. (a) $\frac{21}{2}$ (b) -1

(c) 3 (d) -1

(e) $\frac{3}{8}$

8. $\frac{y+6x}{6y}$

9. Albert

10. $P = 2a + 4$, $Q = a^2 + 3a + 2$, $R = 3$

11. $\frac{3}{5}$

Exercise 1C

1. (a) $y = \frac{k-ax}{b}$

(b) $n = \frac{PV}{RT}$

(c) $d = \frac{5b-3c}{2}$

(d) $a = \frac{R}{m} - g$

2. (a) $a = m(b + c)$

(b) $p = \frac{3}{2}(5q-r)$

(c) $k = \frac{a}{14}$

(d) $b = \frac{2A}{h} - a$

3. (a) $h = m^3 + k$

(b) $D = b^2 - 4ac$

(c) $V = \pm\sqrt{PR}$

(d) $\theta = \frac{360A}{\pi r^2}$

4. ± 4

5. (a) ± 1.15 (b) $\frac{14}{17}$

6. (a) $C = \frac{5}{9}(F-32)$

(b) $l = \frac{A-2\pi r^2}{\pi r}$

(c) $u = \frac{s}{t} - \frac{1}{2}at$

(d) $d = \frac{2S-2an}{n(n-1)}$

7. (a) $h = \frac{3-k}{k-2}$

(b) $z = \frac{y^2}{y-x}$

(c) $p = \frac{q^2}{x-q}$

(d) $b = \frac{a}{a-1}$

8. (i) $h = \frac{V}{\pi r^2} - \frac{2}{3}r$

(ii) 1.83

9. (a) $r = \sqrt[3]{\frac{3V}{4\pi}}$

(b) $u = \pm\sqrt{v^2 - 2as}$

(c) $x = p \pm \sqrt{y-q}$

(d) $z = \frac{t^2(m-3)}{4}$

10. (i) $b = \frac{c+a^2c}{a^2-3}$

(ii) 25 (iii) 3

11. (i) $l = g\left(\frac{T}{2\pi}\right)^2$

(ii) 0.0912 m

12. (i) $v = \sqrt{\frac{2(E-mgh)}{m}}$

(ii) 19.0 m s^{-1}

13. (a) $\frac{40}{7}$ (b) 126

(c) ± 11.5 (d) 0

14. (a) 2210 (b) 19.0

15. (i) (a) 17.6 (b) 688

(c) ± 27.5

(ii) Yes

Chapter 2 Quadratic Equations and Graphs

Practise Now 1

(a) 0 or 2 (b) 0 or -1

(c) 0 or -6 (d) 0 or $\frac{15}{2}$

Practise Now 2

1. (a) -5 or 7 (b) $-\frac{2}{3}$ or $\frac{5}{4}$

(c) -4 (d) $\frac{4}{3}$ or 2

(e) $\frac{5}{2}$ (f) -2 or -8

2. (i) $\frac{4}{3}$ or 2 (ii) $\frac{1}{3}$ or 1

3. (i) 6 (ii) -4

Practise Now 3

1. (a) $\pm\frac{2}{3}$ (b) $\pm\frac{1}{2}$

(c) $\pm\frac{1}{6}$ (d) ± 4

Practise Now 4

(a) -1 or -5 (b) $\frac{1}{3}$ or $\frac{2}{3}$

(c) -1 or $\frac{4}{3}$ (d) $-\frac{1}{5}$ or $-\frac{3}{4}$

Practise Now 5

1. 8, 10

2. 4, 9

Practise Now 6

Length = 6 cm, breadth = 4 cm

Practise Now 7

$A\left(-1\frac{1}{2}, 0\right)$, $B(2, 0)$, $C(0, 6)$

Practise Now 8

(a) 5

(c) (i) 3.5 (ii) 0.4 or 3.6

(iii) $y = 3$, $x = 2$

(d) $x = 2$

Practise Now 9

(b) (i) $(-0.75, 6.1)$

(ii) $x = -0.75$

Practise Now 10

1. (a) 28 m

(c) (i) 65 m; 1.75 m

(ii) 4.05 m

2. (iii) 3.025 m; 9 m

Practise Now 11

1. (i) 576 m (ii) 24 seconds

2. (ii) No (iii) 14.8 m

Practise Now 14

1. (i) $y = -(x-2)^2 - 2$

2. (i) $y = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$

3. (i) $y = -(x+3)^2$

Exercise 2A

1. (a) 0 or 9 (b) 0 or -7

(c) 0 or -5 (d) 0 or $\frac{1}{11}$

(e) 0 or -4 (f) 0 or $\frac{1}{27}$

(g) 0 or $-\frac{3}{2}$ (h) 0 or 5

2. (a) 4 or 9 (b) 3 or -5

(c) -1 or -2 (d) $\frac{6}{7}$ or $\frac{5}{4}$

(e) $-\frac{3}{5}$ or 2 (f) $-\frac{5}{2}$ or $\frac{5}{8}$

3. (a) -3 or -7 (b) 7 or 9

(c) $-\frac{4}{3}$ or -15

(d) $\frac{5}{6}$ or 4

(e) -9 or 3 (f) -3 or $\frac{2}{3}$

(g) 8 (h) -6

(i) $-\frac{7}{5}$ (j) $\frac{3}{2}$

4. (a) ± 4 (b) ± 5

(c) ± 11 (d) $\pm\frac{1}{10}$

(e) $\pm\frac{8}{3}$ (f) $\pm\frac{1}{3}$

5. (a) -3 or $\frac{1}{2}$ (b) $\frac{3}{2}$ or 6

(c) -7 or 5 (d) -8 or 15

(e) -3 or 7 (f) -1 or 3

6. 2

7. 3

8. 7, 8

9. 5, 12

10. 9, 18 or -18, -9

11. 0 or -3

12. (a) -12 or 5 (b) $-\frac{5}{4}$ or $\frac{3}{2}$

(c) $\frac{1}{2}$ or 5 (d) $-\frac{2}{3}$ or $\frac{4}{13}$

13. (i) $-\frac{3}{2}$ or $\frac{5}{3}$ (ii) $\frac{14}{3}$ or $\frac{3}{2}$

14. (a) $\pm\frac{10}{3}$

15. (a) $-\frac{1}{2}$ or 6 (b) -2 or 4

16. Length = 23 m, breadth = 9 m

17. 4 m

18. 5

19. 16 cm; 28 cm

20. (ii) ± 2.51 (iii) 10.0 cm

21. (ii) -5 or 4 (iii) 5 hours

22. $y = \frac{2}{3x}$

23. (i) 7 (ii) 2

24. $-\frac{21}{2}$ or 1

25. 3 and 12

Exercise 2B

1. $A(-6, 0)$, $B(-1, 0)$, $C(0, 6)$
2. (a) $a = 0$, $b = -5$
   (c) (i) 2.45 or −4.45
   (ii) −9
   (d) $x = -1$
3. (a) $p = -7$, $q = -3$
   (c) (i) (−0.75, 3.1)
   (ii) $x = -0.75$
4. (i) $A(-5, 0)$, $B(4, 0)$, $C(0, 20)$
   (ii) 8
5. (b) (i) (−0.5, 10.3)
   (ii) $x = -0.5$
   (c) (ii) −3.45 or 2.45
6. (i) $y = 8x(8 - x)$
   (iii) \$4
7. (iii) 4.5 cm, 4.5 cm
8. (i) 20 m (ii) 2 seconds
9. (b) (i) −0.4 or 2.4
   (ii) −1
   (c) $x = 1$ (d) 0 or 3
10. (ii) \$5000, 1 year
11. (iii) 8 m
12. (i) 11.25 m
    (ii) 3.4 seconds
    (iii) 0.75 m
13. (i) 110.25 m
    (ii) 8 seconds
    (iii) 1 and 7 seconds

Exercise 2C

3. (i) $x\left(x+\frac{3}{4}\right)$
5. (i) $(x + 3)(x - 2)$
7. (a) (i) $y=-\left(x+\frac{7}{2}\right)^2-\frac{11}{4}$
   (b) (i) $y=\left(x-\frac{3}{2}\right)^2+\frac{7}{4}$
8. (a) (i) $y = -(x + 5)^2$
   (b) (i) $y = (x - 4)^2$
9. (a) (i) $y = -(x - 3)^2 + 3$
   (b) (i) $y = (x - 4)^2 - 11$
10. (i) $h=-\frac{1}{2}, k=\frac{3}{4}$

Chapter 3 Quadratic and Fractional Equations

Practise Now 1

(a) −1 or 8
(b) −4 or $\frac{5}{2}$

Practise Now 2

(a) 3 or −17
(b) 4.16 or 0.842

Practise Now 3

(a) $(x + 6)^2 - 36$
(b) $\left(x-\frac{7}{2}\right)^2-\frac{49}{4}$
(c) $(x + 0.8)^2 - 0.64$
(d) $\left(x-\frac{3}{8}\right)^2-\frac{9}{64}$

Practise Now 4

1. (a) $(x + 7)^2 - 44$
   (b) $(x + 3.5)^2 - 13.45$
   (c) $\left(x-\frac{9}{2}\right)^2-\frac{69}{4}$
   (d) $\left(x-\frac{3}{5}\right)^2-\frac{109}{25}$
2. (a) $-(x - 3)^2 + 7$
   (b) $-(x - 4.5)^2 + 16.75$
   (c) $-\left(x+\frac{7}{2}\right)^2+\frac{69}{4}$
   (d) $-\left(x+\frac{2}{9}\right)^2-\frac{77}{81}$

Practise Now 5

1. (a) 0.61 or −6.61
   (b) −0.81 or −6.19
   (c) 1.62 or −0.62
2. 4.72 or −5.72

Practise Now 6

(a) 1.27 or −2.77
(b) 1.72 or −0.117
(c) 1.47 or −1.14
(d) 0.140 or −7.14
(e) 4.73 or 1.27
(f) 5.24 or 0.764

Practise Now 7

(a) 2.35 or −0.851
(b) 4.31 or 2.58
(c) −1 or −6
(d) 11.9 or 0.420

Practise Now 8

(a) 2.59 or −5.79
(b) 3.91 or 2.09
(c) 2.28 or 0.219
(d) −0.370 or −3.38

Practise Now 9

1. (i) 15, 5, −1, −3, −1, 5, 15
   (iii) −0.2 or 2.2
2. −2.3 or 1

Practise Now 10

1. (i) 16, 9, 4, 1, 0, 1, 4, 9
   (iii) 3
2. 4

Practise Now 11

(i) $(9 - x)$ m
(iii) 7.928 or 1.072
(v) 29.7 m^3

Practise Now 12

(ii) 126.16 or −133.16
(iii) 4 hours 30 minutes

Practise Now 13

1. Yes
2. (ii) 7.5

Exercise 3A

1. (a) −9 or 2 (b) $-\frac{7}{2}$ or 1
   (c) $\frac{3}{5}$ or 5 (d) $\pm\frac{7}{2}$
2. (a) 2 or −4 (b) $\frac{3}{2}$ or $-\frac{5}{2}$
   (c) $\frac{13}{5}$ or −1 (d) $\frac{25}{12}$ or $\frac{31}{12}$
   (e) 0.32 or −6.32
   (f) 3.90 or −0.90
   (g) 2.35 or 7.65
   (h) −2.66 or 3.66
3. (a) $(x + 10)^2 - 100$
   (b) $\left(x-\frac{15}{2}\right)^2-\frac{225}{4}$
   (c) $\left(x+\frac{1}{4}\right)^2-\frac{1}{16}$
   (d) $\left(x-\frac{1}{9}\right)^2-\frac{1}{81}$
   (e) $(x + 0.1)^2 - 0.01$
   (f) $(x - 0.7)^2 - 0.49$
   (g) $-(x + 5)^2 + 25$
   (h) $-\left(x-\frac{11}{2}\right)^2+\frac{121}{4}$
4. (a) $(x - 3)^2 - 8$
   (b) $\left(x+\frac{3}{2}\right)^2-\frac{17}{4}$
   (c) $(x + 4.5)^2 - 22.05$
   (d) $\left(x-\frac{1}{7}\right)^2+\frac{342}{49}$
   (e) $-(x - 5)^2 + 23$
   (f) $-\left(x-\frac{13}{2}\right)^2+\frac{143}{4}$
   (g) $-(x + 4.5)^2$
   (h) $-\left(x+\frac{3}{8}\right)^2+\frac{201}{64}$
5. (a) 1.45 or −3.45
   (b) 11.20 or 0.80
   (c) 5.85 or −0.85
   (d) 1.61 or −1.86
   (e) 0.81 or 0.05
   (f) 0.74 or −1.34
6. (i) $\left(x+\frac{17}{2}\right)^2-\frac{409}{4}$
   (ii) 1.6 or −18.6
7. (a) 0.618 or −1.62
   (b) 4.79 or 0.209
   (c) 2.32 or −4.32
   (d) 7.80 or −1.80
8. $\frac{a \pm \sqrt{a^2+24}}{2}$

Exercise 3B

1. (a) −0.268 or −3.73
   (b) 0.155 or −2.15
   (c) 3.36 or −1.69
   (d) 0.922 or −3.25
   (e) 3.19 or 0.314
   (f) 1.77 or 0.225
2. (a) 2.72 or −7.72
   (b) 1.96 or −0.764
   (c) 1.07 or −0.699
   (d) −0.5
   (e) 1.67 or −1.07
   (f) 6.27 or 9.73
3. (a) 1.77 or −2.27
   (b) 8.14 or 0.860
   (c) 0.268 or 3.73
   (d) $-\frac{2}{3}$
   (e) 5.54 or −0.541
   (f) −0.382 or −2.62
4. (a) 0.618 or −1.62
   (b) 2.70 or −0.370
   (c) 3.73 or 0.268
   (d) 4.54 or −1.54
   (e) 1.5
   (f) 0.468 or −0.468
5. (a) $-\frac{1}{3}$
   (b) 15.6 or 1.41

6. 10.6 or –0.141
7. (a) 2 or –4
(b) 2.70 or –3.70
(c) 11.8 or 1.74
(d) $\frac{2}{3}$
(e) 7.84 or –0.510
(f) 6.43 or –2.43
(g) 5.14 or 1.36
(h) 7.16 or 0.838
8. (a) $\frac{1}{2}$ or $-\frac{1}{3}$
(b) 5.12 or –3.12
(c) 7.54 or 1.46
(d) –0.825 or –2.43

Exercise 3C

1. (i) 8, 1, –2, –1, 4, 13
(iii) 2.3 or 0.2
2. (i) –5, 5, 9, 7, –1, –15
(iii) –2.55 or 0.9
3. (i) 4, 1, 0, 1, 4, 9
(iii) –3
4. (ii) –2.1 or 0.8
5. –3.45 or 1.45
6. (ii) –1.5
7. 5

Exercise 3D

1. (i) $(56 - x)$ cm
(iii) 41.67 or 14.33
(iv) 27
2. (i) $\frac{60}{x}$ (ii) $\frac{60}{x+2}$
(iv) $\frac{1}{2}$ or $-\frac{5}{3}$
(v) 6 minutes
3. (i) $\frac{600}{x}$ kg (ii) $\frac{600}{x-0.4}$ kg
(iv) \$2.12
4. 6.25 m
5. (i) $(8x^2 + 20x)$ cm²
(iii) 1.55 or –4.05
(v) 9.61 m²
6. (i) $\left(\frac{2}{x}+\frac{8}{x+1}\right)$ hours
(iii) 3.53 or –4.53
(iv) 10.20 a.m.
7. (ii) 88.08 or 11.92
(iii) 7.95 hours
8. (i) $\frac{1500}{x}$ minutes
(ii) $\frac{1500}{x+50}$ minutes
(iv) 363.10 or –413.10
(v) 3 minutes 38 seconds
9. (i) US$\left(\frac{2000}{x}\right)$
(ii) US$\left(\frac{1000}{x+0.05}\right)$
(iv) 1.3599 or –0.0337
(v) US\$1 = S\$1.41
10. 15 km/h
11. (ii) (a) 202.5 cm
(b) 5.6 m
(iii) 6.4

Chapter 4 Indice, Surds, Exponential Growth and Decay, and Standard Form

Practise Now 1

(a) 7^7 (b) $(-3)^6$
(c) a^{20} (d) $6x^6y^7$

Practise Now 2

(a) 9^4 (b) $(-4)^7$
(c) a^4 (d) $3x^3y$

Practise Now 3

1. (a) 6^{12} (b) k^{45}
(c) $(-4)^{8p}$ (d) 3^{7q}
2. 2

Practise Now 4

(a) 24^7 (b) $125b^{12}$
(c) $-32c^{10}d^{25}$ (d) $-h^2k^{16}$

Practise Now 5

(a) 3^3 (b) $\frac{a^3}{4^3}$
(c) pq^4 (d) x^{11}

Practise Now 6

1. (a) 1 (b) 1
(c) 3 (d) 1
2. (a) 3 (b) 10

Practise Now 7

1. (a) $\frac{1}{36}$ (b) $-\frac{1}{8}$
(c) $\frac{125}{64}$ (d) $\frac{1}{9}$
2. $\frac{d^2}{e^4}$

Practise Now 8

1. (a) a^4 (b) $\frac{2d}{e^2}$
(c) $\frac{6}{g^2}$
2. $-2h^4$

Practise Now 9

(a) 4 (b) 4
(c) $\frac{4}{5}$ (d) $\frac{2}{3}$

Practise Now 10

(a) 6 (b) 5
(c) $\frac{1}{3}$ (d) $-\frac{1}{10}$

Practise Now 11

1. (a) 16 (b) $\frac{1}{8}$
(c) 1000 (d) 113
2. (a) $a^{\frac{n}{3}}$ (b) $x^{-\frac{2}{5}}$

Practise Now 12

1. (a) m^2 (b) $\frac{1}{m^{\frac{7}{15}}}$
(c) $\frac{m}{n^{\frac{5}{3}}}$ (d) $\frac{10y^3}{x^3}$
(e) $\frac{h^{\frac{11}{3}}}{k^{\frac{11}{12}}}$ (f) $\frac{5p^7}{q^{\frac{14}{5}}}$
2. $\frac{9}{16}$

Practise Now 13

1. (a) 3 (b) –2
(c) $\frac{4}{3}$
2. 4

Practise Now 14

(a) 9 (b) 5
(c) $2\sqrt{3}$ (d) $\frac{\sqrt{15}}{2}$

Practise Now 15

(a) $11\sqrt{3}$ (b) $2\sqrt{5}$
(c) $-\sqrt{6}$

Practise Now 16

(a) $29+3\sqrt{3}$ (b) $34-24\sqrt{2}$
(c) –11 (d) $86+48\sqrt{3}$

Practise Now 17

1. (a) $4\sqrt{3}$ (b) $8-2\sqrt{5}$
(c) $\frac{2\sqrt{6}+3}{3}$ (d) $\frac{6\sqrt{3}-48}{11}$
2. $16-10\sqrt{6}$
3. $h=\frac{47}{2}$, $k=\frac{21}{2}$

Practise Now 18

1. $\left(\frac{19}{11}-\frac{6}{11}\sqrt{3}\right)$ cm
2. $(33-18\sqrt{3})$ cm

Practise Now 19

\$141 000

Practise Now 20

\$12 500

Practise Now 21

1. (a) 5.3×10^6
(b) 6×10^8
(c) 4.8×10^{-5}
(d) 1.67×10^{-10}
2. (a) 1 325 000
(b) 0.0044

Practise Now 22

(a) 4.0×10^{12} bytes
(b) 2.54×10^{-5} m
(c) 4.94×10^{-3} m

Practise Now 23

(a) 5.20×10^9 (b) 1.09×10^{-1}
(c) 4×10^3 (d) 2.5×10^{-13}
(e) 1.60×10^5 (f) 2.56×10^6
(g) 6.57×10^{-9} (h) 4.21×10^4

Practise Now 24

800

Exercise 4A

1. (a) 2^{10} (b) $(-4)^{11}$
(c) x^{11} (d) $24y^9$
2. (a) 5^5 (b) $(-7)^7$
(c) $6x^4$ (d) $-3y^5$
3. (a) 3^6 (b) 2^4
4. (a) 9^8 (b) h^{10}
(c) 18^5 (d) 15^{14}
(e) $8k^{18}$ (f) $81x^{24}y^8$

5. (a) 2^{13} (b) 3^{20}
(c) $\frac{m^5}{2^5}$ (d) $\frac{3^5}{n^6}$
(e) $\frac{p^{24}}{q^6}$ (f) $\frac{x^4}{y^8}$

6. (a) $h^{13}k^{10}$ (b) $-4m^{18}n^{12}$
(c) $22p^9q^{17}$ (d) h^4k^2
(e) $5m^6n^6$ (f) $5x^4y$

7. (a) a^{11} (b) b^{41}
(c) $-c^{28}$ (d) $\frac{9d^3}{8}$
(e) e^7f^6 (f) $-8k^9$

8. 1

9. (a) $8a^9b^9$ (b) $25c^4d^8$
(c) $64ef^3$ (d) $-2gh$

10. (a) $\frac{2a^4}{b^5}$ (b) $\frac{c^{12}}{8d^9}$
(c) $3e^3f^5$ (d) $-\frac{27g^9h^3}{8}$

11. (a) $\frac{5x^6y^8}{2}$ (b) $\frac{32x^8y^2}{9}$
(c) $4xy^7$ (d) $2x^2y^2$

12. $a = 2, b = 6$

Exercise 4B

1. (a) 1 (b) 1
(c) 4 (d) –8
(e) 1 (f) 7

2. (a) 16 (b) 7
(c) –63 (d) 25

3. (a) $\frac{1}{8}$ (b) $-\frac{1}{5}$
(c) $\frac{16}{9}$ (d) $\frac{3}{5}$

4. (a) 1 (b) $\frac{24}{25}$
(c) $\frac{8}{3}$ (d) 16

5. (a) 14 (b) 5
(c) $\frac{1}{2}$ (d) $\frac{2}{3}$

6. (a) 9 (b) –3
(c) $\frac{1}{2}$ (d) $-\frac{1}{7}$

7. (a) 36 (b) $\frac{1}{32}$
(c) 64 (d) 10 000
(e) $26\frac{26}{27}$ (f) 100 125

8. (a) $a^{\frac{1}{4}}$ (b) $b^{\frac{4}{5}}$
(c) $c^{\frac{2n}{3}}$ (d) $d^{-\frac{1}{n}}$
(e) $2e^{-\frac{1}{2}}$ (f) $f^{-\frac{5}{3}}$

9. 0.0081

10. (a) 3 (b) –7
(c) $\frac{5}{2}$ (d) –2

11. (a) –4 (b) 8
(c) 8

12. (a) $\frac{b^5}{c^2}$ (b) $\frac{5f^4}{3}$
(c) $\frac{d^6}{c^3}$

13. (a) $15a^9$ (b) $-\frac{8}{3}$
(c) $\frac{2}{e^{24}}$ (d) $\frac{144}{h^6}$
(e) $\frac{k^{12}}{j^{12}}$ (f) m^3n

14. (a) a^3 (b) $61p^3$

15. (a) $a^{\frac{5}{6}}$ (b) $b^{\frac{17}{10}}$
(c) $\frac{1}{c^{\frac{6}{5}}}$ (d) $\frac{m^{\frac{1}{5}}}{n^6}$
(e) $\frac{p}{q^{\frac{6}{5}}}$ (f) $\frac{2y^2}{3x^3}$

16. (a) $\frac{a^{\frac{4}{3}}}{b^{\frac{3}{2}}}$ (b) $\frac{c^{10}}{d^{\frac{16}{5}}}$
(c) $\frac{e^{\frac{11}{3}}}{f^{\frac{11}{12}}}$ (d) $\frac{5g}{h}$
(e) $\frac{j^2k}{h^3}$ (f) $\frac{m^{\frac{56}{5}}n^{\frac{3}{5}}}{2}$

17. (a) $\frac{z^{13}}{x^{53}y^4}$ (b) $\frac{x^2z^{11}}{y^{10}}$
(c) $\frac{ac^{2n+1}}{b^3}$ (d) $\frac{a}{c(a+b)^3}$

Exercise 4C

1. (a) 8 (b) 7
(c) $3\sqrt{7}$ (d) 15

2. (a) $6\sqrt{7}$ (b) $\sqrt{5}$
(c) $3\sqrt{3}$ (d) $-2\sqrt{11}$
(e) $-2\sqrt{15}$ (f) $\frac{1}{2}$

3. (a) $24-9\sqrt{2}$
(b) $33+12\sqrt{6}$
(c) $115-24\sqrt{11}$
(d) $210-42\sqrt{21}$
(e) 61
(f) –17

4. (a) $2\sqrt{5}$ (b) $\frac{\sqrt{10}}{30}$
(c) $14-7\sqrt{3}$ (d) $\frac{72+9\sqrt{6}}{58}$
(e) $\frac{5+10\sqrt{3}}{22}$ (f) $\frac{8\sqrt{7}-12}{19}$

5. (a) $\frac{4\sqrt{3}-\sqrt{15}}{22}$
(b) $\frac{8\sqrt{5}-36}{11}$

6. (a) $\frac{23\sqrt{2}}{12}$
(b) $\frac{64\sqrt{3}-27\sqrt{2}}{36}$
(c) $\frac{10}{3}$ (d) $-\frac{23}{2}$

7. (a) $\frac{19-8\sqrt{3}}{13}$ (b) $\frac{76+6\sqrt{2}}{289}$
(c) $2\sqrt{2}-3$ (d) $\frac{16-7\sqrt{6}}{19}$

8. $75-17\sqrt{3}$

9. $6\sqrt{2}$

10. $(4+2\sqrt{6})$ cm

11. $(3-\sqrt{3})$ cm

12. (i) $3+2\sqrt{2}$ (ii) $4\sqrt{2}$

13. $a = \frac{199}{4}$, $b = 15, c = 1$

Exercise 4D

1. 3200

2. 4740

3. (i) $P = 1000(1.01)^4$
(ii) 1350

4. (i) \$2000 (ii) \$5190

5. (i) \$25 000 (ii) 14%

6. (i) 30 (ii) 52
(iii) 2028

7. (i) 4800 g (ii) 0.0732 g
(iii) 9 hours

8. 40 895

Exercise 4E

1. (a) 8.53×10^4
(b) 5.27×10^7
(c) 2.3×10^{-4}
(d) 9.04×10^{-8}

2. (a) 9600 (b) 400 000
(c) 0.000 28 (d) 0.000 001

3. 7×10^6 g

4. (i) 3×10^2 MHz
(ii) 3×10^5 MHz

5. (i) 7×10^{-11} m
(ii) 7.4×10^{-11} m
(iii) 35 : 37

6. 3.94×10^3%

7. (a) 1.67×10^2
(b) 1.41×10^{-8}
(c) 3.35×10^{-1}
(d) 3.33×10^3
(e) 3.36×10^4
(f) 3.04×10^7
(g) 1.53×10^{-1}
(h) 3.35×10^{-5}

8. (a) 2.46×10^{-12}
(b) 6.94×10^7
(c) 1.1×10^4
(d) 2.1×10^2
(e) 5.41×10^{-2}
(f) 1.99×10^5

9. (a) 3.15×10^9
(b) 4.5×10^4

10. 7.6×10^{-3}

11. (a) 1.6×10^{14}
(b) 6.4×10^{-2}

12. 3.33×10^{-7}

13. (i) 3×10^8 m/s
(ii) 43 minutes 15 seconds

14. (i) 1.44×10^6 km
(ii) 400 days

15. (i) 1.25×10^8
(ii) 2.15
(iii) 1.98

Chapter 5 Coordinate Geometry

Practise Now 1

(a) 5 units
(b) 11.4 units
(c) 6 units

Practise Now 2

(a) $\left(0, 2\frac{1}{4}\right)$ (b) $(-3, 0)$

$3\frac{3}{8}$ units^2

Practise Now 3

1. $\angle DEF$
2. No

Practise Now 4

(a) $\frac{2}{3}$ (b) –1
(c) 0

Practise Now 5

1. 12
2. 0 or -7

Practise Now 6

1. 7
2. 32

Practise Now 7

(a) $y=\frac{2}{7}x+\frac{11}{7}$ (b) $y=4$

(c) $x=-3$

Practise Now 8

(a) (i) $(2, 5)$ (ii) $(4, 3)$

(iii) $(1, 1)$

(b) $p=-4$, $q=3$

Practise Now 10

1. (i) $D(-1, 4)$ (ii) 4.24 units
2. (i) $\left(2\frac{5}{6}, 2\frac{5}{6}\right)$ (ii) 1.70 units

Practise Now 11

(i) -13 (ii) 2 or 3

Practise Now 12

1. No
2. (i) $M(3, 2)$ (ii) $\frac{3}{2}$

Practise Now 13

$p=2$, $q=-10$

Practise Now 14

1. $y=2x+2$; $y=-\frac{1}{2}x+3$
2. (a) $y=-x+9$
   (b) $y=2x+7$
   (c) $3y=-2x-19$
   (d) $y=3x+13$

Practise Now 15

$y=-2x+2$

Practise Now 16

(i) l_{DC}: $y=\frac{1}{2}x+\frac{15}{2}$,

l_{BC}: $y=-2x+40$

(ii) $C(13, 14)$ (iii) 8.94 units

Exercise 5A

1. (a) 8.06 units
   (b) 8.54 units
   (c) 11.4 units
   (d) 10.8 units
2. ± 7.07
3. (a) $\left(0, -8\frac{1}{3}\right)$
   (b) $\left(2\frac{3}{11}, 0\right)$
4. $\left(0, -3\frac{21}{26}\right)$
5. (i) 32 units; 48 units²
   (ii) 9.6 units
6. (i) 4.5 units²
   (ii) 5.83 units
   (iii) $(0, 4)$
   (iv) -5 or 11
7. (i) $AB=3$ units, $BC=4.12$ units, $AC=4.47$ units
   (ii) 6 units²
   (iii) -8 or 6
8. -2 or 1
9. (ii) 9 units²
10. 2.57 units
11. 3.48 units

Exercise 5B

1. (a) $-\frac{1}{2}$ (b) -10
   (c) $-\frac{4}{3}$ (d) -3
   (e) $\frac{11}{4}$ (f) 0
2. Gradient of $AB=0$,
   Gradient of $AE=\frac{1}{2}$,
   Gradient of $DC=-\frac{5}{6}$,
   Gradient of $DE=-\frac{1}{6}$.
3. $-2\frac{4}{5}$
4. $1\frac{5}{6}$
5. -6 or 3
6. 9
7. 3
8. -1 or $1\frac{1}{2}$
9. (i) Gradient of $AB=-\frac{5}{2}$,
   Gradient of $BC=\frac{2}{5}$,
   Gradient of $CD=-\frac{5}{2}$,
   Gradient of $AD=\frac{2}{5}$.
   (ii) They are equal.

Exercise 5C

1. 3
2. 15
3. (a) $y=-x$
   (b) $y=2x+1$
   (c) $y=\frac{1}{4}x+\frac{7}{2}$
   (d) $y=\frac{9}{10}x+\frac{2}{5}$
   (e) $y=-x-6$
   (f) $y=\frac{2}{3}x-\frac{1}{3}$
   (g) $y=0$
   (h) $x=0$
4. (a) $y=\frac{1}{3}x$
   (b) $y=3x-2$
   (c) $y=-3x+1$
   (d) $y=-\frac{1}{2}x+\frac{19}{2}$
   (e) $y=4$
   (f) $y=ax+a$
5. $y=2x$
6. (a) 0, 1; $y=1$
   (b) undefined, not applicable; $x=1.5$
   (c) 1, -1; $y=x-1$
   (d) $-\frac{1}{2}$, 1; $y=-\frac{1}{2}x+1$
7. (i) 4 units²
   (ii) $\frac{1}{2}$ (iii) $y=x$
8. -4 or 3
9. (i) $y=-\frac{2}{3}x+2$
   (ii) $-\frac{2}{3}$
   (iii) $(3, 0)$
10. (i) $y=-\frac{2}{3}x+3$
    (ii) 0
11. $y=\frac{5}{2}x-\frac{19}{2}$
12. (i) $y=3x-8$
    (ii) $(4, 4)$
13. (i) $(-6, 0)$
    (ii) $\left(2, -6\frac{2}{3}\right)$
    (iii) $y=-\frac{5}{6}x+\frac{3}{2}$
    (iv) $y=-1$
14. (a) (i) 3
    (ii) $y=3x+3$
    (b) $(6, 3)$
15. 0; $n=0$
16. (i) $A(8, 0)$, $B(0, 6)$
    (ii) 10 units
    (iii) $y=x$
    (iv) $\left(3\frac{3}{7}, 3\frac{3}{7}\right)$
17. (i) $(-8.5, 0)$
    (ii) $(2, 7)$
    (iii) 7.28 units
    (iv) 3.85 units

Exercise 5D

1. (a) $(4, 2)$ (b) $\left(\frac{1}{2}, 2\frac{1}{2}\right)$
   (c) $(6, 4)$ (d) $(0, 2)$
   (e) $\left(\frac{1}{2}a+b, \frac{1}{2}a+b\right)$
   (f) $\left(\frac{ah^2+ak^2}{2}, ah+2ak\right)$
2. (a) $Q(15, 0)$ (b) $Q(3, -4)$
   (c) $Q(11, 3)$ (d) $Q(-4, -5)$
3. (i) $(4, -1)$ (ii) $(4, -11)$
5. $S(3, 2)$
6. $(-2, 0)$
7. $\left(-\frac{1}{6}, 2\frac{7}{12}\right)$; 1.86 units
8. (b) $\left(\frac{h+m-2k}{2}, \frac{n-k}{2}\right)$
9. (i) $R\left(4\frac{3}{5}, 0\right)$ (ii) No
10. $P(-11, -3)$, $Q(7, 9)$, $R(3, -11)$

Exercise 5E

1. (i) $\pm\frac{1}{3}$ (ii) ± 1
2. $\frac{26}{3}$
3. 1
4. (i) 17 (ii) -9
5. (a) Parallel
   (b) Perpendicular
   (c) Perpendicular
   (d) Neither
6. $m=1$, $n=5$
7. (i) $P(4, 2)$
8. Yes
9. Yes
10. (a) $C(4, 0)$ or $C(8, 0)$
    (b) $D(0, 24)$
11. (a) Perpendicular
    (b) Parallel
13. (i) 1 (ii) $\frac{4}{7}$
    (iii) 29.7°

14. $a = 2, b = 3, c = 3, d = -2$

15. Yes

16. Yes

Exercise 5F

1. (a) $y = 3x + 14$, $y = -\frac{1}{3}x + 5$
   (b) $y = -\frac{1}{2}x + \frac{1}{2}$; $y = -\frac{1}{2}x - 4$
2. (a) $y = 2x - 5$
   (b) $y = -10x + 6$
   (c) $y = 3$ (d) $x = -9$
3. (a) $y = 5$ (b) $x = -1$
   (c) $y = -3x + 20$
   (d) $y = \frac{1}{2}x - 4$
4. $y = -1$
5. (a) $y = -x$ (b) $y = \frac{1}{5}x + \frac{26}{5}$
   (c) $x = 3$ (d) $y = \frac{5}{2}$
6. $y = \frac{1}{2}x - 3$
7. $26y + 13x - 25 = 0$
8. $y = -2x + 17$; $y = \frac{1}{2}x + 2$
9. (i) $y = \frac{5}{2}x - \frac{27}{4}$
   (ii) $B\left(2\frac{7}{10}, 0\right)$
   (iii) $D\left(4\frac{3}{10}, 4\right)$
10. (i) $y = -\frac{4}{3}x + \frac{16}{3}$
   (iii) $D(5, 7)$ (iv) 50 units^2
11. (i) $y = -2x + 7$
   (ii) $P(2, 3)$
   (iii) $AP = 2.24$ units, $BC = 4.47$ units, $AC = 7.07$ units
   (iv) 5 units^2 (v) 1.41 units
12. (i) $m_{BC} = -1$, $m_{AB} = \frac{2}{9}$
   (ii) $H(0, 9)$ (iii) $D(-8, 6)$
   (iv) $y = x + 2$
13. (a) $y = -x + 8$
   (b) $y = \frac{1}{5}x + \frac{16}{5}$; (4, 4)
14. (i) $y = \frac{1}{2}x + \frac{27}{4}$
   (ii) 15.1 units
   (iii) $R\left(14\frac{1}{2}, 14\right)$

Chapter 6 Graphs of Functions and Graphical Solution

Practise Now 1

(a) −17.5 (b) −2.55

Introductory Problem Revisited

(a) No (b) 11.5

Practise Now 2

(a) 29

(c) (ii) 1
(iii) $A = 2, B = -3$

Practise Now 5

(a) (i) 1.2 (ii) −2.5

(b) (i) $y = 10x + 1$
(ii) 0.5 or −0.6

Practise Now 7

(a) −0.9 (b) −0.8 or 0.8

Practise Now 8

(a) −8.6 (b) 6.6

Practise Now 9

(a) −0.8 (b) 3.1

Practise Now 10

(a) 0.25 (b) −0.325

Practise Now 12

1. (i) $y = 0$
   (iii) (a) −0.1
   (b) −0.8 or 0.4
2. (i) −3.3, −1.6, 1.2, 1.3, −14.9
   (ii) $x = 5$
   (iv) (a) −9.3
   (b) −4.5 or 3.5
3. (i) 0.8, 0.63, 0, −1.4, 7.86, 5, 4.2, 4, −4.33, −1.4, 0.73
   (ii) $x = -1, x = 3, y = 1$
   (iv) (a) −2.6
   (b) −4.3 or 6.3

Practise Now 13

(a) $a = 5, b = -3$

(c) 1.6

(d) (ii) $h = 2, k = -4$

Practise Now 14

(ii) 3.5 minutes

(iii) 27 km/h

Practise Now 15

(i) $2\frac{2}{3}$ m/s^2 or 2.67 m/s^2

(ii) 4.89 m/s (iii) 2 m/s^2

Practise Now 16

(i) $a = 2, b = 17$

(iii) (a) 0.65, 3.85
(b) 2.25 seconds
(c) 9 m/s^2
(d) $0.25 < t < 4.25$

Practise Now 17

(i) 60 beats/minute

(ii) 6 beats/minute^2

(iii) 1 beat/minute^2

Exercise 6A

1. (i) −3, 0.1, 0.4, −2.4, −2.1, 1, 7.1
   (iii) (a) −1.625
   (b) −0.55, 0.65, 2.9
3. (i) 8, 4, 1.3, 0.8
   (iii) (a) 1.1 (b) 2.65
5. (i) $a = 1.1, b = 0.4$
   (iii) (a) 1.3 (b) 1.5
6. (a) 13.6 (b) 4
7. (a) −2 (b) 2.3
8. (i) (a) 9.5 (b) 15
   (c) 27
   (ii) (a) 0.8 (b) 3.5
   (c) 4.15
10. (i) −1.8 (ii) 3.325
11. (i) $h = 2, k = 5.5$
   (iii) (a) −0.3 (b) 0.9
12. (i) 5.45 (ii) 1 or 2.75
13. (a) −1, −0.6, 1.6
   (b) (i) −1.55, −0.35, 1.9
   (ii) −1.5, −0.35 or 1.9
14. (b) (ii) −4, 2
   (iii) $A = -12, B = 16$
15. (i) 1.7
   (iii) No solution
16. (i) −4.2
   (ii) (a) 1.45 or 4
   (b) 1.6 or 5.8
   (c) 0.6 or 4.3
17. (i) $A = \frac{2600}{x} + 2 + 0.0001x$
   (ii) 5100 units; \$3.02

Exercise 6B

1. (i) 0.5, 8, 16, 32
   (iii) (a) 12.2 (b) −0.65
2. (i) 3, 4.2, 8.5
   (iii) (a) 4.9; 14.8
   (b) −0.275; 1.3
4. (i) $y = \frac{1}{2}$
   (iii) (a) −0.3
   (b) −0.8 or 0
5. (i) 1.47, 1.45, 1.4, 1.33, 1, 2.33, 2, 1.67, 1.57, 1.55
   (ii) $x = \frac{5}{2}, y = \frac{3}{2}$
   (iv) (a) 1.8
   (b) 2.4
6. (i) −0.22, −0.29, −0.4, −0.67, −0.8, −1, −2, −10, 10, 2, 1, 0.8, 0.4, 0.29, 0.18
   (ii) $x = 7, y = 0$
   (iv) (a) 0.4
   (b) 6.4
7. (i) $a = 2.5, b = 7.7$
   (iii) (a) 2.6; 8.5
   (b) 2.45
8. (a) 1.6
   (b) (i) (−1.175, 0.3)
   (ii) −1.175
10. 3
11. (i) −0.36, −0.44, −0.56, −0.75, −1, −0.99, 1.25, 15, 3, 1.78, 1.25, 0.56, 0.44
   (ii) $x = 3, y = 0$
   (iv) (a) −0.89
   (b) −0.5, 2.7 or 3.8
12. (a) 5, 8, 5 (c) 4
   (d) (ii) $h = 1, k = 9$
13. (ii) −14 (iii) 10
14. (a) (ii) −0.875
15. (iii) (1, 2)
16. (ii) 15 years
17. (i) $S = 0.85^x$ (iii) 1.75 hours
18. (i) −0.4, −0.8, −3.8, 1.4, 0.4, 0, −0.3, 1.4, 0.5, 0.2
   (ii) $x = 0, x = -4, y = 0$
   (iv) −5.3, −2.9, 0.1

Exercise 6C

1. (ii) 19.8 km/h
2. (i) 15 km
   (iii) 65 minutes
3. (i) 1 hour (ii) 30 km/h
   (iii) 34.3 km/h
4. (i) 5 m/s² (ii) 7.5 m/s
5. (ii) 7 m/s
6. (ii) 4 m/s (iii) 25.5
7. (ii) (a) 2.3 minutes
   (b) 0.462 km/min
   (c) 2.75 minutes
8. (ii) (a) 1 hour 11 minutes
   (b) 1 hour and 1 hour 22 minutes
9. (ii) (a) 1011 hours
   (b) 8.2 km
10. (i) 9 (ii) 30 m/s
11. (i) 1.5 m/s²
    (ii) 100 seconds
12. $6\frac{2}{3}$ m/s or 6.67 m/s
13. (i) $a = 4$, $b = 10$
    (iii) (a) 1.7, 5.3
    (b) 3.5 minutes
    (c) –3 m/min²
    (d) $2.4 \leqslant t \leqslant 4.6$
14. (ii) 9.5 m/s; 34 m/s
    (iii) 2.5 m/s²; 5 m/s²
15. (i) $h = 8$, $k = 10$
    (iii) (a) 2.85 hours
    (b) 10 km/h²
    (c) $1.65 \leqslant t \leqslant 4$
    (iv) 0.4
16. (i) 25 °C
    (ii) 4 °C/minute
    (iii) 150 (iv) 2 hours

Chapter 7 Volume, Surface Area, and Symmetry of Pyramids, Cones and Spheres

Practise Now 1

1. 84 cm³
2. 2 550 000 m³

Practise Now 2

9 m

Practise Now 3

504 m²

Practise Now 4

(i) 8 cm (ii) 117 cm³

Practise Now 5

1. 1140 cm³
2. 7 m

Practise Now 6

1. 44.0 cm²
2. 5.92 m

Practise Now 7

1. 427 m²
2. 500 cm³

Practise Now 8

1. 1890 g
2. 11.0 cm

Introductory Problem Revisited

Pyramid, cone, sphere, cylinder

Practise Now 9

1. 4.52 m²
2. 1960 cm²

Practise Now 10

5.64 cm

Practise Now 11

50 *l*

Practise Now 12

(a) (i) 15 300 cm³
    (ii) 3210 cm²
(b) (i) 31.2 cm
    (ii) 1092.5π cm²

Exercise 7A

1. 20 cm³
2. 46 cm³
3. $23\frac{1}{3}$ m³ or 23.3 m³
4. 7.5 cm
5. 5 m
6. 7824 cm²
7. (a) 4 (b) 6
8. (a) (i) 6 (ii) 4
   (b) (i) 4 (ii) 1
9. 1041.6 g
10. (i) 14.1 cm (ii) 636 cm³
11. $8\frac{1}{3}$ cm or 8.33 cm
12. Height = 5 cm, dimensions of base = 10 cm × 6 cm
13. (i) 6.75 cm (ii) 226 cm²
14. (i) 9.375 m (ii) 584 m²
15. 46.0 m³
17. $28\frac{8}{9}$ cm or 28.9 cm
18. *VA* is shorter than *VB*.
19. (i) 6.93 cm (ii) 60.3 cm³
20. (i) 4 (ii) 1
    (iii) 4

Exercise 7B

1. (a) 528 cm³
   (b) $256\frac{2}{3}$ cm³ or 257 cm³
   (c) 180 cm³ (d) 12 900 mm³
2. 24 m
3. $r = 5$, 38.4 cm
4. 3 cm
5. (a) 138 cm² (b) 1940 mm²
   (c) 3040 cm²
6. 14 mm
7. Radius = 15 cm, slant height = 6.22 cm
8. 16.0 m
9. Infinite
10. (i) Infinite (ii) 1
11. 29 cm
12. 8192
14. (i) 5 cm (ii) 39.3 cm²
15. 204 cm²
16. 1230 cm³
17. (i) 2710 mm³
    (ii) 1290 mm²
19. (i) 660 cm² (ii) 1360 cm³
20. 1540 m³

Exercise 7C

1. (a) 2140 cm³
   (b) 11 500 mm³
   (c) 268 m³
2. (a) 6.97 cm (b) 14.3 mm
   (c) 5.71 m (d) 9 cm
   (e) 7.20 mm (f) 2.25 m
3. (a) 1810 cm²
   (b) 1020 mm²
   (c) 113 m²
4. 462 cm²
5. (a) 4.09 cm (b) 24.0 mm
   (c) 15.9 m (d) 4 cm
   (e) 15.1 mm (f) 3.5 m
6. 13.5 cm
7. 709
8. 215 kg
9. 3 cm
10. 24.0 cm
11. 14.86 cm
12. 434 m²
13. 7240 cm³
14. (i) 182 cm²
    (ii) $2\frac{4}{15}$ cm or 2.27 cm
15. 2 : 1

Exercise 7D

1. 1980 m²
2. 1560 cm³
3. (i) 2260 cm³
   (ii) 902 cm²
4. (i) 32 300 cm³
   (ii) 5080 cm²
5. 42 *l*
6. (i) 352 m² (ii) 552 m³
7. 273 m³
8. 6.75 cm
9. (i) 84 cm (ii) 5635π cm²
10. (i) 44 400 cm³
    (ii) 7610 cm²

Chapter 8 Averages of Statistical Data

Practise Now 1

1.61 m

Practise Now 2

56

Practise Now 3

1. (i) 77 (ii) 9, 9
2. 168 cm
3. 12

Practise Now 4

9.48

Practise Now 5

(i) 200 (ii) $19 600

(iii) $98

Practise Now 6

35 minutes

Practise Now 7

1. 47.3 years old
2. 43.5 mm

Practise Now 8

(a) 15 (b) 17.3

Practise Now 9

1. (a) 17.5 (b) 8.4
2. Any number ≥ 7

Practise Now 10

1. 7 minutes
2. $5 < x \leq 10$
3. 3

Practise Now 11

(i) 60 cm and 110 cm

(ii) 60 cm

Practise Now 12

(a) PKR 3000

(b) $45 \leq x < 60$

(c) 1

(d) Caramel and hazelnut

Practise Now 13

(a) 5 (b) 3

(c) 1, 2, 3, 4, 5

Practise Now 14

0, 1, 1, 2, 3, 5

Practise Now 15

52 kg; 54 kg; 57.3 kg

Exercise 8A

1. 38.1
2. $35.86
3. 8
4. 54 kg
5. (i) 96 (ii) 13
6. $3\frac{4}{7}$ or 3.57
7. $51\frac{2}{3}$ kg or 51.7 kg
8. (i) 30 (ii) 60
   (iii) 2
9. 1.35
10. (a) 9 (b) $8.90
    (c) 4.85 years
11. (i) 128 (ii) 21
12. (i) 27 cm (ii) 42 cm
13. $1110
14. (a) 4; 5
    (b) (i) $37\frac{1}{2}$% or 37.5%
    (ii) 40%
15. (i) $6\frac{7}{15}$ or 6.47 hours
    (ii) 6.88 hours
16. (i) 28 cm (ii) $\frac{3}{4}$
17. (i) 123.52 minutes
    (ii) $\frac{29}{50}$
18. (i) 52.7 km/h
    (ii) 2 : 3
19. (i) 7; 0; 1; 1; 8; 3
    (ii) 22.55 million km
20. 11
21. (i) 172.1 hours
    (iii) 172.7 hours

Exercise 8B

1. (a) 5 (b) 2.8
   (c) 29.5 (d) 12.75
2. (a) 400
   (b) $15 < x \leq 20$
   (c) 70 (d) 5.5
3. (a) 3 (b) 7.7 and 9.3
   (c) No mode
4. (i) 27 °C
   (ii) 22 °C and 27 °C
5. (a) No mode
   (b) 30
   (c) White bread
6. (a) Blue and purple
   (b) $0 \leq x < 10$
   (c) $32 (d) Japan
7. 5
8. (i) Albert: 5; Bernard: $4\frac{2}{3}$
   (ii) Albert; No
   (iii) Albert: 3; Bernard: 4
   (iv) Albert: 2; Bernard: 6
   (v) Mode
9. (ii) 7; 7 (iii) 6; 8
   (iv) Mode
10. (a) (iii) $x = 12$, $y = 6$
    (b) (i) 7 (ii) 10
11. (a) 12
    (b) Possible values: 3, 4, 5
12. 1, 2, 2, 8, 10, 10, 16
13. Possible sets:
    −3, 5, 9, 9, 9, 13;
    −3, 7, 9, 9, 9, 11;
    −9, 9, 9, 9, 11, 13
14. 16 minutes; 16 minutes; 18 minutes
15. $40; $30; $25
16. (i) $x = 10$, $y = 4$
    (ii) (a) 2 (b) 2
    (iii) 1
17. (a) 7 (b) 7
    (c) 7
18. (i) 4 (ii) 2

Answer Keys

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